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Classification of the Lie bialgebra structures on the Witt and Virasoro algebras

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Abstract

We prove that all the Lie bialgebra structures on the one sided Witt algebra W_1, on the Witt algebra W and on the Virasoro algebra V are triangular coboundary Lie bialgebra structures associated to skew-symmetric solutions r of the classical Yang-Baxter equation of the form $r = a \wedge b$. In particular, for the one-sided Witt algebra $W_1 = Der\, k[t]$ over an algebraically closed field k of characteristic zero, the Lie bialgebra structures discovered in Michaelis (Adv. Math. 107 (1994) 365–392) and Taft (J. Pure Appl. Algebra 87 (1993) 301–312) are all the Lie bialgebra structures on W_1 up to isomorphism. We prove the analogous result for a class of Lie subalgebras of W which includes W_1. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let k be a field. The Witt algebra W over k can be identified with the Lie algebra of derivations on the Laurent polynomial algebra $k[t, t^{-1}]$. The one-sided Witt algebra W_1 is the Lie algebra of derivations on the polynomial algebra $k[t]$. One may write

$$e_n = t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}.$$

Then $\{e_n \mid n \in \mathbb{Z}\}$ is a basis for W and

$$[e_n, e_m] = (m - n)e_{n+m}$$

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for $n, m \in \mathbb{Z}$. Moreover, W_1 is then the subalgebra of W spanned by $\{e_n \mid n \geq -1\}$. The Virasoro algebra is a central extension of W spanned by an element c and by the elements L_n, $n \in \mathbb{Z}$ whose bracket is defined by

$$[L_n, c] = 0 \quad \text{and} \quad [L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m+n,0} \cdot c$$

for $m, n \in \mathbb{Z}$, where δ_{ij} is the Kronecker delta function and $char\, k \neq 2, 3$. The Virasoro algebra is in fact the universal central extension of W (see [6]).

The problem of classifying the Lie bialgebra structures on Witt algebra was motivated by Witten when he was studying the left invariant symplectic structures of the Virasoro group. In the paper [20], Witten studied the Poisson structures on the Virasoro group which in turn are related to the Lie bialgebra structures on the Virasoro algebra. He attempted to construct Lie bialgebra structures on $Vect_{\mathbb{R}}(S^1)$ by inverting 2-cocycles on $Vect_{\mathbb{R}}(S^1)$. The rigorous formulation of the construction on $Vect_{\mathbb{C}}(S^1)$, the complex vector fields on S^1, was done by Beggs and Majid [2]. By considering Gelfand–Fuks cocycles on the Lie algebra $Vect_{\mathbb{C}}(S^1)$, Grabowski constructed another Lie bialgebra structure on $Vect_{\mathbb{C}}(S^1)$ by applying the procedure described by Belavin and Drinfel'd in the finite dimensional case (cf. [1,7]). The choice of different function spaces for the Lie algebra of vector fields sheds a different light on the problem. In [11], Leitenberger considered the formal vector fields of the circle and obtained a class of Lie bialgebra structures on it. Related results are also studied independently by Kupershmidt and Stoyanov in a different context [10].

In [16,18], the sequence $\{e_0 \wedge e_n \mid n \geq -1\}$ in W_1 is shown to consist of solutions of the classical Yang-Baxter equation. Moreover, the Lie bialgebra structures associated to these solutions are distinct when characteristic $k = 0$ [18]. In this paper, we show that these Lie bialgebra structures on W_1 are all of the Lie bialgebra structures on W_1 up to isomorphism when k is algebraically closed of characteristic zero. In fact, the result holds for a class of Lie subalgebras of W. We will prove the following result in Section 9

Theorem 1.1. *Let $\mathfrak{g}$ be an infinite dimensional Lie subalgebra of W over an algebraically closed field of characteristic zero such that $e_0 \in \mathfrak{g}$ and $\mathfrak{g} \ncong W$ as Lie algebras. Let $\mathfrak{g}^{(n)}$ be the Lie bialgebra structure on $\mathfrak{g}$ associated to the solution $e_0 \wedge e_n$ of the CYBE for any $e_n \in \mathfrak{g}$. Every Lie bialgebra structure on $\mathfrak{g}$ is isomorphic to $\mathfrak{g}^{(n)}$ for some n with $e_n \in \mathfrak{g}$ where the Lie cobracket δ on $\mathfrak{g}^{(n)}$ is given by*

$$\delta(x) = e_0 \wedge [x, e_n] + e_n \wedge [e_0, x] \quad (x \in \mathfrak{g}).$$

However, a similar result does not hold for the Witt algebra W. There are many more Lie bialgebra structures on W distinct from those associated to the solutions $e_0 \wedge e_n$ of the CYBE ($n \in \mathbb{Z}$) [18]. It was proved in [16] that for every 2-dimensional subalgebra spanned by a, b, of a Lie algebra, $a \wedge b$ is a solution of the CYBE. The converse holds for Witt and Virasoro algebras when characteristic $k = 0$ [16]. We will prove the following theorem in Section 6.

Theorem 1.2. *Let $\mathfrak{g}$ be the one sided Witt algebra, Witt algebra or Virasoro algebra over a field of characteristic zero. Every Lie bialgebra structure on $\mathfrak{g}$ is a triangular coboundary Lie bialgebra associated to a solution r of the CYBE of the form $r=a\wedge b$ for some $a,b\in\mathfrak{g}$. In particular, for any solution $r\in\mathfrak{g}\wedge\mathfrak{g}$ of the CYBE, we have $r=a\wedge b$ for some $a,b\in\mathfrak{g}$.*

2. Definition and preliminary results

Throughout this paper k denotes a field of characteristic zero; and all vector spaces and tensor products are over k. If E is a k-vector space, we will denote by $\langle S\rangle$ the linear subspace spanned by $S\subseteq E$. Moreover, we will view $E\wedge E$ and $E\wedge E\wedge E$ as subspaces of $E\otimes E$ and $E\otimes E\otimes E$ in the natural way. In particular, $a\wedge b$ corresponds to $\frac{1}{2}(a\otimes b-b\otimes a)$. Let $U\mathfrak{g}$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. For $r=\sum_i a_i\otimes b_i\in\mathfrak{g}\otimes\mathfrak{g}$ define $C(r)\in U\mathfrak{g}\otimes U\mathfrak{g}\otimes U\mathfrak{g}$ to be

$$[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}]$$

where $r^{12}=r\otimes 1$, $r^{23}=1\otimes r$, $r^{13}=\sum_i a_i\otimes 1\otimes b_i$. It is clear that if $r\in\mathfrak{g}\wedge\mathfrak{g}$, $C(r)\in\bigwedge^3\mathfrak{g}$.

Definition 2.1. (i) An element $r\in\mathfrak{g}\otimes\mathfrak{g}$ is a solution to the classical Yang–Baxter equation (CYBE) if $C(r)=0$.

(ii) An element $r\in\mathfrak{g}\otimes\mathfrak{g}$ is a solution to the modified classical Yang–Baxter equation (MCYBE) if $C(r)$ is a $\mathfrak{g}$-invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ given by

$$[x,a\otimes b\otimes c]=[x,a]\otimes b\otimes c+a\otimes[x,b]\otimes c+a\otimes b\otimes[x,c].$$

Definition 2.2. A *Lie coalgebra* is a pair (M,δ), where M is a vector space and $\delta: M\to M\wedge M$ is a linear map satisfies the *co-Jacobi identity*

$$(id+\sigma+\sigma^2)\circ(1\otimes\delta)\circ\delta=0,$$

where σ is the permutation (123) in S_3 acting in the usual way on $M\otimes M\otimes M$. δ is called the *comultiplication* or *cobracket* of M. If (M_1,δ_1) and (M_2,δ_2) are Lie coalgebras, then a *a Lie coalgebra map* f from M_1 to M_2 is a linear map such that

$$\delta_2\circ f=(f\otimes f)\circ\delta_1.$$

For additional properties about Lie coalgebras, the reader may refer to [13–15].

Definition 2.3. A Lie algebra $\mathfrak{g}$ which is simultaneously a Lie coalgebra is called a *Lie bialgebra* if the cobracket $\delta\in Z^1(\mathfrak{g},\mathfrak{g}\wedge\mathfrak{g})$, where $\mathfrak{g}$ acts on $\mathfrak{g}\wedge\mathfrak{g}$ by the adjoint action $[x,a\wedge b]=[xa]\wedge b-[xb]\wedge a$. That means $\delta:\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}$ is a derivation. If $\delta\in B^1(\mathfrak{g},\mathfrak{g}\wedge\mathfrak{g})$, that is $\delta=dr$ for some $r\in\mathfrak{g}\wedge\mathfrak{g}$, $\mathfrak{g}$ is called a *coboundary Lie bialgebra* (cf. [3]), where $dr(x)=[x,r]$ for $x\in\mathfrak{g}$. Let $\mathfrak{g}_1,\mathfrak{g}_2$ be Lie bialgebras and $\phi:\mathfrak{g}_1\to\mathfrak{g}_2$ be a linear map. ϕ is called a *Lie bialgebra map* if ϕ is a Lie algebra map and a Lie coalgebra map.

There is a correspondence between the skew-symmetric solutions of MCYBE and the coboundary Lie bialgebra structures on a given Lie algebra. The result was stated in [3]. A complete proof of this result can be found in [18]. (see also [12,16]). We will state this as the following proposition.

Proposition 2.4. *Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \wedge \mathfrak{g}$. Then dr gives a Lie bialgebra structure on $\mathfrak{g}$ if and only if r is a solution of the MCYBE.*

By this proposition, we may denote by $\mathfrak{g}^{(r)}$ the Lie bialgebra structure on the Lie algebra $\mathfrak{g}$ for any solution r of the MCYBE in $\mathfrak{g} \wedge \mathfrak{g}$. The coboundary Lie bialgebra $\mathfrak{g}^{(r)}$ is said to be *triangular* if r is a solution of the CYBE.

Proposition 2.5. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras and $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra map. Let $r_1 \in \mathfrak{g}_1 \wedge \mathfrak{g}_1$ be a solution of the CYBE. Then $r_2 = (\phi \otimes \phi)(r_1)$ is a solution of the CYBE. Moreover, ϕ is a Lie bialgebra map from $\mathfrak{g}_1^{(r_1)}$ into $\mathfrak{g}_2^{(r_2)}$. If ϕ is surjective and r_1 is a solution of the MCYBE, then r_2 is also a solution of the MCYBE.*

Proof. Let $r_1 = \sum a_i \otimes b_i$. Then

$$\begin{aligned} 0 = C(r_1) &= [r_1^{12}, r_1^{13}] + [r_1^{12}, r_1^{23}] + [r_1^{13}, r_1^{23}] \\ &= \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + a_i \otimes [b_i, a_j] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j]. \end{aligned}$$

Acting $\phi \otimes \phi \otimes \phi$ on above equation, we have

$$\begin{aligned} 0 &= (\phi \otimes \phi \otimes \phi)(C(r_1)) \\ &= \sum_{i,j} [\phi(a_i), \phi(a_j)] \otimes \phi(b_i) \otimes \phi(b_j) + \phi(a_i) \otimes [\phi(b_i), \phi(a_j)] \otimes \phi(b_j) \\ &\quad + \phi(a_i) \otimes \phi(a_j) \otimes [\phi(b_i), \phi(b_j)] \\ &= C\left(\sum_i \phi(a_i) \otimes \phi(b_i) \right) = C(r_2) \end{aligned}$$

Therefore, r_2 is also a solution of the CYBE. If ϕ is surjective, and r_1 is a solution of the MCYBE, i.e., $C(r_1)$ is a $\mathfrak{g}_1$ invariant, then $C(r_2)$ is a $\mathfrak{g}_2 = \phi(\mathfrak{g}_1)$ invariant, i.e., r_2 is a solution of the MCYBE. It remains to prove that ϕ is also a Lie coalgebra map. For $x \in \mathfrak{g}_1$,

$$(\phi \otimes \phi) d(r_1)(x) = \sum_i \phi([x, a_i]) \otimes \phi(b_i) + \phi(a_i) \otimes \phi([x, b_i]) = d(r_2)(\phi(x)). \quad \square$$

Lemma 2.6. *Let V be a vector space and r a non-zero element in $V \wedge V$. Then there exist linearly independent vectors $u_1, \ldots, u_{2s} \in V$ such that $r = u_1 \wedge u_2 + \cdots + u_{2s-1} \wedge u_{2s}$.*

Proof. Let $r = \sum_{i,j=1}^n \alpha_{ij} x_i \otimes x_j$ where $x_1, \ldots, x_n$ are linearly independent in V. As $r \in V \wedge V$, $\alpha_{ij} = -\alpha_{ji}$ and $\alpha_{ii} = 0$. Let V' be the vector subspace spanned by $x_1, \ldots, x_n$

and let $B(x_i,x_j)=\alpha_{ij}$. Then, B defines an alternating form on V' (see [9]). There exists a basis $\{u_1,\dots,u_n\}$ for V' such that the matrix $(B(u_i,u_j))_{ij}$ is of the form

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & & \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}.$$

Let $x_i=\sum_{j=1}^{n}\beta_{ji}u_j$ for $i=1,\dots,n$. Then

$$\begin{aligned} r &= \sum_{i,j,k,l}\alpha_{ij}\beta_{ki}\beta_{lj}u_k\otimes u_l \\ &= \sum_{k,l}B(u_k,u_l)u_k\otimes u_l \\ &= u_1\wedge u_2+\cdots+u_{2s-1}\wedge u_{2s} \end{aligned}$$

where $2s=rank(B'(u_i,u_j))_{ij}$. $\square$

Definition 2.7. Let $\mathfrak{g}$ be a Lie algebra over k. For any $r\in\mathfrak{g}\wedge\mathfrak{g}$, define

$$\mathfrak{h}_r=\{(1\otimes f)(r)\mid f\in\mathfrak{g}^*\}.$$

Let $r\in\mathfrak{g}\wedge\mathfrak{g}$ be a solution of the CYBE. By Lemma 2.6, there exist linearly independent vectors, $u_1,\dots,u_{2s}$ such that $r=u_1\wedge u_2+\cdots+u_{2s-1}\wedge u_{2s}$. Hence, $\mathscr{B}=\{u_1,\dots,u_{2s}\}$ is a basis for $\mathfrak{h}_r$. One may write $r=\sum_{j=1}^{2s}Au_j\otimes u_j$, where A is the linear map on the subspace $\mathfrak{h}_r$ and the matrix of A with respect to $\mathscr{B}$ is

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{pmatrix}$$

Clearly, A is an automorphism on $\mathfrak{h}_r$. Moreover,

$$C(r)=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}]=0$$

implies that

$$\sum_{1\le i,j\le 2s} [Au_i,Au_j]\otimes u_i\otimes u_j + u_i\otimes [Au_i,Au_j]\otimes u_j + u_i\otimes u_j\otimes [Au_i,Au_j] = 0.$$

Hence, $[Au_i,Au_j]\in \mathfrak{h}_r$ for $i,j\le 2s$. Therefore, $\mathfrak{h}_r$ is a Lie subalgebra of $\mathfrak{g}$. This proves the following proposition.

Proposition 2.8. *If $r\in\mathfrak{g}\wedge\mathfrak{g}$ is a solution of the CYBE, then $\mathfrak{h}_r$ is an even dimensional Lie subalgebra of $\mathfrak{g}$ such that r is in $\mathfrak{h}_r\wedge\mathfrak{h}_r$.*

Remark 2.9. If $\mathfrak{h}$ is a 2-dimensional Lie algebra and $\{x,y\}$ is a basis for $\mathfrak{h}$, then one can easily see that $x\wedge y\in\mathfrak{h}\wedge\mathfrak{h}$ is a solution of the CYBE. If $\mathfrak{h}$ is a 2-dimensional Lie subalgebra of a Lie algebra $\mathfrak{g}$, then $r\in\mathfrak{h}\wedge\mathfrak{h}$ is a solution of the CYBE. Therefore, by Proposition 2.4, $\mathfrak{g}^{(r)}$ is a triangular coboundary Lie bialgebra. Conversely, if $r=x\wedge y$ and $\mathfrak{g}^{(r)}$ is a triangular coboundary Lie bialgebra, then the vector subspace $\mathfrak{h}$ generated by x,y is a Lie subalgebra by Proposition 2.8.

Definition 2.10. Let $\mathfrak{g}$ be a Lie algebra and G the group of Lie algebra automorphisms on $\mathfrak{g}$. By Proposition 2.5, G defines a group action on $\mathscr{M}(\mathfrak{g})$, the set of all solutions of the MCYBE in $\mathfrak{g}\wedge\mathfrak{g}$, namely $\phi\cdot r=(\phi\otimes\phi)(r)$ for any $r\in\mathscr{M}(\mathfrak{g})$ and $\phi\in G$. We say that $r_1,r_2\in\mathscr{M}(\mathfrak{g})$ are *conjugate* if $G\cdot r_1=G\cdot r_2$ or $r_1=\phi\cdot r_2$ for some $\phi\in G$. We abbreviate conjugacy as $r_1\sim r_2$.

Proposition 2.11. *Let $\mathfrak{g}$ be a Lie algebra such that $H^0(\mathfrak{g},\mathfrak{g}\wedge\mathfrak{g})=0$. Then for any $r_1,r_2\in\mathscr{M}(\mathfrak{g})$, $r_1\sim r_2$ if and only if $\mathfrak{g}^{(r_1)}\cong\mathfrak{g}^{(r_2)}$ as Lie bialgebra.*

Proof. If $r_1\sim r_2$, $r_2=(\phi\otimes\phi)(r_1)$ for some $\phi\in Aut_{Lie}(\mathfrak{g})$. By Proposition 2.5, $\phi:\mathfrak{g}^{(r_1)}\to\mathfrak{g}^{(r_2)}$ is a Lie bialgebra isomorphism. Conversely, let $\phi:\mathfrak{g}^{(r_1)}\to\mathfrak{g}^{(r_2)}$ be a Lie bialgebra isomorphism. Then for $x\in\mathfrak{g}$, then

$$\phi\otimes\phi\circ d(r_1)(x)=d(r_2)\circ\phi(x).$$

Let $\tilde r_2=(\phi\otimes\phi)(r_1)$. Then by Proposition 2.5,

$$\phi\otimes\phi\circ d(r_1)(x)=d(\tilde r_2)\circ\phi(x).$$

Thus, for $x\in\mathfrak{g}$, $d(r_2)\circ\phi(x)=d(\tilde r_2)\circ\phi(x)$. As ϕ is an isomorphism, $d(r_2)=d(\tilde r_2)$. Since $H^0(\mathfrak{g},\mathfrak{g}\wedge\mathfrak{g})=0$, $r_2=\tilde r_2$ and hence $r_2\sim r_1$. □

Proposition 2.12. *Let $\mathfrak{g}$ be a Lie algebra such that every finite dimensional non-zero ideal is of odd dimension. Then, $H^0(\mathfrak{g},\mathfrak{g}\wedge\mathfrak{g})=0$.*

Proof. Let $r\in\mathfrak{g}\wedge\mathfrak{g}$ such that $d(r)=0$. Suppose $r\neq 0$. Then by Lemma 2.6, there exist linearly independent elements $x_1,\dots,x_{2s}\in\mathfrak{g}$ such that $r=x_1\wedge x_2+\dots+x_{2s-1}\wedge x_{2s}$. For $x\in\mathfrak{g}$,

$$0=d(r)(x)=\sum_{k=1}^{s}[x,x_{2k-1}]\wedge x_{2k}-[x,x_{2k}]\wedge x_{2k-1}. \tag{1}$$

Let $\mathfrak{h}$ be the vector subspace spanned by $x_1,\dots,x_{2s}$. By (1), $[x,x_i]\in\mathfrak{h}$ for $i=1,\dots,2s$. Hence, $\mathfrak{h}$ is a non-zero ideal of $\mathfrak{g}$ and $\dim\mathfrak{h}=2s$. $\square$

Proposition 2.13. *For any simple Lie algebra over* $\mathbb{C}$, $H^0(\mathfrak{g},\mathfrak{g}\wedge\mathfrak{g})=0$.

Proof. If $\mathfrak{g}$ is infinite dimensional, it follows directly from Proposition 2.12. If $\mathfrak{g}$ is finite dimensional, $(\mathfrak{g}\wedge\mathfrak{g})^{\mathfrak{g}}\cong(\mathfrak{g}^*\wedge\mathfrak{g}^*)^{\mathfrak{g}}=H^2(\mathfrak{g},\mathbb{C})=0$ by the Whitehead Lemma. $\square$

Remark 2.14. In the Virasoro algebra, the center Z is the only finite dimensional nonzero ideal, and $\dim Z=1$. By Proposition 2.12, $H^0(V,V\wedge V)=0$. Actually, $H^0(V,\bigwedge^n V)=0$ for all $n\geq 2$ (see Lemma 5.4).

3. Classification of finite dimensional subalgebras of Witt and Virasoro algebras

For any $x\in W\setminus\{0\}$, $x=\sum_{i\in\mathbb{Z}}\alpha_ie_i$ for some $\alpha_i\in k$. Let $\deg_1 x$ denote the largest integer n such that $\alpha_n\neq 0$, and let $\deg_2 x$ denote the least integer m such that $a_m\neq 0$. One can easily see that $\deg_i[x,y]=\deg_i x+\deg_i y$ for $i=1,2$ as long as $x,y\neq 0$ and $\deg_i x\neq\deg_i y$. Moreover, $\deg_1 x\geq\deg_2 x$.

Proposition 3.1. *Let* $\mathfrak{h}$ *be a finite dimensional Lie subalgebra of* W. *Then* $\dim\mathfrak{h}\leq 3$.

Proof. Let $n=\max\{\deg_1 x\,|\,x\in\mathfrak{h}\setminus\{0\}\}$. Suppose $\dim\mathfrak{h}=k\geq 4$. By means of Gaussian elimination, one can find a basis $\{x_1,\dots,x_k\}$ such that $\deg_1 x_i>\deg_1 x_{i+1}$. Clearly, $\deg_1 x_1=n$ and $\deg_1 x_k\leq\deg_1 x$ for $x\in\mathfrak{h}\setminus\{0\}$. Then $\deg_1[x_1,x_2]=\deg_1 x_1+\deg_1 x_2\leq n$. Therefore, $\deg_1 x_2\leq 0$. Hence, $\deg_1 x_3<0$. Then, $\deg_1[x_3,x_k]=\deg_1 x_3+\deg_1 x_k<\deg_1 x_k$. This is a contradiction. Therefore, $\dim\mathfrak{h}\leq 3$. $\square$

Lemma 3.2. *If* $\mathfrak{h}$ *is a Lie subalgebra of* W *such that* $e_0\in\mathfrak{h}$, *then* $\mathfrak{h}=\langle e_n\,|\,n\in S\rangle$ *for some* $S\subseteq\mathbb{Z}$ *containing zero.*

Proof. For $x\in\mathfrak{h}$, there exists $\alpha_0\in k$ such that $x-\alpha_0e_0\in\langle e_n|n\neq 0\rangle$. Therefore, there exists distinct $n_1,\dots,n_r\in\mathbb{Z}\setminus\{0\}$ such that

$$x-\alpha_0e_0=x'=\sum_{i=1}^r\alpha_ie_{n_i}$$

with $\alpha_i\neq 0$ for $i=1,\dots,r$. Thus,

$$ad(e_0)^j(x')=\sum_{i=1}^r\alpha_in_i^je_{n_i}\in\mathfrak{h}$$

for $j = 0, 1, \ldots, r-1$. As

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \alpha_1 n_1 & \alpha_2 n_2 & \cdots & \alpha_r n_r \\ \vdots & \vdots & & \vdots \\ \alpha_1 n_1^{r-1} & \alpha_2 n_2^{r-1} & \cdots & \alpha_r n_r^{r-1} \end{vmatrix} = \alpha_1 \cdots \alpha_r \begin{vmatrix} 1 & 1 & \cdots & 1 \\ n_1 & n_2 & \cdots & n_r \\ \vdots & \vdots & & \vdots \\ n_1^{r-1} & n_2^{r-1} & \cdots & n_r^{r-1} \end{vmatrix} \neq 0.$$

$\langle x', [e_0, x'], \ldots, ad(e_0)^{r-1}(x')\rangle = \langle e_{n_1}, \ldots, e_{n_r}\rangle$. Therefore, $V_x = \langle e_0, e_{n_1}, \ldots, e_{n_r}\rangle \subseteq \mathfrak{h}$. Clearly, $\mathfrak{h} = \sum_{x \in \mathfrak{h}} V_x$ where we define $V_0 = 0$. Therefore, $\mathfrak{h} = \langle e_n \mid n \in S\rangle$ for some $S \subseteq \mathbb{Z}$ containing zero. □

Lemma 3.3. *If $\langle x, y\rangle$ is a 2-dimensional Lie subalgebra of W and* $\deg_1 x = 0$ *and* $\deg_1 y < 0$, then $\langle x, y\rangle = \langle e_0, e_n\rangle$ for some integer $n < 0$.

Proof. Notice that there is a scalar α such that $\deg_2(x - \alpha y) \neq \deg_2 y$. Let $x' = x - \alpha y$. Then $\deg_1 x' = 0$ and $\langle x', y\rangle = \langle x, y\rangle$. Since $\dim \langle x, y\rangle = 2$, $\deg_2[x', y]$ is either $\deg_2 x'$ or $\deg_2 y$. As $\deg_2[x', y] = \deg_2 x' + \deg_2 y$, either $\deg_2 x' = 0$ or $\deg_2 y = 0$. Since $0 > \deg_1 y \geq \deg_2 y$, $\deg_2 x' = 0$. Therefore, $x' \in \langle e_0\rangle$. By Lemma 3.2, $\langle x, y\rangle = \langle e_0, e_n\rangle$ for some integer $n \neq 0$. □

Proposition 3.4. *Let $\mathfrak{h}$ be a 3-dimensional Lie subalgebra of W. Then $\mathfrak{h} = \langle e_n, e_0, e_{-n}\rangle$ for some integer $n > 0$.*

Proof. Let x_1, x_2, x_3 be a basis for $\mathfrak{h}$. By Gaussian elimination, one can assume that

$$\deg_1 x_1 > \deg_1 x_2 > \deg_1 x_3.$$

Hence, $\deg_1[x_1, x_2] = \deg_1 x_1 + \deg_1 x_2$. Since $\dim \mathfrak{h} = 3$, $\deg_1[x_1, x_2] \leq \deg_1 x_1$. Thus, $\deg_1 x_2 \leq 0$. By a similar argument, one can obtain the relations $\deg_1[x_2, x_3] = \deg_1 x_2 + \deg_1 x_3$ and $\deg_1[x_2, x_3] \geq \deg_1 x_3$. This implies that $\deg_1 x_2 \geq 0$. Therefore, $\deg_1 x_2 = 0$ and so $\deg_1[x_2, x_3] = \deg_1 x_3$. If $[x_2, x_3]$ and x_3 are linearly independent, there is a $\beta \in k$ such that $\deg_1([x_2, x_3] - \beta x_3) < \deg_1 x_3$. This contradicts the assumption that $\dim \mathfrak{h} = 3$. Therefore, $[x_2, x_3] = \alpha x_3$ for some $\alpha \in k$. Hence $\langle x_2, x_3\rangle$ is a 2-dimensional Lie subalgebra of W. By Lemma 3.3, $\langle x_2, x_3\rangle = \langle e_0, e_n\rangle$ for some $n < 0$. Hence, by Lemma 3.2, $\mathfrak{h} = \langle e_m, e_0, e_n\rangle$. Since $\dim \mathfrak{h} = 3$, $m \neq n$. Moreover, $(m - n)e_{m+n} = [e_n, e_m] \in \mathfrak{h}$. By considering the dimension of $\mathfrak{h}$, one can easily see that $m + n = 0$. □

Corollary 3.5. *For any finite dimensional Lie subalgebra $\mathfrak{h}$ of the Virasoro algebra V,* $\dim \mathfrak{h} \leq 4$. *In particular, if* $\dim \mathfrak{h} = 4$, $\mathfrak{h} = \langle c, e_n, e_0, e_{-n}\rangle$ *for some nonzero integer n.*

Proof. Let η be the natural surjection from V onto W and let $i : \langle c\rangle \to V$ be the inclusion map. Then we have the exact sequence of Lie algebras

$$0 \to \langle c\rangle \xrightarrow{i} V \xrightarrow{\eta} W \to 0. \tag{2}$$

The result follows from Propositions 3.4 and 3.1. □

Corollary 3.6. *Let $\mathfrak{g}$ be one of the Lie algebras V, W or W_1. Then* $\dim \mathfrak{h}_r = 2$ *for any solution $r \in \mathfrak{g} \wedge \mathfrak{g}$ of the CYBE.*

Proof. For $\mathfrak{g} = W$ or W_1, the statement follows immediately from Propositions 2.8 and 3.1. Suppose there is a solution of the CYBE $r \in V \wedge V$ such that $\dim \mathfrak{h}_r = 4$. By Corollary 3.5, $\mathfrak{h}_r = \langle c, e_n, e_0, e_{-n} \rangle$ for some non-zero integer n. The exact sequence (2) induces the exact sequence

$$0 \to \langle c \rangle \xrightarrow{i} \mathfrak{h}_r \to \eta(\mathfrak{h}_r) \to 0.$$

Notice that $\eta(\mathfrak{h}_r) = \langle e_n, e_0, e_{-n} \rangle \subseteq W$ is isomorphic to $\mathfrak{sl}_2$ and $H^2(\mathfrak{sl}_2, k) = 0$. Therefore, $\mathfrak{h}_r$ is isomorphic to a trivial extension of $\mathfrak{sl}_2$. Hence $H^2(\mathfrak{h}_r, k) = 0$. By Proposition 2.4 of [1], there exists a non-degenerate skew-symmetric bilinear form B on $\mathfrak{h}_r$ which is also a 2-cocycle. As $H^2(\mathfrak{h}_r, k) = 0$, B is then a coboundary. Thus, there exists a linear map $f : \mathfrak{h}_r \to k$ such that

$$B(x, y) = f([x, y])$$

for $x, y \in \mathfrak{h}_r$. However, $B(x, c) = f([x, c]) = 0$ for all $x \in \mathfrak{h}_r$. This contradicts the nondegeneracy of B. Therefore, $\dim \mathfrak{h}_r \neq 4$. □

4. The Lie bialgebra structures on W_1

The Lie algebra W_1 is isomorphic to the Lie algebra of derivations on the polynomial ring $k[t]$ under the identification

$$e_n = t^{n+1} \frac{\mathrm{d}}{\mathrm{d}t}.$$

Hence, $k[t]$ is naturally a W_1-module by the action $e_n \cdot t^m = m t^{n+m}$ for $m \geq 0$ and $n \geq -1$.

Proposition 4.1. *The set of all solutions of the CYBE in $W_1 \wedge W_1$ is given by*

$$\begin{aligned} \mathscr{R} &= \left\{ \alpha(t+\beta)\frac{\mathrm{d}}{\mathrm{d}t} \wedge (t+\beta)^n \frac{\mathrm{d}}{\mathrm{d}t} \,\middle|\, \alpha, \beta \in k,\ n \geq 0 \right\} \\ &= \left\{ \alpha(e_0 + \beta e_{-1}) \wedge \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i \,\middle|\, \alpha, \beta \in k,\ n \geq -1 \right\}. \end{aligned}$$

Proof. For any $\alpha, \beta \in k$ and $n \neq 0$,

$$\left[e_0 + \beta e_{-1}, \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i \right] = n \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i.$$

By [16], $r = (e_0 + \beta e_{-1}) \wedge \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i$ is a solution of the CYBE and so is αr. Conversely, let $r \in W_1 \wedge W_1$ be a solution of the CYBE. By Corollary 3.6, $\dim \mathfrak{h}_r = 2$, say $\mathfrak{h}_r = \langle x, y \rangle$. Without loss of generality, we may assume that

$$[x, y] \in \langle y \rangle \tag{$*$}$$

and $y = \sum_{i=-1}^{n} \alpha_i e_i$ with $\alpha_n = 1$. We may further assume that $\deg_1 x \neq \deg_1 y$, for otherwise we replace x by $x - \gamma y$, where γ is the leading coefficient of x. Then $\deg_1[x,y]=\deg_1 x+\deg_1 y$, and by $(*)$ we have $\deg_1[x,y]=\deg_1 y$. Therefore, $\deg_1 x=0$. We may take $x = e_0 + \beta e_{-1}$ for some $\beta \in k$. If $0 > n$, then $y = e_{-1}$ and hence r is a scalar multiple of $x \wedge y = e_0 \wedge e_{-1}$. Therefore, $r \in \mathscr{R}$. Assume $n > 0$ and consider the action of W_1 on the polynomial ring $k[t]$. Then we have

$$\alpha P(t) = \alpha y \cdot t = [x,y] \cdot t = x \cdot (y \cdot t) - y \cdot (x \cdot t) = (t+\beta)P'(t) - P(t),$$

where $P(t) = t^{n+1} + \alpha_{n-1}t^n + \cdots + \alpha_0 t + \alpha_{-1}$. By comparing the coefficient of t^n, $\alpha = n$ and so $(t+\beta)P'(t) = (n+1)P(t)$. Using this equation, one can easily prove that $P(t) = (t+\beta)^{n+1}$ and hence $\alpha_i = \binom{n+1}{i+1} \beta^{n-i}$ for $i = -1, \ldots, n$. Therefore, $y = \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i$. As r is a scalar multiple of $x \wedge y$, $r \in \mathscr{R}$. □

Related results for the set of solutions of the CYBE for $L\tilde{\wedge}L$, for L the Lie subalgebra of W_1 with basis $\{e_n \mid n \geq 0\}$, are stated in [10] in Theorem 4. See also [17].

Proposition 4.2. *Let k be an algebraically closed field. For any $\alpha, \beta \in k$ with $\alpha \neq 0$,*

$$e_0 \wedge e_n \sim \alpha(e_0 + \beta e_{-1}) \wedge \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i \text{ in } W_1 \wedge W_1$$

for $n \geq -1$.

Proof. (i) $e_0 \wedge e_n \sim \alpha e_0 \wedge e_n$ for $\alpha \neq 0$ and $n \geq -1$ and $n \neq 0$. Let ρ be a root of the equation $t^n - \alpha = 0$. Consider the linear endomorphism $\phi : e_i \mapsto \rho^i e_i$ for $i \geq -1$; it is clearly a Lie algebra automorphism on W_1 and $(\phi \otimes \phi)(e_0 \wedge e_n) = \alpha e_0 \wedge e_n$.

(ii) By step (i), the statement is obvious for $n = -1, 0$.

(iii) Assume $n \geq 1$. Define the algebra map $\xi : k[t] \to k[t]$ by $\phi(t) = t - \beta$.

Clearly, ξ is an automorphism and ϕ^{-1} is defined by $\xi^{-1}(t) = t + \beta$. ξ induces an automorphism T_ξ on $\operatorname{Der} k[t]$ defined by $T_\xi(D) = \xi^{-1} \circ D \circ \xi$ for $D \in \operatorname{Der} k[t]$. Then $T_\xi(\mathrm{d}/\mathrm{d}t)(t) = t+\beta$ and hence $T_\xi(\mathrm{d}/\mathrm{d}t) = (t+\beta)\mathrm{d}/\mathrm{d}t$. Moreover, $T_\xi(t^{n+1}\mathrm{d}/\mathrm{d}t)(t) = (t+\beta)^{n+1}$ and so

$$T_\xi\left(t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}\right) = (t+\beta)^{n+1}\frac{\mathrm{d}}{\mathrm{d}t} = \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i.$$

Therefore,

$$\alpha(e_0 + \beta e_{-1}) \wedge \sum_{i=-1}^{n} \binom{n+1}{i+1} \beta^{n-i} e_i \sim \alpha e_0 \wedge e_n \sim e_0 \wedge e_n. \qquad \square$$

5. Some cohomology results

In this section will establish some homological results for the remaining sections. In particular, we will prove that $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g}) = 0$ for $\mathfrak{g} = W_1$, W or V (Corollary 5.8).

The result was stated in the paper [4] but there are gaps in the proof. For the sake of completeness a short and complete proof of the result will be given.

Let $\mathfrak{L}$ be the Lie algebra W_1 or W and consider the set

$$I = \begin{cases} \mathbb{Z} & \text{if } \mathfrak{L} = W, \\ \{i \in \mathbb{Z} \mid i \geq -1\} & \text{if } \mathfrak{L} = W_1. \end{cases}$$

In both cases, $\{e_i \mid i \in I\}$ is a basis for $\mathfrak{L}$ and $\{i \in \mathbb{Z} \mid i \geq -1\} \subseteq I$ and $\mathfrak{L}$ is a $\mathbb{Z}$-graded Lie algebra.

A (complete) $\mathbb{Z}$-graded $\mathfrak{L}$-module N is inner $\mathbb{Z}$-graded if the degree n homogeneous space N_n satisfies $[L_{(0)}, x] = nx$ for $x \in N_n$. Notice that for $n \geq 1$, $\bigwedge^n \mathfrak{g}$ is an inner $\mathbb{Z}$-graded $\mathfrak{L}$-module. Moreover, the completion $\mathfrak{L}\tilde{\wedge}\mathfrak{L}$ is a complete inner $\mathbb{Z}$-graded $\mathfrak{L}$-module. Let $C^*(\mathfrak{L}, N)$ be the standard complex for an inner $\mathbb{Z}$-graded $\mathfrak{L}$-module N.

$$C^q_{(0)}(\mathfrak{L}, N) = \{c \in C^q(\mathfrak{L}, N) \mid c(e_{i_1}, \ldots, e_{i_q}) \in N_m \text{ where } m = i_1 + \cdots + i_q\}.$$

$C^*_{(0)}(\mathfrak{L}, N)$ is a subcomplex of $C^*(\mathfrak{L}, N)$ and the inclusion map induces an isomorphism of the homology groups (cf. [5]). We will denote by

$$Z^q_{(0)}(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L}), \quad B^q_{(0)}(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L}), \quad H^q_{(0)}(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L})$$

the q-cocycles, q-coboundaries and qth homology group of the complex $C^*_{(0)}(\mathfrak{L}, N)$.

Consider $\mathfrak{L}\widetilde{\wedge}\mathfrak{L}$ to be the vector space of formal sums of the form

$$\sum_{\substack{i<j \\ i,j\in I}} \lambda_{ij} e_i \wedge e_j.$$

Define an $\mathfrak{L}$-action on $\mathfrak{L}\widetilde{\wedge}\mathfrak{L}$ by the formula

$$\left[e_n, \sum_{\substack{i<j \\ i,j\in I}} \lambda_{ij} e_i \wedge e_j \right] = \sum_{\substack{i<j \\ i,j\in I}} \lambda_{ij}(i-n) e_{i+n} \wedge e_j + \lambda_{ij}(j-n) e_i \wedge e_{j+n},$$

where $e_s \wedge e_r = -e_r \wedge e_s$ if $r < s$ and $e_s \wedge e_s = 0$. One can easily see that $\mathfrak{L}\widetilde{\wedge}\mathfrak{L}$ is an $\mathfrak{L}$-module and $\mathfrak{L} \wedge \mathfrak{L}$ is a $\mathfrak{L}$-submodule of $\mathfrak{L}\widetilde{\wedge}\mathfrak{L}$. One can view $\mathfrak{L}\widetilde{\wedge}\mathfrak{L}$ as the "completion" of the exterior square of $\mathfrak{L}$.

For $n \in \mathbb{Z}$, define

$$M_n = \left\{ \sum_{\substack{i,j\in I, i<j \\ i+j=n}} \lambda_{ij} e_i \wedge e_j \;\middle|\; \lambda_{ij} \in k \right\}.$$

Clearly, $\mathfrak{L}\widetilde{\wedge}\mathfrak{L} = \prod_{n\in I} M_n$. Thus, for $r \in \mathfrak{L}\widetilde{\wedge}\mathfrak{L}$, there exist unique $r_n \in M_n$ such that $r = \sum_{n\in I} r_n$. Moreover, $\{e_i \wedge e_j \mid i < j,\, i+j = n \text{ and } i,j \in I\}$ form a pseudo-basis of M_n. Therefore, any $r \in M_n$ can be uniquely written in the form

$$r = \sum_{\substack{i\in I \\ i>-n/2}} \lambda_i e_{-i} \wedge e_{i+n}.$$

If $\mathfrak{L} = W_1$, this sum is clearly finite. One can also notice that $[e_m, M_n] \subseteq M_{m+n}$ for $m \in I$ and $n \in \mathbb{Z}$. Therefore, $\mathfrak{L}\widetilde{\wedge}\mathfrak{L}$ is the total completion of the graded $\mathfrak{L}$-module $\oplus_{n\in I} M_n$.

Lemma 5.1. *If* $H^1(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L}) = 0$, *then* $H^1(\mathfrak{L}, \mathfrak{L}\wedge\mathfrak{L}) = 0$.

Proof. By above remark, it suffices to show $H^1_{(0)}(\mathfrak{L}, \mathfrak{L}\wedge\mathfrak{L}) = 0$. Let $\delta \in Z^1_{(0)}(\mathfrak{L}, \mathfrak{L}\wedge\mathfrak{L})$. Then $\delta \in Z^1_{(0)}(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L})$ and so there exists $r_0 \in M_0$ such that $\delta = d(r_0)$. To prove that $\delta \in B^1_{(0)}(\mathfrak{L}, \mathfrak{L}\wedge\mathfrak{L})$, it is enough to show that $r_0 \in \mathfrak{L}\wedge\mathfrak{L}$. For $m \in I$,

$$\alpha(e_m) = [e_m, r_0] + \sum_{n\in I_0\setminus\{0\}} [e_m, r_n] \in \mathfrak{L}\wedge\mathfrak{L}.$$

Therefore, $[e_m, r_0] \in M_m \cap \mathfrak{L}\wedge\mathfrak{L}$ for $m \in I$. Let $r_0 = \sum_{i>0} \alpha_i e_{-i}\wedge e_i$. Notice that

$$[e_1, r_0] = -2a_1 e_0 \wedge e_1 + \sum_{i>0}((i-1)\alpha_i - (i+2)\alpha_{i+1})e_{-i}\wedge e_{i+1},$$

$$[e_2, r_0] = -4a_2 e_0 \wedge e_2 + \sum_{i>0}(\alpha_i(i-2)\alpha_i - (i+4)\alpha_{i+1})e_{-i}\wedge e_{i+2}.$$

There exists a positive integer N such that for $i \ge N$,

$$(i-1)\alpha_i - (i+2)\alpha_{i+1} = 0,$$
$$(i-2)\alpha_i - (i+4)\alpha_{i+2} = 0.$$

Hence, for $i \ge N$,

$$\begin{aligned}(i-1)i\alpha_i - (i+2)(i+3)\alpha_{i+2} = 0,\\ (i-2)\alpha_i - (i+4)\alpha_{i+2} = 0.\end{aligned} \tag{3}$$

One can easily see that for large enough i, the system of equations (3) has only the trivial solution. Hence, $r_0 \in \mathfrak{L}\wedge\mathfrak{L}$. □

Lemma 5.2. *Let* $\mathfrak{g}$ *be a Lie algebra such that* $\mathfrak{g}^{\mathfrak{g}}$, $(\mathfrak{g}\wedge\mathfrak{g})^{\mathfrak{g}}$ *and* $H^1(\mathfrak{g},\mathfrak{g}) = 0$. *Then for any* 1-*dimensional central extension* $\tilde{\mathfrak{g}}$ *of* $\mathfrak{g}$, *there is a linear embedding from* $H^1(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})$ *into* $H^1(\mathfrak{g}, \mathfrak{g}\wedge\mathfrak{g})$. *In particular, if* $H^1(\mathfrak{g}, \mathfrak{g}\wedge\mathfrak{g}) = 0$, *then* $H^1(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}}) = 0$.

Proof. Let $\eta : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be a surjective Lie algebra map such that $\ker\eta = \mathfrak{h}$ is in the center of $\tilde{\mathfrak{g}}$ and $\dim\mathfrak{h} < \infty$. Consider the Hochschild–Serre spectral sequence (see for example [8,19]) with respect to $\tilde{\mathfrak{g}}$ and $\mathfrak{h}$. Then

$$E_2^{01} = H^0(\mathfrak{g}, H^1(\mathfrak{h}, \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})) = (\tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})^{\mathfrak{g}},$$
$$E_2^{10} = H^1(\mathfrak{g}, H^0(\mathfrak{h}, \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})) = H^1(\mathfrak{g}, \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})$$

where $\mathfrak{h}$ is considered as a trivial $\mathfrak{g}$-module. We have the following exact sequence for the low degree terms in the spectral sequence

$$0 \to E_2^{10} \to H^1(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}}) \to E_2^{01}.$$

The map η induces a $\mathfrak{g}$-module epimorphism $\eta\wedge\eta : \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}} \to \mathfrak{g}\wedge\mathfrak{g}$ and

$$\ker(\eta\wedge\eta) = \{x\wedge h \mid x \in \tilde{\mathfrak{g}}\}$$

where $\langle h\rangle = \mathfrak{h}$. One can easily see that $K = \ker(\eta\wedge\eta)$ is isomorphic to $\mathfrak{g}$ as $\mathfrak{g}$-module. By the long exact sequence

$$0 \to K^{\mathfrak{g}} \to (\tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})^{\mathfrak{g}} \to (\mathfrak{g}\wedge\mathfrak{g})^{\mathfrak{g}} \to H^1(\mathfrak{g}, K) \to H^1(\mathfrak{g}, \tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}}) \to H^1(\mathfrak{g}, \mathfrak{g}\wedge\mathfrak{g}),$$

we have $E_2^{01}=0$ and so $E_2^{10}\cong H^1(\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})$. By the same exact sequence, we have the exact sequence

$$0\to E_2^{10}\to H^1(\mathfrak{g},\mathfrak{g}\wedge\mathfrak{g}). \qquad \square$$

Remark 5.3. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ satisfies the conditions in Lemma 5.2 by Proposition 2.12 and the Whitehead lemma. Hence, for any 1-dimensional central extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$, $H^1(\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}\wedge\tilde{\mathfrak{g}})=0$. In particular, $H^1(\mathfrak{gl}_n,\mathfrak{gl}_n\wedge\mathfrak{gl}_n)=0$ as $\mathfrak{gl}_n$ is a 1-dimensional central extension of $\mathfrak{sl}_n$.

Lemma 5.4. *If* $\mathfrak{g}=W_1, W$ *or* V, *then* $(\bigwedge^n\mathfrak{g})^{\mathfrak{g}}=0$ *for* $n\geq 2$. *Moreover,* $\mathfrak{g}^{\mathfrak{g}}=0$ *if* $\mathfrak{g}=W_1$ *or* W.

Proof. Since W_1 and W are simple Lie algebras, the second statement is obvious. We will prove the first statement for $\mathfrak{g}=V$. Let $\hat{\mathbb{Z}}=\mathbb{Z}\cup\{-\infty\}$ and

$$S=\{(j_1,\dots,j_n)\,|\,j_1<\cdots<j_n \text{ and } j_1,\dots,j_n\in\hat{\mathbb{Z}}\}.$$

We assign the lexicographical ordering to S. For $J=(j_1,\dots,j_n)\in S$, define $L_J=L_{j_1}\wedge\cdots\wedge L_{j_n}\in\bigwedge^n V$ where $L_{-\infty}=c$ is the central charge. Then, $\mathscr{B}=\{L_J\,|\,J\in S\}$ is a basis for $\bigwedge^n V$ and the ordering on S induces a totally ordering on $\mathscr{B}$. Notice that for $(j_1,\dots,j_n)\in S$, $j_n\neq-\infty$ for $n\geq 2$. Moreover, for $J_1=(j_1,\dots,j_n)<J_2=(i_1,\dots,i_n)$ and $t>|j_n|$, the highest term in $[L_t,L_{J_1}]$ is strictly less than the highest term in $[L_t,L_{J_2}]$.

We may write any non-zero element $x\in\bigwedge^n V$ as

$$x=\alpha_1L_{J_1}+\cdots+\alpha_lL_{J_l}$$

for some $J_1,\dots,J_l\in S$ and $\alpha_1,\dots,\alpha_l\in k\setminus\{0\}$, where $J_1<\cdots<J_l$. Hence, for $x\neq 0$ there exist $t\in\mathbb{Z}$ such that $[L_t,x]\neq 0$. Therefore, $(\bigwedge^n V)^V=0$. By the same argument, one can prove that $(\bigwedge^n\mathfrak{g})^{\mathfrak{g}}=0$ for $n\geq 2$ and $\mathfrak{g}=W_1$ or W. $\square$

Remark 5.5. Let $\mathfrak{g}$ be a Lie algebra such that $H^0(\mathfrak{g},\bigwedge^3\mathfrak{g})=0$. Then any solution $r\in\mathfrak{g}\wedge\mathfrak{g}$ of the MCYBE is actually a solution of CYBE. This, in particular, holds for W_1, W and V over a field k of characteristic zero.

Proposition 5.6. $H^1(\mathfrak{L},\mathfrak{L})=0$, *where* $\mathfrak{L}=W$ or W_1.

Proof. As $\mathfrak{L}$ is an inner $\mathbb{Z}$-graded $\mathfrak{L}$-module, $H^1(\mathfrak{L},\mathfrak{L})=H^1_{(0)}(\mathfrak{L},\mathfrak{L})$. Let $\delta\in Z^1_{(0)}(\mathfrak{L},\mathfrak{L})$. We have for $n\in I$, $\delta(e_n)=\beta_ne_n$ for some $\beta_n\in k$. We claim that $\delta=d(\beta_1e_0)$. By the bracket rule of $\mathfrak{L}$, for $n\neq m$, $\beta_{m+n}=\beta_m+\beta_n$. In particular, $\beta_{n-1}=\beta_n+\beta_{-1}$ for $n\geq 0$. By induction $\beta_n=-n\beta_{-1}$ for $n\geq 0$. In particular $\beta_1=-\beta_{-1}$ and so $\beta_n=n\beta_1$ for $n\geq -1$. If $\mathfrak{L}=W_1$, then $\delta=d(\beta_1e_0)$. If $\mathfrak{L}=W$, we consider one more equation $\beta_{n+1}=\beta_n+\beta_1$. Hence, for $n\leq 0$, $\beta_n=n\beta_1$. Thus, $\beta_n=n\beta_1$ for $n\in\mathbb{Z}$. Therefore, $\delta=d(\beta_1e_0)$. $\square$

Theorem 5.7. $H^1(\mathfrak{L},\mathfrak{L}\hat{\wedge}\mathfrak{L})=0$ *where* $\mathfrak{L}=W$ or W_1.

Proof. By above remark, it suffices to show that $H^1_{(0)}(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L}) = 0$. Let $\alpha \in Z^1_{(0)}(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L})$. Then, $\alpha(e_{-1}) \in M_{-1}$. There exist unique λ_i such that $\alpha(e_{-1}) = \sum_{i\geq 1} \lambda_i e_{-i} \wedge e_{i-1}$. Define

$$\mu_1 = \lambda_1/2,$$

$$\mu_i = \frac{1}{(i+1)i(i-1)} \sum_{n=2}^{i} \lambda_n n(n-1) \quad \text{for } i \geq 2,$$

$$r = \sum_{i\geq 1} \mu_i e_{-i} \wedge e_i.$$

Then $[e_{-1}, r] = \alpha(e_{-1})$. Let $\delta = \alpha - d(r)$. Then $\delta(e_{-1}) = 0$. As $r \in M_0$, both $d(r)$ and hence δ belong to $Z^1_{(0)}(\mathfrak{L}, \mathfrak{L}\tilde{\wedge}\mathfrak{L})$. We will prove that $\delta = 0$ in four steps.

(i) $\delta(e_0) = 0$. For $r \in M_n$, $[e_0, r] = nr$, and for $n \in I, [e_0, e_n] = ne_n$. Therefore,

$$n\delta(e_n) = [e_0, \delta(e_n)] - [e_n, \delta(e_0)].$$

As $\delta(e_n) \in M_n$, $[e_0, \delta(e_n)] = n\delta(e_n)$. Thus, $[e_n, \delta(e_0)] = 0$ for $n \in I$. By Lemma 5.4, $\delta(e_0) = 0$.

(ii) $\delta(e_i) = 0$ for $i = -1, \dots, 5$. Since $\delta(e_1) \in M_1$, there exist unique β_i such that $\delta(e_1) = \sum_{i\geq 0} \beta_i e_{-i} \wedge e_{i+1}$. Since $[e_{-1}, e_1] = 2e_0$, $[e_{-1}, \delta(e_1)] = \delta(e_0) = 0$ and so we have

$$\begin{aligned} 0 &= \left[e_{-1}, \sum_{i\geq 0} \beta_i e_{-i} \wedge e_{i+1} \right] \\ &= \sum_{i\geq 0} \beta_i(-i+1)e_{-i-1} \wedge e_{i+1} + \sum_{i\geq 0} \beta_i(i+2)e_{-i} \wedge e_i \\ &= \sum_{i\geq 1} \beta_{i-1}(-i+2)e_{-i} \wedge e_i + \sum_{i\geq 0} \beta_i(i+2)e_{-i} \wedge e_i \\ &= \sum_{i\geq 1} (\beta_{i-1}(-i+2) + \beta_i(i+2))e_{-i} \wedge e_i. \end{aligned}$$

Therefore, $\beta_i = \beta_{i-1}(i-2)/(i+2)$ for $i \geq 1$. By this recursive relation, one can easily see that $\beta_i = 0$ for $i \geq 2$. Moreover, $-3\beta_1 = \beta_0$. Thus,

$$\delta(e_1) = \beta_1(e_{-1} \wedge e_2 - 3e_0 \wedge e_1).$$

As $[e_{-1}, e_2] = 3e_1$, we have $[e_{-1}, \delta(e_2)] = 3\delta(e_1)$. As $\delta(e_2) \in M_2$, there exist unique γ_i such that $\delta(e_2) = \sum_{i\geq 0} \gamma_i e_{-i} \wedge e_{i+2}$ and we have

$$[e_{-1}, \delta(e_2)] = -\sum_{i\geq 1} (\gamma_{i-1}(i-2) - \gamma_i(3+i))e_{-i} \wedge e_{i+1} + 3\gamma_0 e_0 \wedge e_1.$$

Thus,

$$3\beta_1(e_{-1} \wedge e_2 - 3e_0 \wedge e_1) = -\sum_{i\geq 1} (\gamma_{i-1}(i-2) - \gamma_i(3+i))e_{-i} \wedge e_{i+1} + 3\gamma_0 e_0 \wedge e_1.$$

Hence, $\gamma_0 = -3\beta_1$ and $-\gamma_0 - 4\gamma_1 = -3\beta_1$ and $\gamma_i = 0$ for $i \geq 2$. One can easily compute that $\gamma_1 = \frac{3}{2}\beta_1$. Thus

$$\delta(e_2) = \beta_1 \left(\tfrac{3}{2} e_{-1} \wedge e_3 - 3e_0 \wedge e_2 \right) = \tfrac{3}{2}[e_1, \delta(e_1)].$$

In particular, $\delta(e_1), \delta(e_2) \in W_1 \wedge W_1$. For $x \in W_1 \wedge W_1$, $x = \sum_{r=1}^{s} \alpha_r e_{i_r} \wedge e_{j_r}$ where $j_r > i_r$ and $(i_r, j_r) < (i_{r+1}, j_{r+1})$, using the lexicographical ordering with $\alpha_r \neq 0$ for $r = 1, \ldots, s$. We will call $\alpha_s e_{i_s} \wedge e_{j_s}$ the leading terms of x. By the equation $[e_1, \delta(e_n)] - [e_n, \delta(e_1)] = (n-1)\delta(e_{n+1})$, the leading terms of $\delta(e_3)$, $\delta(e_4)$ and $\delta(e_5)$ are $3\beta_1 e_{-1} \wedge e_4$, $5\beta_1 e_{-1} \wedge e_5$ and $\frac{22}{3}\beta_1 e_{-1} \wedge e_6$. However, by the equation $\delta(e_5) = [e_2, \delta(e_3)] - [e_3, \delta(e_2)]$, we see that the leading term of $\delta(e_5)$ is $6\beta_1 e_{-1} \wedge e_6$. Therefore, $\beta_1 = 0$ and hence $\delta(e_1) = \delta(e_2) = 0$.

(iii) Since W_1 is generated by e_{-1} and e_2 and δ is a derivation, $\delta = 0$ on W_1, i.e. $\delta(e_i) = 0$ for $i \geq -1$.

(iv) $\delta = 0$: If $\mathfrak{L} = W_1$, then $\delta = 0$ by step (iii). Suppose $\mathfrak{L} = W$. Then $I = \mathbb{Z}$ and the map $T : \mathfrak{L} \to \mathfrak{L}$ defined by $T(e_n) = -e_{-n}$ is a Lie algebra automorphism on $\mathfrak{L}$. Moreover, T induces an $\mathfrak{L}$-module automorphism T_2 on $\mathfrak{L} \tilde{\wedge} \mathfrak{L}$ defined by $T_2(e_i \wedge e_j) = -e_{-j} \wedge e_{-i}$ for any integers $i < j$. Let $\delta' = T_2 \circ \delta \circ T$. Since $T^2 = id_{\mathfrak{L}}$, $\delta' \in Z^1_{(0)}(\mathfrak{L}, \mathfrak{L} \tilde{\wedge} \mathfrak{L})$ and $\delta'(e_{-1}) = T_2(\delta(e_1)) = 0$ by (iii). Hence, by repeating the argument of (i)–(iii), one can prove that $\delta'(e_i) = 0$ for $i \geq -1$. Hence, for $i \geq 1$, $T_2\delta(e_{-i}) = 0$. As T_2 is an automorphism on $\mathfrak{L} \tilde{\wedge} \mathfrak{L}$, $\delta(e_{-i}) = 0$ for $i \geq 1$ and so $\delta = 0$. $\square$

A similar result for L the Lie subalgebra of W_1 with basis $\{ e_n \mid n \geq 0 \}$ is stated in [10] in Theorem 3.

Corollary 5.8. *For* $\mathfrak{g} = W, W_1$ *or* V, $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g}) = 0$.

Proof. By Lemma 5.1 and Theorem 5.7, $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g}) = 0$ for $\mathfrak{g} = W_1$ or W. As V is a 1-dimensional central extension of W, by Lemmas 5.2, 5.4 and Proposition 5.6, $H^1(V, V \wedge V) = 0$. $\square$

6. Lie bialgebra structures on the Witt and Virasoro algebras

For $\mathfrak{g} = W_1, W$ or V, by Corollary 5.8, $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g}) = 0$ and hence all the Lie bialgebra structures on $\mathfrak{g}$ are of coboundary type. As $(\bigwedge^3 \mathfrak{g})^{\mathfrak{g}} = 0$ (Lemma 5.4), all the solutions of the MCYBE are solutions of the CYBE. Therefore, any coboundary Lie bialgebra structure on $\mathfrak{g}$ is triangular. Consequently, every Lie bialgebra structure on $\mathfrak{g}$ is isomorphic to $\mathfrak{g}^{(r)}$ for some solution of the CYBE $r \in \mathfrak{g} \wedge \mathfrak{g}$. By Corollary 3.6, $\mathfrak{h}_r = 2$ and hence $r = a \wedge b$ for some $a \in b \in \mathfrak{g}$. This proves Theorem 1.2. A special case of Theorem 1.1 also follows from a similar argument.

Theorem 6.1. *Let* $W_1^{(n)}$ *be the Lie bialgebra structure on* W_1 *associated to the solution* $e_0 \wedge e_n$ *of the CYBE. Every Lie bialgebra structure on* W_1 *is isomorphic to* $W_1^{(n)}$ *for some* $n \geq -1$ *when* k *is algebraically closed of characteristic zero, where the Lie cobracket* δ *on* $W_1^{(n)}$ *is given by*

$$\delta(e_i) = (n-i)e_0 \wedge e_{n+i} - ie_i \wedge e_n \quad (i \geq -1).$$

Proof. Follow the preceding discussion, it suffices to show that for every triangular coboundary Lie bialgebra $W_1^{(r)}$, the triangular coboundary Lie bialgebra structure associated to a solution of the CYBE $r \in W_1 \wedge W_1$, is isomorphic to $W_1^{(n)}$ for some $n \geq -1$. By Propositions 4.1 and 4.2, there exists $n \geq -1$ such that $r \sim e_0 \wedge e_n$. By the proof of Proposition 2.11, $W_1^{(r)} \cong W_1^{(n)}$. $\square$

Remark 6.2. The analogue of Theorem 6.1 for W does not hold. If one has enough knowledge about the Lie algebra automorphisms of W, this can be seen easily. Clearly, for any $\beta \in k^*$, the multiplicative group of k, $T_\beta : e_n \mapsto \beta^n e_n$ defines a Lie algebra automorphism on W. The map $A : e_n \mapsto -e_{-n}$ also defines a Lie algebra automorphism on W. Actually, these maps generate the group $Aut_{Lie}(W)$.

Theorem 6.3. *For $T \in Aut_{Lie}(W)$, $T = A^r \circ T_\beta$ for some $\beta \in k^*$ and $r = 0, 1$.*

Proof. One can observe easily that all the semisimple elements of W are of the form αe_0 for some $\alpha \in k^*$. Let $T \in Aut_{Lie}(W)$. Then, $Te_0 = \alpha e_0$ for some $\alpha \in k^*$. Let $n \neq 0$ and $Te_n = \sum_l \beta_{ln} e_l$, then

$$nTe_n = T[e_0, e_n] = [Te_0, Te_n] = \alpha[e_0, Te_n].$$

Therefore,

$$\sum_l n\beta_{ln} e_l = \sum_l \alpha l \beta_{ln} e_l.$$

Hence,

$$(n - \alpha l)\beta_{ln} = 0 \tag{4}$$

for $l \in \mathbb{Z}$. As T is injective, there exists l_n such that $\beta_{l_n n} \neq 0$ by (4), $\alpha = n/l_n$ for $n \in \mathbb{Z}$. In particular, $\alpha = 1/l_1$. Therefore, for $n \in \mathbb{Z}$,

$$l_n = n/\alpha = nl_1. \tag{5}$$

As T is surjective, for any prime number p there exists $n \neq 0$ such that $\beta_{pn} \neq 0$. Hence by Eq. (5), $p = nl_1$. Thus, l_1 divides all primes and hence $l_1 = \pm 1$. Then, it follows from Eq. (5) that, $\alpha = \pm 1$ and

$$\begin{aligned} Te_n &= \beta_{-n} e_{-n} \quad \text{if } \alpha = -1, \\ Te_n &= \beta_n e_n \quad \text{if } \alpha = 1. \end{aligned} \tag{6}$$

If $\alpha = 1$, then for $n \in \mathbb{Z}$, $Te_n = \beta_n e_n$ by 6. Notice that for $n_1, n_2 \in \mathbb{Z}$,

$$(n_2 - n_1)e_{n_1+n_2} = [e_{n_1}, e_{n_2}]$$

and so

$$(n_2 - n_1)Te_{n_1+n_2} = [Te_{n_1}, Te_{n_2}] = \beta_{n_1}\beta_{n_2} e_{n_1+n_2}(n_2 - n_1).$$

Hence, if $n_1 \neq n_2$, $\beta_{n_1+n_2} = \beta_{n_1}\beta_{n_2}$. For $m > |2n|$,

$$\beta_{2n+m} = \beta_n\beta_{n+m} = \beta_n\beta_n\beta_m,$$
$$\beta_{2n+m} = \beta_{2n}\beta_m.$$

This implies that $\beta_{2n} = (\beta_n)^2$. As a result, $\beta_n = (\beta_1)^n$ for $n \in \mathbb{Z}$. Consequently, $T = T_{\beta_1}$.

If $\alpha = -1$, then $AT(e_0) = e_0$. By the above, $AT = T_\gamma$ for some $\gamma \in k^*$. Hence, $T = A \circ T_\gamma$ since $A^2 = I$. □

The analogue of Theorem 6.1 does not hold in W. For instance, $e_0 \wedge e_1$ is no longer equivalent to $(e_0 + e_{-1}) \wedge (e_{-1} + 2e_0 + e_1)$ in W. Hence, by Proposition 2.11, the associated Lie bialgebra structures are not isomorphic. Moreover, by Theorem 6.3, for any $n, m \in \mathbb{Z}$, $e_0 \wedge e_n \sim e_0 \wedge e_m$ if and only if $n = \pm m$. Therefore, by Proposition 2.11, we have the following corollary.

Corollary 6.4. *For any $n, m \in \mathbb{Z}$, $W^{(n)} \cong W^{(m)}$ as Lie bialgebras if and only if $n = \pm m$.*

7. Saturated subalgebras of the Witt algebra

Definition 7.1. Let $I \subseteq \mathbb{Z}$ such that $0 \in I$. I is called *saturated* if for any $i \neq j \in I$, $i + j \in I$. I is of *Type I* if I is non-negative or I is non-positive and I is of *Type II* if it is not of Type I. A Lie subalgebra L of W is called saturated if $L = L(I) := \langle e_i \mid i \in I \rangle$ for some saturated set in I. Since $e_0 \in L$, every saturated Lie subalgebra is inner graded.

Lemma 7.2. *Let L be a Lie subalgebra of W. L is saturated if and only if $e_0 \in L$.*

Proof. If L is saturated, $e_0 \in L$ by definition. Conversely, if $e_0 \in L$, then L is a graded Lie subalgebra of W by Lemma 3.2. Therefore, if I denotes $\{i \in \mathbb{Z} \mid e_i \in L\}$, $L = \langle e_i \mid i \in I \rangle$. Since L is closed under Lie bracket, I is saturated. □

Remark 7.3. The intersection of saturated sets is saturated. For any subset S of $\mathbb{Z}$, write $\langle S \rangle$ for the intersection of all saturated subsets containing S. If $I = \langle S \rangle$, we will say that S generates I.

Definition 7.4. Let I be a saturated set of $\mathbb{Z}$. An element $n \in I$ is called *irreducible* if for any $n_1 \neq n_2 \in I$ such that $n = n_1 + n_2$, then $n_1 = 0$ or $n_2 = 0$.

Lemma 7.5. *Let I be a Type I saturated set. Then I is generated by the set I_0 of all irreducible elements of I. Hence, the set $\{e_i \mid i \in I_0\}$ generates $L(I)$ as Lie algebra.*

Proof. Without loss of generality, we may assume I is non-negative. If $|I| < \infty$, then $I = \{0\}$ or $\{0, n\}$ for some $n \in \mathbb{N}$ and the lemma holds. Suppose $|I| = \infty$. Since I consists of non-negative integers, 0 is clearly irreducible. We will show by induction on n that $I \cap [0, n] \subseteq \langle I_0 \rangle$. Suppose $n \in I$ is such that $I \cap [0, n) \subseteq \langle I_0 \rangle$. If $n \in I_0$, done. If $n \notin I_0$, there exist non-zero $n_1 \neq n_2 \in I$ such that $n = n_1 + n_2$. Thus, $0 < n_1,\ n_2 < n$. By our induction assumption, $n_1, n_2 \in \langle I_0 \rangle$. Hence $n = n_1 + n_2 \in \langle I_0 \rangle$. Therefore, $I = \langle I_0 \rangle$. $\square$

Lemma 7.6. *Let $I \subseteq \mathbb{Z}$ be a Type II saturated set. Then $I = \{n, 0, -n\}$, $n\mathbb{Z}$ or $n\mathbb{Z}_1$ for some non-zero integer n, where $\mathbb{Z}_1$ denotes the set $\{r \in \mathbb{Z} \mid r \geq -1\}$.*

Proof. Let $-n_1$ be the largest non-zero negative integer in I and n_2 be the smallest non-zero positive integer in I, and set $n = \min\{|n_1|, |n_2|\}$. If $|n_1| \neq |n_2|$, then $0 < |n_1 - n_2| < |n_1|, |n_2|$, contradiction ! Therefore, $|n_1| = |n_2| = n$ and so $\{-n, 0, n\} \subseteq I$.

Clearly, $\{-n, 0, n\}$ is a saturated set. For each $m \in I \setminus \{n, 0, -n\}$ such that $m > 0$, then $m = ln$ for some $l \geq 2$. For otherwise, $m = ln + r$ for some $0 < r < n$ and $l \geq 1$. Then

$$r = \underbrace{((m - n) - \cdots - n) - n}_{l \text{ terms}} \in I$$

which contradicts the choice of n. Since

$$2n = \underbrace{((m - n) \cdots) - n}_{l-2 \text{ terms}} \in I, \quad n\mathbb{N} \subseteq I.$$

Hence, $I \cap \mathbb{N} = n\mathbb{N}$. Similarly, if there is $m \in I \setminus \{n, 0, -n\}$ such that $m < 0$, then $I \cap (-\mathbb{N}) = -n\mathbb{N}$. Hence, I is either $\pm n\mathbb{Z}_1$ or $n\mathbb{Z}$. $\square$

Remark 7.7. One can observe that for any saturated set I, the bijection $I \leftrightarrow -I$ induces a Lie algebra isomorphism $L(I) \cong L(-I)$.

Corollary 7.8. *If I is a Type II saturated set, $L(I) \cong \mathfrak{sl}_2(k)$, W_1 or W.*

Proof. By Lemma 7.6, $I = \{n, 0, -n\}$, $n\mathbb{Z}$ or $n\mathbb{Z}_1$ for some non-zero integer n. For $I = \{n, 0, -n\}$, let

$$e = 2e_0/n, \ f = e_n/n, \ g = e_{-n}/n.$$

Then e, f, g satisfy the well-known relations for $\mathfrak{sl}_2(k)$. For $I = n\mathbb{Z}$ (or $n\mathbb{Z}_1$), let

$$g_r = e_{rn}/n$$

for $r \in \mathbb{Z}$ (or $r \in \mathbb{Z}_1$). Then

$$[g_r, g_l] = (l - r) g_{r+l}.$$

Therefore $L(I) \cong W$ if $I = n\mathbb{Z}$ and $L(I) \cong W_1$ if $I = n\mathbb{Z}_1$. $\square$

Corollary 7.9. *The finite dimensional saturated Lie subalgebras of W are k, $\mathfrak{sl}_2(k)$ and $\langle e_0, e_1 \rangle$ up to isomorphism.*

8. Calculations of the cohomology group $H^1(L(I), L(I) \wedge L(I))$

In this section, we consider a saturated Lie subalgebra $L = L(I)$ of the Witt algebra W, and calculate $H^1(L, L \wedge L)$. By Corollary 7.9 we may assume I is infinite.

Lemma 8.1. *Let $\mathfrak{g}$ be a Lie algebra and $\delta \in Z^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g})$. Then $\mu \circ \delta \in Z^1(\mathfrak{g}, \mathfrak{g})$ where μ is the Lie multiplication.*

Proof. Direct verification. □

Lemma 8.2. *Let I be an infinite saturated set and set $L = L(I)$. Then $H^1(L, L) = 0$.*

Proof. If I is of Type II, then by Corollary 7.8, $L \cong W$ or W_1 as Lie algebras. However, $H^1(W, W) = H^1(W_1, W_1) = 0$ (see Proposition 5.6). If I is of type I, we may assume I is non-negative. Since L is an inner graded Lie algebra, it suffices to show $H^1_{(0)}(L, L) = 0$. Let $\delta \in Z^1_{(0)}(L, L)$. Then $\delta(e_n) = \lambda_n e_n$ for $n \in I$ for some $\lambda_n \in k$. Since

$$n\delta e_n = \delta[e_0, e_n] = [e_0, \delta e_n] - [e_n, \delta e_0] = n\delta e_n + n\lambda_0 e_n$$

for $n \in I$, $\lambda_0 = 0$. As δ is a derivation, one can easily see that

$$\lambda_{n_1+n_2} = \lambda_{n_1} + \lambda_{n_2} \tag{7}$$

for $n_1 \neq n_2 \in I$. Therefore, if $r \in I$ and $lr \in I$ for some $l > 1$, then $(l+s)r \in I$ for all $s \geq 0$. By induction, one can prove that

$$\lambda_{(l+s)r} = s\lambda_r + \lambda_{lr}\,. \tag{8}$$

In particular, $\lambda_{(2l+1)r} = (l+1)\lambda_r + \lambda_{lr}$. By Eqs. (7) and (8),

$$\lambda_{(2l+1)r} = \lambda_{(l+1)r} + \lambda_{lr} = \lambda_r + 2\lambda_{lr}\,.$$

Therefore, $\lambda_{lr} = l\lambda_r$. Let n_0 be the smallest non-zero integer in I. Then, for $n_0 < m \in I$, $m+n_0 \in I$. Assume $l(m+n_0) \in I$ for some $l \geq 1$. As $l(m+n_0) > n_0$, $l(m+n_0)+n_0 \in I$. Moreover, $l(m+n_0)+n_0 > m$ and $(l+1)(m+n_0) \in I$. Hence, by induction, $l(m+n_0) \in I$ for $l \geq 1$. In particular, $mn_0(m+n_0) \in I$. Thus,

$$\lambda_{mn_0(m+n_0)} = (m+n_0)m\lambda_{n_0} = n_0(m+n_0)\lambda_m\,.$$

Therefore,

$$\lambda_m = m\lambda_{n_0}/n_0.$$

Take $\lambda = \lambda_{n_0}/n_0$. Then, $\delta = \lambda\, ad e_0 \in B^1_{(0)}(L, L)$. □

Theorem 8.3. *For any infinite saturated set I, $H^1(L, L \wedge L) = 0$.*

Proof. For simplicity, write L for $L(I)$. If I is of Type II, $H^1(L, L \wedge L) = 0$ by Corollary 7.8 and Theorem 5.7. If I is of Type I, we can assume I is non-negative (replacing I by $-I$ if necessary). Let $\delta \in Z^1_{(0)}(L, L \wedge L)$ and $\theta = \mu \circ \delta$ where μ is the Lie multiplication on L. Then, $\theta \in Z^1_{(0)}(L,L)$ by Lemma 8.1. The subgroup $B^1_{(0)}(L,L)$ is generated by $ad(e_0)$. By Lemma 8.2, there exists $\lambda \in k$ such that $\theta = \lambda\, ad(e_0)$. Let I_0 be the set of irreducible elements in I. Notice that $L \wedge L$ is also an inner graded L-module and the degree m-homogeneous space is

$$(L \wedge L)_m = \langle e_0 \wedge e_m \rangle$$

for $m \in I_0$. Therefore, $\delta(e_m) = \delta_m e_0 \wedge e_m$ for $m \in I_0$ and $\delta_m \in k$. Since

$$m\lambda e_m = \theta(e_m) = \mu \circ \delta(e_m) = m\delta_m e_m\,,$$

$\lambda = \delta_m$ for $m \in I_0 \setminus \{0\}$. It is enough to show that $\delta = 0$. Since $\{e_m \mid m \in I_0\}$ generates L as a Lie algebra and δ is a derivation, it suffices to prove that $\lambda = 0$. As $|I| = \infty$, there exist $m, n \in I \setminus \{0\}$ such that $m > n$. Then

$$\begin{aligned}(m-n)\delta e_{m+n} &= [e_n, \delta e_m] - [e_m, \delta e_n]\\ &= \lambda([e_n, e_0 \wedge e_m] - [e_m, e_0 \wedge e_n])\\ &= \lambda(2(m-n)e_0 \wedge e_{m+n} - (m+n)e_n \wedge e_m)\,.\end{aligned}$$

Therefore,

$$\delta e_{m+n} = \lambda\left(2e_0 e_{m+n} - \frac{m+n}{m-n} e_n \wedge e_m\right). \tag{9}$$

By using the usual iteration, one can obtain

$$\begin{aligned}\delta e_{2m+n} &= \lambda\left(3e_0 \wedge e_{2m+n} - \frac{4m+2n}{n} e_m \wedge e_{m+n}\right),\\ \delta e_{3m+n} &= \lambda\left(4e_0 \wedge e_{3m+n} - \frac{9m+3n}{m+n} e_m \wedge e_{2m+n}\right).\end{aligned}$$

Considering the two different ways $(3m+n)+n$ and $(2m+n)+(m+n)$ to sum up to $3m+2n$, one can obtain

$$\begin{aligned}\delta e_{3m+2n} = \lambda\Bigg(& 5e_0 \wedge e_{3m+2n} - \frac{3m+5n}{3m} e_n \wedge e_{3m+n} - \frac{2(3m+n)}{m+n} e_m \wedge e_{2m+2n}\\ & - \frac{(3m+n)(m-n)}{(m+n)m} e_{m+n} \wedge e_{2m+n}\Bigg)\end{aligned}$$

and

$$\begin{aligned}\delta e_{3m+2n} = \lambda\Bigg(& 5e_0 \wedge e_{3m+2n} - \frac{(m+n)^2}{(m-n)m} e_n \wedge e_{3m+n} - (11m+7n) e_{m+n} \wedge e_{2m+n}\\ & + \frac{2(m+n)}{m-n} e_m \wedge e_{2m+2n}\Bigg).\end{aligned}$$

Comparing the coefficient of $e_n \wedge e_{2m+n}$, we have

$$\frac{\lambda(m+n)^2}{(m-n)m} = \frac{\lambda(3m+5n)}{3m}.$$

Hence, $4\lambda n(m+2n) = 0$. Since $n, m > 0$, $\lambda = 0$ as required. $\square$

9. Proof of Theorem 1.1

Lemma 9.1. *Let I be an infinite saturated set of Type I. For any $T \in Aut_{Lie}(L(I))$, $Te_n \in \langle e_n \rangle$* for $n \in I$.

Proof. We may assume I is non-negative since $L(I) \cong L(-I)$. Since every semisimple element in $L(I)$ is a scalar multiple of e_0, $Te_0 = \alpha_0 e_0$ for some $\alpha_0 \in k^*$. By the proof of Proposition 4.1, $\langle e_0, e_n \rangle$ $(n \in I \setminus \{0\})$ are all the two dimensional Lie subalgebras of $L(I)$. Therefore, $T(\langle e_0, e_n \rangle) = \langle e_0, e_{n'} \rangle$ for some $n' \in I$. Thus $Te_n = \alpha_n e_{n'}$ for some $\alpha_n \in k^*$. Moreover,

$$n\alpha_n e_{n'} = T[e_0, e_n] = [Te_0, Te_n] = n'\alpha_0\alpha_n e_{n'}$$

for any nonzero $n \in I$. Therefore, $n = \alpha_0 n'$. Let n_0 be the smallest non-zero integer in I. Then $Te_{n_0} = \alpha_{n_0} e_{n_0/\alpha_0} \in L(I)$. Hence, $\alpha_0 \leq 1$. As $T^{-1}e_0 = \alpha_0^{-1} e_0$, $\alpha_0^{-1} \leq 1$. Therefore, $\alpha_0 = 1$ and $Te_n = \alpha_n e_n$ for $n \in I$. $\square$

Remark 9.2. Let I be an infinite saturated set. By the same argument as Lemma 5.4, one can prove that $H^0(L(I), \bigwedge^n L(I)) = 0$ for $n \geq 1$. Therefore, by Theorem 8.3, the Lie bialgebra structures on $L(I)$ are of triangular coboundary type.

Proof of Theorem 1.1. Let $\mathfrak{g}$ be an infinite dimensional Lie subalgebra of W such that $e_o \in \mathfrak{g}$ and $\mathfrak{g} \ncong W$ as Lie algebras. Then $\mathfrak{g} = L(I)$ for some infinite saturated set I of $\mathbb{Z}$. If I is of Type II, then $\mathfrak{g} \cong W_1$ under the isomorphism given in the proof of Corollary 7.8. Hence, the result follows from Theorem 6.1. If I is of Type I, one may assume I is non-negative by Remark 7.7. Therefore, we may assume $\mathfrak{g}$ is a Lie subalgebra of W_1. Let $r \in L(I) \wedge L(I)$ be a solution of the CYBE. The Lie subalgebra $\mathfrak{h}_r$ associated to r is also a Lie subalgebra of W_1 of even dimension. Therefore, $\mathfrak{h}_r = \langle e_0, e_n \rangle$ for some $n \in I$ by the proof of Proposition 4.1. Hence, $r = \alpha e_0 \wedge e_n$ for some scalar α. Notice that $Te_m = \beta^m e_m$ is a Lie algebra automorphism on $L(I)$ where β is an nth root of α. Therefore,

$$r \sim e_0 \wedge e_n\,.$$

By Proposition 2.11, $L(I)^{(r)} \cong L(I)^{(n)}$ as Lie bialgebras. Therefore, by Remark 9.2, every the Lie bialgebra structure on $\mathfrak{g}$ is isomorphic to $\mathfrak{g}^{(n)}$ for some $n \in I$. It remains to prove that $L(I)^{(n)}$ $(n \in I)$ are distinct Lie bialgebras up to isomorphism. Let $m, n \in I$ such that $L(I)^{(n)} \cong L(I)^{(m)}$ as Lie bialgebras. Since $H^0(L(I), L(I) \wedge L(I)) = 0$, by Proposition 2.11, there is a Lie algebra automorphism T on $L(I)$ such that $T(e_0) \wedge T(e_n) = e_0 \wedge e_m$. This implies that $m = n$ by Lemma 9.1. $\square$

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