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Regular Sequences in $\mathbb{Z}_2$-Graded Commutative Algebra

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We investigate regular sequences consisting of even and odd elements in the context of $\mathbb{Z}_2$-graded commutative algebra (or "superalgebra"). We associate to every finite module A over a $\mathbb{Z}_2$-commutative local ring a coherent sheaf $\mathbb{P}\mathscr{C}_A$ on a projective space $\mathbb{P}^n$, and its support contains all information on odd regular sequences on A. We introduce the $\mathbb{Z}_2$-version of the classical notion of regular local rings, and it turns out that over such rings the common length of all maximal odd A-regular sequences is connected with the growth of the ranks of the terms in a minimal free resolution of A. We also investigate properties of flat local homomorphisms and a partial analogue of the theorem of Vasconcelos on conormally free ideals.

CONTENTS

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1. Introduction; Preliminaries

1.1. In the context of classical commutative algebra, the concept of regular sequences is well investigated (cf., e.g., [13, 6]). Two of the central results we would like to mention are the common length of all maximal regular sequences on a finite module A over a local ring R, which is called *depth of* A, and, in case that R is regular, its connection with the projective dimension of A which is given by the formula of Auslander and Buchsbaum

$$\operatorname{pd} A + \operatorname{depth} A = \dim R. \tag{1.1.1}$$

On the other hand, recent developments in quantum field theory have raised the interest of mathematicians into *superalgebra* and *supergeometry* (cf., e.g., [2, 9–12, 15]). Roughly speaking, the basic ideology of "super" consists in generalizing the classical law $ab = ba$ to the $\mathbb{Z}_2$-*graded commutative law*

$$ab = (-1)^{|a||b|}\, ba \tag{1.1.2}$$

(cf. 1.2 for precise definitions). The standard example for this is the Grassmann algebra $\Lambda_R(\xi_1, ..., \xi_m)$ over a commutative ring R, and it is in fact a kind of prototype: It is nothing but the "odd analogon" of the polynomial algebra $R[x_1, ..., x_n]$ over R. The general super polynomial algebra over R will be $\Lambda_R(\xi_1, ..., \xi_m) \otimes_R R[x_1, ..., x_n]$, and we will consequently denote it by

$$R[\xi \mid x] = R[\xi_1, ..., \xi_m \mid x_1, ..., x_n].$$

Now it is not difficult to observe that large parts of the formal machinery of commutative algebra and algebraic geometry can be carried over to the more general $\mathbb{Z}_2$-graded situation. In particular, without further ado one can associate to every $\mathbb{Z}_2$-graded ring which satisfies (1.1.2) its *affine spectrum* $X = \operatorname{Spec} R$; the only difference is that $\mathcal{O}_X$ is now a sheaf of $\mathbb{Z}_2$-graded commutative algebras. Thus, there is nothing dangerous about "super" in general and the rule (1.1.2) in particular: We may and we will use mainly the techniques of classical commutative algebra and algebraic geometry, and anyone who is acquainted with them is invited to read this paper.

Now if we accept the thesis that $K[\xi]$ is the odd analogue of $K[x]$ we are faced with the fact that while $\operatorname{Spec} K[x]$ has a quite nontrivial structure even as topological space, $\operatorname{Spec} K[\xi]$ is simply a point with a structure sheaf. This is a general phenomenon: Roughly speaking, one can say that while even dimensions exist both infinitesimally (as directions in the tangent space of any point) and globally (as directions in the underlying topological space of a scheme), odd dimensions are invisible in the underlying topological space; they have "no topology": They are coded into the structure sheaf, and from there they emerge geometrically in an infinitesimal way, namely, as the directions in the odd part of the tangent space of any point.

Nevertheless, the odd directions have their own geometry which interferes with the geometry of the even ones, and which deserves further study. In particular, they exhibit specific aspects which have no classical pendant, and in this paper we want to investigate some of them.

In [14], the authors proposed an "odd pendant" to the classical notion of regular sequences: If A is a module over a $\mathbb{Z}_2$-graded commutative ring R, an odd element $\rho \in R_1$ is called *A-regular* if the sequence

$$A \xrightarrow{\rho} A \xrightarrow{\rho} A$$

is exact. That this definition is reasonable is shown both by the Koszul complex characterization of regularity (cf. 2.3) and the characterization by the bijectivity of the Hironaka–Grothendieck map (cf. 3.1).

But the most striking property of this notion is the fact (apparently hitherto unnoticed) that if (R, M) is a local ring and $\rho \in R_1$ is regular on a finitely generated R-module A then it remains so after adding any odd element of M^2. In other words, the A-regularity of ρ depends only on the residue class of ρ in $(M/M^2)_1 := \Phi$. Once this is proved it generalizes to sequences, and we may call a subspace $V \subseteq \Phi$, *A-regular* if some—and hence any—sequence of elements $\rho_1, ..., \rho_k$ the residues of which in Φ form a base of V is A-regular.

Moreover, it turns out that V is A-regular iff $\mathbb{P}(\Pi V)$ does not intersect a certain closed subset Sing A in $\mathbb{P}(\Pi\Phi)$ (which, however, may have no K-rational point). Therefore any two maximal odd A-regular sequences have the same length which is given by

$$\operatorname{odpth} A := \dim_K \Phi - 1 - \dim \operatorname{Sing} A$$

and which we call the *odd depth* of A. Thus, the odd analogue of the classical fact that any two maximal A-regular sequences have the same length is true—but it holds by quite other reasons.

Moreover, if R is a regular local ring in the sense to be defined in 3.3, Sing A is the support of two coherent sheaves $\mathscr{C}_A$ and $\mathscr{C}_{A'}$ which live on the projective superspace $\mathbb{P}(\Pi(M/M^2))$ (the underlying space of the latter is just $\mathbb{P}(\Pi\Phi)$), and the ith graded piece of the corresponding $\mathbb{Z}$-graded modules is given by

$$\operatorname{Ext}^i_R(K, A) \tag{1.1.3}$$

and

$$\mathrm{Ext}^i_R(A, K) \tag{1.1.4}$$

respectively. It follows from the Hilbert–Serre theorem that the dimensions of the K-vector spaces (1.1.3) and (1.1.4) grow polynomially with i, and the common degree of these polynomials is just $n - \mathrm{odpth}\ A$.

Thus the odd depth is intimately connected with the *growth* of a minimal free resolution of A. This is a remarkable odd pendent of the formula (1.1.1) which links the classical depth with the *length* of the (in this case finite) minimal free resolution.

A further remarkable consequence of this is that a module A over a regular local ring R possesses a *finite* free resolution iff $n = \mathrm{odpth}\ A$. If this is the case the classical formula (1.1.1) holds again.

The main technical tool for the construction of $\mathscr{C}_A$, $\mathscr{C}_{A'}$, and Sing A will be a certain functor which we call the *Koszul transform* because of its intimate connection with the Koszul complex. If $Y \subseteq X$ is a subscheme of a (super) scheme such that the conormal sheaf $\mathscr{N}_{Y/X}$ is locally free then the latter determines a vector bundle (in the super sense) $\Pi(N_{Y/X})^*$, and the Koszul transform assigns to every coherent sheaf $\mathscr{A}$ on X a coherent sheaf $\mathbb{P}\mathscr{C}_{\mathscr{A}}$ on the associated projective bundle $\mathbb{P}(\Pi(N_{Y/X})^*)$.

Although we cannot claim to have this functor well understood, it is a remarkable object in itself since it transforms "odd things" into even ones and vice versa. Moreover, it seems to have connexions (up to now not well understood, either) with the construction of [3], where a certain correspondence is established between coherent sheaves on the projective space $\mathbb{P}^n$ and certain modules over the Grassman algebra. It would be interesting here to have more clarity.

I wish to thank Dr. Thomas Zink for many valuable discussions.

1.2. Concerning the framework of $\mathbb{Z}_2$-graded linear algebra, we will generally follow [8], cf. also [12]; in the following we recall some main points and fix some further notations and conventions.

Generally, all abelian groups A (in particular, all rings, modules, vector spaces,...) are equipped with a $\mathbb{Z}_2$-grading:

$$A = A_0 \oplus A_1.$$

If $a \in A_i$ for $i = 0, 1$ we call *a homogeneous* and write $|a| = i$; and we call a *even* or *odd* if $i = 0$ and $i = 1$, respectively.

The parity assignments are always made in such a way that the *parity rule* holds: The parity of a multilinear expression is the sum modulo 2 of the parities of the factors.

We also note the *exchange rule*: Whenever in a multilinear expression two adjacent terms a, b have to be interchanged the sign factor $(-1)^{|a||b|}$ has to be introduced (cf. also [12 p. 179] for further comment).

By abuse of language, we will use the term *ring* to denote an associative, $\mathbb{Z}_2$-graded ring with unit element $R = R_0 \oplus R_1$ which satisfies

$$|ab| = |a| + |b| \qquad \text{for} \quad a, b \in R_0 \cup R_1,$$

according to the parity rule, and which is commutative in the sense of the exchange rule:

$$ab = (-1)^{|a||b|}\, ba. \tag{1.2.1}$$

Moreover, we require

$$a^2 = 0 \qquad \text{for} \quad a \in R_1. \tag{1.2.2}$$

It is usually required in the literature that 2 is a unit in R; then (1.2.2) is implied by (1.2.1). However, since we do not want this restriction we will show that the rules (1.2.1) and (1.2.2) are "consistent" without imposing it.

LEMMA. *Let R' be a ring, and let R be an associative, $\mathbb{Z}_2$-graded algebra over R' which is generated as R'-algebra by a set S of homogeneous elements. Assume that the laws (1.2.1) and (1.2.2) are satisfied for all $a, b \in R'_0 \cup R'_1 \cup S$. Then they are satisfied for all homogeneous $a, b \in R$, i.e., R is a ring again.*

Proof. It is clear that we can assume that S is finite. Moreover, by induction we are reduced to the case that S consists of a single element, and the result then follows by elementary computations. ∎

It follows from the lemma that the tensor product $R \otimes_{R'} R''$ of two rings over a third one if equipped with $\mathbb{Z}_2$-grading according to the parity rule and multiplication according to the exchange rule (cf. [8, 1.1.4]) is a ring again. Note that a *ring homomorphism* $\varphi: R \to R'$ is always required to respect the parity. Then $\ker \varphi$ is an ideal in the sense of the definition below.

By an *R-module A*, where R is a ring, we will understand a $\mathbb{Z}_2$-graded left unitary module over the associative ring R which, according to the parity rule, satisfies

$$|ra| = |r| + |a| \qquad \text{for} \quad r \in R_0 \cup R_1,\ a \in A_0 \cup A_1.$$

Setting in accordance with the exchange rule

$$ar := (-1)^{|r||a|}\, ra,$$

A becomes a bimodule over R (i.e., $(ra)\,r' = r(ar')$). If A, B are R-modules then $A \otimes_R B$, $A \oplus B$, $\operatorname{Hom}_R(A, B)$ are R-modules in an obvious way (cf. [8]). Besides this, we have the *parity shift functor* Π: $A \mapsto \Pi A$. The set ΠA consists of all symbols Πa, $a \in A$, and the module structure on ΠA is determined by the requirement that the map Π: $A \to \Pi A$, $a \mapsto \Pi a$, be an odd, bijective R-linear map (i.e., $\Pi(ra) = (-1)^{|r||\Pi|}\, r\Pi a = (-1)^{|r|}\, r\Pi a$). Briefly, we put

$$A^{k|l} := \bigoplus^{k} A \oplus \bigoplus^{l} \Pi A.$$

Thus $A^{k|l} = A \otimes_R R^{k|l}$. One can think of $A^{k|l}$ also as the set $\mathsf{X}^{k+l} A$ equipped with componentwise addition and right multiplication with scalars and $\mathbb{Z}_2$-grading

$$A_i^{k|l} = \mathsf{X}^{k} A_i \times \mathsf{X}^{l} A_{1-i} \qquad (i = 0, 1).$$

Thus we will use $(a\,|\,\alpha) = (a_1, ..., a_k\,|\,\alpha_1, ..., \alpha_l) \in A_0^{k|l}$ as shorter notation of $a_1, ..., a_k \in A_0$, $\alpha_1, ..., \alpha_l \in A_1$. We call an R-module P *free of rank* $k\,|\,l$ if it is isomorphic to $R^{k|l}$; then $k\,|\,l$ are uniquely determined by P. For the construction of the *symmetric power* $S_R P$ cf. [8]; $S_R P$ is a ring again. Note that if V is an odd vector space over a field K (i.e., $V_0 = 0$) then $S_K V$ is the classical exterior algebra over V.

By an *ideal* I in a ring R we mean a submodule of the R-module R. Thus I is homogeneous:

$$I = I \cap R_0 + I \cap R_1,$$

and R/I is a ring again. Generally, in $\mathbb{Z}_2$-commutative algebra only homogeneous things are of interest (else one would have to do genuinely non-commutative algebra).

We call an R-module A *finite* if there exists a surjection $R^{k|l} \to A$ for suitable k, l; and we call R *Noetherian* if all ideals in R are finite. In that case, any submodule of a finite R-module is finite again.

If we speak of *exact sequences* $0 \to A' \to^{\alpha} A \to^{\beta} A'' \to 0$ it is tacitly understood that α, β are homogenous A-homomorphisms (but they may be either even or odd). On the other hand, it is natural to require—and we will do so throughout—that in a *complex* of R-modules the differential is always an odd map (cf. [12, Chap. 3, Section 4, No. 1]). In the Koszul complex (cf. 2.3) this is automatically the case; on the other hand, given a complex, the differential of which is in each degree homogeneous (and other complexes are not natural to consider), then it takes only some applications of the functor Π to convert it into a complex with an odd differential.

This convention also prescribes the $\mathbb{Z}_2$-grading of the R-modules $\mathrm{Ext}^i_R(A, A')$ and $\mathrm{Tor}_i^R(A, A')$. In particular, if $R_1 = A_1 = A'_1 = 0$ (so that we are in the situation of classical commutative algebra) it follows that

$$\mathrm{Ext}^i_R(A, A') = \mathrm{Ext}^i_R(A, A')_j$$

and likewise with the Tor's; here $j = 0, 1$ denotes the residue of i modulo 2.

If we speak of a *classical ring* R we mean a ring with $R_1 = 0$. In particular, every ring without zero divisors is classical. Moreover, the category of all classical rings is a full subcategory of the category of all rings in the sense of the definition above.

If R is a ring we will denote by R^1 the ideal which is generated by the odd part R_1 of R, and by R^i its ith power. It is easily seen that all elements of R^1 are nilpotent. In fact, if R is Noetherian then R^1 is generated by finitely many, say n, odd elements, and it follows that $R^{n+1} = 0$.

We can associate to every ring R the classical ring $\tilde{R} := R/R^1$; we will denote the projection $R \to \tilde{R}$ by $r \mapsto \tilde{r}$. If $I \subseteq R$ is an ideal in a ring and A is an R-module we put as usual

$$\mathrm{gr}^i_I A := I^i A / I^{i+1} A, \qquad \mathrm{gr}_I A := \bigoplus_{i \geqslant 0} \mathrm{gr}^i_I A.$$

Then $\mathrm{gr}_I A$ is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\mathrm{gr}_I R$. By $\sigma_i \colon I^i A \to \mathrm{gr}^i_I A$ we will denote the projection.

We call an ideal P in a ring R *prime* if R/P has no zero divisors. Then $P \supseteq R^1$, and it is easy to see that the assignments $P \mapsto \tilde{P}$, $P \mapsto P_0$ yield 1–1-correspondences between the primes of the rings R, $\tilde{R}$, and R_0.

As usual, we call a Noetherian ring R *local* if it has a unique maximal ideal. We will also use the standard notation (R, M, K) to indicate that R is a local ring with maximal ideal M and residue field K. The Grassman algebra $K[\xi_1, ..., \xi_n]$ over a field K is a non-classical example of a local ring.

Like in the classical situation, we can associate to every ring R its *affine spectrum* $\mathrm{Spec}\, R$ (cf. [10]). By abuse of language, we call a locally ringed space $X = (\mathrm{space}(X), \mathcal{O}_X)$ a *scheme* iff it is locally isomorphic to an affine scheme (no separability condition required). Generally, we will freely use the terminology and techniques of the classical scheme language, e.g., in forming relative Spec's and Proj's.

If X is a scheme let $\mathcal{O}^1$ denote the ideal subsheaf of $\mathcal{O} = \mathcal{O}_X$ generated by the odd part of $\mathcal{O}$. We put $\tilde{\mathcal{O}} := \mathcal{O}/\mathcal{O}^1$, $\tilde{X} := (\mathrm{space}(X), \tilde{\mathcal{O}})$, $X_0 := (\mathrm{space}(X), \mathcal{O}_0)$. These are classical schemes, and we have natural morphisms

$$\tilde{X} \hookrightarrow X \to X_0.$$

We call $\tilde{X}$ the *underlying classical scheme* of X.

Now if $\varphi: X \to Y$ is a morphism of schemes it is advisable to introduce into the usual definition of the cotangential sheaf $\Omega^1_{X/Y}$ an additional parity shift: We set $\Omega^1_{X/Y} := \Pi(\mathscr{I}/\mathscr{I}^2)$, where $\mathscr{I}$ is the ideal sheaf which cuts out the diagonal in $X \times_Y X$. Then the differential of the de-Rham-complex $0 \to \mathcal{O}_X \to^d \Omega^1_{X/Y} \to^d \Omega^2_{X/Y} \to \cdots$ is odd, in accordance with our requirements from above, and, moreover, $\Omega_{X/Y} := \bigoplus_{i \geqslant 0} \Omega^i_{X/Y}$ is a sheaf of $\mathbb{Z}_2$-commutative algebras again. In fact, if $X \to k$ is smooth over a field k, then $\operatorname{Spec} \Omega_{X/k}$ is nothing but the total space ΠT_X of the vector bundle over X with graded section sheaf $\Pi \mathscr{T}_X$, and the exterior derivative d turns into an odd vector field on ΠT_X. (Cf. [15] for the situation in the C^∞ category.)

2. Elementary Theory of Regular Sequences

2.1. Let A be a module over a ring R and $\rho \in R_1$. Because of $\rho^2 = 0$ we have a complex

$$A \xrightarrow{\rho} A \xrightarrow{\rho} A$$

the cohomology of which we will denote by $\mathscr{C}(\rho, A)$. We call ρ *regular on* A or A*-regular* if $\mathscr{C}(\rho, A) = 0$; obviously, this is equivalent with the assertion that

$$0 \to \rho A \subsetneq A \xrightarrow{\rho} \rho A \to 0 \tag{2.1.1}$$

is an exact sequence.

We call a sequence $(\rho) = (\rho_1, ..., \rho_n)$ of odd elements of R *regular on* A or A*-regular* if for each i the element ρ_i is regular on $A/(\rho_1, ..., \rho_{i-1}) A$. If this is the case one has by (2.1.1) and induction,

$$A/(\rho_1, ..., \rho_n) A = \rho_1 \cdots \rho_n A.$$

It also follows from (2.1.1) and induction that this module is not zero whenever $(\rho_1, ..., \rho_n)$ is A-regular and $A \neq 0$.

Remark. Later on, we will also consider regular sequences which consist of even and odd elements, and we will show that the corresponding theories "decouple."

Lemma 2.1.1. *Let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{2.1.2}$$

be an exact sequence of R-modules, let $\rho \in R_1$. *Then we have an exact triangle*

$$\begin{tikzcd} & \mathscr{C}(\rho, B) \arrow[dr] & \\ \mathscr{C}(\rho, A) \arrow[ur] & & \mathscr{C}(\rho, C) \arrow[ll, "\delta"] \end{tikzcd} \tag{2.1.3}$$

In particular, if ρ *is regular on two members of* (2.1.2) *then it is so on the third, too.*

Proof. By applying Π if necessary we may assume α, β to be even. Assign to any R-module A the infinite complex

$$A_\rho \colon \cdots \xrightarrow{\rho} A \xrightarrow{\rho} A \xrightarrow{\rho} \cdots .$$

Then (2.1.3) arises by taking cohomology in $0 \to A_\rho \to B_\rho \to C_\rho \to 0$. ∎

Remarks. (3) In case that ρ is R-regular, the complex R_ρ truncated at the zeroth term will be a free resolution of $\rho R = R/\rho R$. Hence $\mathscr{C}(\rho, A) = \operatorname{Tor}_i^R(A, \rho R) = \operatorname{Ext}^i_R(\rho R, A)$ for any $i \geqslant 1$, and the corresponding long exact sequences arising from (2.1.2) collapse to (2.1.3).

(4) If A is an R-module, $\rho \in R_1$, and $\sigma(\rho) \in \operatorname{gr}^1_{R^1}(R)$ is $\operatorname{gr}_{R^1}(A)$-regular then ρ is A-regular (Consider the spectral sequence associated to the complex A_ρ equipped with the R^1-adic filtration. Then the assumption says $E_2^{p,q} = 0$ for all p, q). The converse is false, as the following counterexample shows: Consider over $R := \mathbb{C}[\xi, \alpha, \beta]$ the module $A = Re_1 + Re_2$ with defining relations $|e_1| = 1$, $|e_2| = 0$, $\alpha\beta e_1 = \xi e_2$, $\alpha e_2 = \beta e_2 = 0$. Then A has the vector space basis $\{e_1, e_2, \xi e_1, \alpha e_1, \beta e_1, \xi e_2, \xi\alpha e_1, \xi\beta e_1\}$, and ξ is A-regular. On the other hand,

$$0 \to \operatorname{gr}^1_{R^1}(A) \xrightarrow{\sigma(\xi)} \operatorname{gr}^2_{R^1}(A)$$

is not exact: We have $\sigma_0(e_2) = 0$ but

$$\sigma(\xi)\,\sigma_0(e_2) = \sigma_1(\xi e_2) = \sigma_1(\alpha\beta e_1) = 0.$$

Hence $\sigma(\xi)$ cannot be regular on $\operatorname{gr}_{R^1}(A)$.

2.2.

PROPOSITION. *Let*

$$0 \to A \to B_1 \to \cdots \to B_m \to C \to 0 \tag{2.2.1}$$

be an exact sequence of R-modules; suppose that $(\rho_1, ..., \rho_n)$ *is a regular sequence on each* B_i. *Then* $(\rho_1, ..., \rho_n)$ *is regular on A iff it is regular on C. Moreover, if this is the case, the sequence*

$$0 \to \rho A \to \rho B_1 \to \cdots \to \rho B_m \to \rho C \to 0 \tag{2.2.2}$$

with $\rho := \rho_1 \rho_2 \cdots \rho_n$ *is exact, too.*

Proof. It suffices obviously to consider the case $m = 1$. We make induction on n:

Let $n = 1$. Then the first assertion follows from Lemma 2.1.1. We are left to show that if ρ_1 is regular on A, B_1, and C then

$$0 \to \rho_1 A \to \rho_1 B_1 \to \rho_1 C \to 0 \tag{2.2.3}$$

is again exact. Without loss of generality we may assume $A \subseteq B_1$. Then if $\rho_1 b \in \rho_1 B_1$ maps to zero we have by the exactness of (2.2.1) at least $\rho_1 b \in A$. On the other hand, since $\rho_1(\rho_1 b) = 0$ and since ρ_1 is regular on A, we get $\rho_1 b \in \rho_1 A$, and (2.2.3) is proved exact.

Now let $n \geq 1$, let the assertion be proved for the sequence $(\rho_1, ..., \rho_{n-1})$, put $\rho' := \rho_1 \rho_2 \cdots \rho_{n-1}$. Then

$$\begin{aligned}
&(\rho_1, ..., \rho_n) \ A\text{-regular} \\
\leftrightarrow\ &(\rho_1, ..., \rho_{n-1}) \ A\text{-regular, and } \rho_n \ \rho' A\text{-regular} \\
\leftrightarrow\ &(\rho_1, ..., \rho_{n-1}) \ C\text{-regular, } \rho_n \ \rho' A\text{-regular, and} \\
&0 \to \rho' A \to \rho' B_1 \to \rho' C \to 0 \text{ is exact} \qquad (2.2.4) \\
\leftrightarrow\ &(\text{by the case } n = 1) \ (\rho_1, ..., \rho_{n-1}) \ C\text{-regular, } \rho_n \\
&\rho' A\text{-regular, and (2.2.4) is exact} \\
\leftrightarrow\ &(\rho_1, ..., \rho_n) \ C\text{-regular.}
\end{aligned}$$

Moreover, if this is true, the exactness of (2.2.4) and the case $n = 1$ applied onto ρ_n yield the exactness of (2.2.2). ▮

2.3. Let R be a ring. In accordance with classical commutative algebra we call an even element $r \in R_0$ *regular* on an R-module A if the map $A \to^r A$ is injective but not bijective. A sequence $(s_1, ..., s_k)$ of even and odd elements of R is called *regular on A* or *A-regular* if for any $i = 1, ..., k$ s_i is regular on $A/(s_1, ..., s_{i-1}) A$.

Now let $(r \mid \rho) = (r_1, ..., r_m \mid \rho_1, ..., \rho_n) \in R_0^{m \mid n}$ be a fixed sequence of even and odd elements of R, let

$$(\eta \mid y) = (\eta_1, ..., \eta_m \mid y_1, ..., y_n)$$

be a sequence of m odd and n even variables, and consider the polynomial algebra $R[\eta|y] = \bigoplus_{i \geqslant 0} R[\eta|y]_i$ in its natural $\mathbb{Z}$-grading. The derivation

$$d = \sum_{i=1}^{m} r_i \frac{\partial}{\partial \eta_i} + \sum_{j=1}^{n} \rho_j \frac{\partial}{\partial y_j}$$

acts on it, and we get a complex of R-modules

$$K(r \mid \rho)\colon \cdots \xrightarrow{d} R[\eta \mid y]_1 \xrightarrow{d} R[\eta \mid y]_0 \to 0.$$

For any R-module A we call $K(r \mid \rho; A) := K(r \mid \rho) \otimes_R A$ the *Koszul complex of A with respect to* $(r \mid \rho)$ (cf. also [12, Chap. 3, Section 4, Nos. 2, 3, 6]).

THEOREM. *Suppose that either $m = 0$, or that R is noetherian, A is finite, and all r_i lie in the radical of R. Then the following assertions are equivalent*:

(i) $(r \mid \rho)$ *is A-regular*;

(ii) $H_i(K(r \mid \rho; A)) = 0$ *for all* $i \geqslant 1$;

(iii) $H_1(K(r \mid \rho; A)) = 0$.

We first note:

LEMMA. *Let $\rho \in R_1$, let $C.\colon \cdots \to^d C_1 \to^d C_0 \to 0$ be any chain complex of R-modules. If*

$$H_1(C. \otimes_R K(\rho; R)) = 0 \tag{2.3.1}$$

then

$$H_1(C.) = 0, \tag{2.3.2}$$

and

$$\rho \text{ is regular on } H_0(C.). \tag{2.3.3}$$

Proof. At any rate, we have

$$\begin{aligned} &H_1(C. \otimes_R K(\rho; R)) \\ &\quad = \{(c_1, c_0) \in C_1 \oplus C_0 \colon dc_1 + \rho c_0 = 0\}/ \\ &\qquad \{(dc_2 + \rho c_1, dc_1 + \rho c_0); \quad (c_2, c_1, c_0) \in C_2 \oplus C_1 \oplus C_0\}. \end{aligned} \tag{2.3.4}$$

Suppose that (2.3.1) holds. In order to prove (2.3.2) we suppose $dc_1 = 0$, $c_1 \in C_1$. Then $(c_1, 0)$ lies in the numerator of (2.3.4), hence (2.3.1) implies

$$(c_1, 0) = (dc'_2 + \rho c'_1, dc'_1 + \rho c'_0). \tag{2.3.5}$$

In particular, (c_1', c_0') lies in the numerator of (2.3.4), hence (2.3.1) implies

$$(c_1', c_0') = (dc_2'' + \rho c_1'', dc_1'' + \rho c_0'').$$

From this and (2.3.5) we get

$$c_1 = dc_2' + \rho(dc_2'' + \rho c_1'') = d(c_2' - \rho c_2''),$$

and (2.3.2) is proved.

For the proof of (2.3.5) suppose that ρ annihilates the homology class $[c_0]$, i.e., $\rho c_0 = dc_1$ with $c_1 \in C_1$. Then we get again $(-c_1, c_0) = (dc_2' + \rho c_1', dc_1' + \rho c_0')$, in particular $c_0 - \rho c_0' = dc_1'$, i.e., $[c_0] = \rho[c_0']$. ▮

Proof of the Theorem. The implication (i) → (ii) was first observed in [14] for the case $A = R$; the general case can be done in the same way. (ii) → (iii) is trivial. In order to prove (iii) → (i) we do induction on n. The case $n = 0$ is the classical one (cf. [1]). For $n > 0$ we may write $K(r \mid \rho; A) = K(r \mid \rho'; A) \otimes_R K(\rho_n; R)$ with $\rho' := (\rho_1, ..., \rho_{n-1})$. Now $H_1(K(r \mid \rho; A)) = 0$ implies by the lemma above that $H_1(K(r \mid \rho'; A)) = 0$, and hence, by hypothesis of induction, $(r \mid \rho')$ is A-regular. Moreover, the lemma says that ρ_n is regular on $H_0(K(r \mid \rho'; A)) = A/(r \mid \rho') A$, and hence $(r \mid \rho)$ is A-regular. ▮

2.4. Now we may "decouple" the odd and even elements in regular sequences.

THEOREM. *Let A be a finite module over a noetherian ring R.*

(i) *Let $(s_1, ..., s_k)$ be a sequence of even and odd elements which is A-regular, and suppose that all s_i lie in the radical of R. Then for any permutation $\pi \in S_k$ the sequence $(s_{\pi(1)}, ..., s_{\pi(k)})$ is A-regular, too.*

(ii) *If $(r \mid \rho) \in R_0^{m \mid n}$, and all r_i lie in the radical of R, then $(r \mid \rho)$ is A-regular iff both (r) and (ρ) are A-regular.*

Proof. Ad (i). Since any permutation is a product of transpositions of adjacent elements, we may assume $k = 2$. The case that s_1, s_2 are both odd follows from Theorem 2.3, while if they are both even the assertion is classical. Therefore we are left to prove:

LEMMA. *Let $r \in R_0$, $\rho \in R_1$. Then $(r \mid \rho)$ is regular on A iff $(\rho \mid r)$ is also.*

Proof. Assume that $(\rho \mid r)$ is A-regular. Then the sequence $0 \to \rho A \to A \to^{\rho} \rho A \to 0$ is exact, hence (cf. [4, Chap. IV, Section 1, No. 1, Proposition 3])

$$\mathrm{Ass}_{R_0} \rho A = \mathrm{Ass}_{R_0} A. \tag{2.4.1}$$

Hence r is a non-zero divisor on A, too. Moreover, Lemma 2.1.1 applied onto

$$0 \to A \xrightarrow{r} A \to A/rA \to 0 \tag{2.4.2}$$

shows that ρ is A/rA-regular, i.e., $(r \mid \rho)$ is A-regular.

Now assume that $(r \mid \rho)$ is A-regular. Then (2.4.2) and Lemma 2.1.1 show that $\mathscr{C}(\rho, A) \to^{r} \mathscr{C}(\rho, A)$ is an isomorphism. Since r lies in the radical of R the Nakayama lemma implies that $\mathscr{C}(\rho, A) = 0$, i.e., ρ is regular on A, and (2.4.1) now implies that r is regular on $\rho A = A/\rho A$. ∎

Continuation of Proof. Assertion (i) is proved. In view of it, we are left to show that if (r) and (ρ) are A-regular then (ρ) is regular on $A/(r)A$. This is easily done by induction on m, using Lemma 2.1.1 applied onto

$$\begin{aligned} 0 \longrightarrow A/(r_1, ..., r_i)\, A &\xrightarrow{r_{i+1}} A/(r_1, ..., r_i)\, A \\ &\longrightarrow A/(r_1, ..., r_{i+1})\, A \longrightarrow 0. \end{aligned}$$

The theorem is proved. ∎

2.5. Let A be an R-module, let $\rho, \alpha \in R_1$, $r \in R_0$, and suppose that ρ is A-regular. We look for conditions which guarantee that $\rho' := \rho + \alpha r$ is A-regular, too. By Lemma 2.1.1, the exact sequence

$$0 \longrightarrow \rho A \hookrightarrow A \xrightarrow{\rho} \rho A \longrightarrow 0$$

induces an exact triangle

$$\begin{tikzcd} \mathscr{C}(\rho', A) \arrow[rr, bend left, "\delta"] & & \mathscr{C}(\rho', \rho A). \arrow[dl] \\ & \mathscr{C}(\rho', A) \arrow[ul] & \end{tikzcd}$$

Hence:

LEMMA 2.5.1. *ρ' is A-regular iff δ is an isomorphism.*

Now any element of $\mathscr{C}(\rho', \rho A)$ has the form $[\rho a]$ with $\rho' \rho a = 0$; and δ acts by

$$\delta\colon [\rho a] \mapsto [\rho' a] = [\rho a] + [\alpha r a] =: (1 + N)[\rho a].$$

LEMMA 2.5.2. *N^2 maps $\mathscr{C}(\rho', \rho A)$ into $r\mathscr{C}(\rho', \rho A)$.*

Proof. Let $[\rho a] \to \mathscr{C}(\rho', \rho A)$, i.e., $\rho'\rho a = \alpha r\rho a = 0$. Then $N[\rho a] = [\alpha r a]$, and we have

$$\alpha r a = \rho a' \tag{2.5.1}$$

with suitable $a' \in A$. Thus

$$N^2[\rho a] = N[\rho a'] = [\alpha r a']. \tag{2.5.2}$$

Now we have, due to (2.5.1),

$$\rho(\alpha a') = -\alpha\rho a' = -\alpha(\alpha r a) = 0,$$

and

$$\rho'(\alpha a') = (\rho + \alpha r)(\alpha a') = 0;$$

hence $[\alpha a'] \in \mathscr{C}(\rho', \rho A)$. Now (2.5.2) yields

$$N^2[\rho a] = r[\alpha a']. \quad \blacksquare$$

Now, if $r^k = 0$ for some $k > 0$ then $N^{2k} = 0$, and hence δ is invertible. But in fact, we can do much more:

LEMMA 2.5.3. *Let (R, M) be a local ring, A a finite R-module, and let $\rho, \rho' \in R_1$, $\rho - \rho' \in M_1^2$. Assume that ρ is A-regular. Then ρ' is A-regular, too.*

Proof. Obviously, it suffices to do the case $\rho - \rho' = \alpha r$ where $\alpha \in R_1$, $r \in M_0$. Then, in the notations from above, we claim that the map $(1 - N^2)$: $\mathscr{C}(\rho', \rho A) \circlearrowleft$ is an isomorphism.

It is injective: $(1 - N^2)\,c = 0$ implies by Lemma 2.5.2 and induction that $c = N^{2k}c \in r^k\mathscr{C}(\rho'; \rho A)$ for all $k \geqslant 1$; and since $\mathscr{C}(\rho', \rho A)$ is finite and $r \in M_0$, we get $c = 0$. It is surjective due to

$$\mathscr{C}(\rho', \rho A) = (1 - N^2)\,\mathscr{C}(\rho', \rho A) + r\mathscr{C}(\rho', \rho A)$$

and the Nakayama lemma.

Now we have $(1 - N^2) = (1 + N)(1 - N)$; hence $\delta = 1 + N$ is an isomorphism, too, and Lemma 2.5.1 yields that ρ' is A-regular. $\blacksquare$

2.6. Let (R, M) be a local ring. We introduce the notation

$$\Phi := \Phi_R := (M/M^2)_1 = R_1/R_1 M_0.$$

This is a finite dimensional vector space over $K = R/M$; let $\varphi\colon M_1 \to \Phi$ denote the projection.

Remark. The geometric meaning of Φ is that it is the odd part of the conormal sheaf of the embedding of the closed point into Spec R.

THEOREM. *Let A be a finite, non-zero R-module.*

(1) *Let $(\rho_1, ..., \rho_n) \in R_1^n$ be an A-regular sequence. Then the elements $\varphi(\rho_1), ..., \varphi(\rho_n) \in \Phi$ are linearly independent.*

(2) *Let $V \subseteq \Phi$ be any K-subspace. The following assertions are equivalent:*

- (i) *There exists a basis $\varphi(\rho_1), ..., \varphi(\rho_n)$ of V with $\rho_1, ..., \rho_n \in R_1$ such that $(\rho_1, ..., \rho_n)$ is A-regular;*
- (ii) *For every basis $\varphi(\rho_1), ..., \varphi(\rho_n)$ of V with $\rho_1, ..., \rho_n \in R_1$, the sequence $(\rho_1, ..., \rho_n)$ is A-regular.*

If they are true we call the subspace $V \subseteq \Phi$ A-regular.

Proof. Let $\sum_{i=1}^n d_i \varphi(\rho_i) = 0$ be a relation of linear dependence with $d_1, ..., d_{k-1} \in R_0$, $d_k \in R_0 \backslash M_0$, $d_{k+1}, ..., d_n \in M_0$. Since d_k is a unit, we get

$$\rho_k = -\sum_{i=1}^{k-1} d_i d_k^{-1} \rho_i + \rho'' \qquad \text{with} \quad \rho'' \in M_1^2.$$

Now ρ_k is regular on $A' := \rho_1 \cdots \rho_{k-1} A$ and, by Theorem 2.4, $\rho_k - \rho''$ will be, too. But this element annihilates A' although $A' \neq 0$, which is a contradiction. Thus (i) is proved. Before proceeding we note:

LEMMA 2.6.1. *Let $(\rho_1, ..., \rho_n)$ be regular on A, let $\rho'_1, ..., \rho'_n \in R_1$ such that $\rho_i - \rho'_i \in M^2$ for all i. Then $(\rho'_1, ..., \rho'_n)$ is regular on A, too.*

Proof. By induction on n, using Lemma 2.5.3 and Theorem 2.4. ▮

LEMMA 2.6.2. *Let $(\rho_1, ..., \rho_n)$ be regular on A, let for $i = 1, ..., n$, $\rho'_i = \sum_{j=1}^n c_{ij} \rho_j$, where (c_{ij}) is an $n \times n$ matrix with* $\det(c_{ij}) \notin M$. *Then $(\rho'_1, ..., \rho'_n)$ is regular on A, too.*

Proof. By standard techniques, one has an isomorphism of the Koszul complexes $K((\rho); A) = K((\rho'); A)$, and Theorem 2.3 yields the assertion. ▮

Combining these lemmata, one easily accomplishes the proof of the theorem. ▮

It is remarkable that the theorem ceases to be valid for non-finite A: In order to construct a counterexample, we take a sequence $(x \mid \xi, \eta)$ of $2 \mid 1$ variables over a field K and consider the local ring $R = K[x]_{xK[x]}[\xi, \eta]$. It is embedded into its full ring of quotients $Q = K(x)[\xi, \eta]$. Now $(x\eta, \xi)$ is a

Q-regular sequence in Q, and it follows (one can apply the theorem for this) that $(\xi + x\eta, \xi)$ is Q-regular, too.

Now if we consider $A := Q/(\xi + x\eta)\,Q$ as an R-module, it follows that ξ is A-regular while $\xi + x\eta$ is not. One notes that this is possible only due to the fact that A possesses subfactors for which the Nakayama lemma fails.

The following "global" property of odd regular sequences is also remarkable (although we will not use it in the sequel):

COROLLARY. *Let R be a noetherian ring and A a finite R-module with*

$$\operatorname{supp} A = \operatorname{Spec} R. \tag{2.6.1}$$

Let $(\rho) = (\rho_1, ..., \rho_n)$ be an odd A-regular sequence, and suppose that $\tilde{R}$ (cf. 1.2) *is reduced. Then the homomorphism*

$$\bigoplus^{n} \tilde{R} \xrightarrow{(\rho)} R^1/R^2$$

is injective.

Proof. We have to show that any relation

$$\sum_{i=1}^{n} c_i \rho_i = 0 \qquad \text{with} \quad c_i \in R_0 \tag{2.6.2}$$

implies

$$c_1, ..., c_n \in R^1. \tag{2.6.3}$$

Due to (2.6.1) we have $A_P \neq 0$ for any $P \in \operatorname{Spec} R$. Applying the theorem onto the R_P-module A_P we get that the images of the ρ_i in $\Phi_{R_P} = (R_P)_1/P_0(R_P)_1$ are linearly independent. Therefore (2.6.2) implies $c_1/1, ..., c_n/1 \in PR_P$, and hence

$$c_1, ..., c_n \in P \qquad \text{for all} \quad P \in \operatorname{Spec} R. \tag{2.6.4}$$

Since $\tilde{R}$ is reduced, the intersection of all $P \in \operatorname{Spec} R$ is just R^1. Therefore (2.6.4) implies (2.6.3). ▮

3. REGULAR IDEALS AND REGULAR LOCAL RINGS

3.1. It is certainly no surprise that the Hironaka–Grothendieck characterization of regular sequences (cf. [6, (15.1.9)] or [7, Chap. III, Section 1]) can be extended to the $\mathbb{Z}_2$-graded case; but we will need it later on. Let R be a fixed noetherian ring. First we note:

LEMMA. *Let α: $A \to B$ be a surjection of R-modules, suppose that $\rho \in R_1$ is regular on B and that the induced map $A/\rho A \to B/\rho B$ is an isomorphism. Then α is an isomorphism, too.*

Proof. The hypothesis implies that $\operatorname{Ker}\alpha \subseteq \rho A$, and we are to show that $\operatorname{Ker}\alpha = 0$. Now if $\rho a \in \operatorname{Ker}\alpha$ we get from $\rho\alpha(a) = 0$, the regularity of ρ on B, and the surjectivity of α that $\alpha(a) = \rho\alpha(a')$ with some $a' \in A$, i.e., $a - \rho a' \in \operatorname{Ker}\alpha \subseteq \rho A$. This implies $\rho a \in \rho^2 A = 0$. ∎

Let A be an R-module which is equipped with a filtration $A = F_0 \supseteq F_1 \supseteq \cdots$, put as usual $\operatorname{gr}_F A := \bigoplus_{i \geqslant 0} F_i/F_{i+1}$. Let $(r_1, ..., r_m \mid \rho_1, ..., \rho_n) \in R_0^m \times R_1^n$ and let $I \subseteq R$ be the ideal generated by these elements. We define a new filtration of A by $F_i' := F_i + IF_{i-1} + \cdots + I^i A$. Then we have a surjection of $\mathbb{Z}$-graded R-modules

$$\psi\colon \operatorname{gr}_F A \otimes_R R/I[x_1, ..., x_m \mid \xi_1, ..., \xi_n] \to \operatorname{gr}_{F'} A$$

defined by

$$\sigma_i(a) \otimes_R [s]\, x^\mu \xi^\nu \mapsto \sigma_{i+|\mu|+|\nu|}(asr^\mu \rho^\nu)$$

($x_1, ..., x_m$ and $\xi_1, ..., \xi_n$ are even and odd indeterminates, respectively).

THEOREM. (i) *If the sequence* $(r \mid \rho) = (r_1, ..., r_m \mid \rho_1, ..., \rho_n)$ *is regular on* $\operatorname{gr}_F A$ *(in the sense of* 2.3*) then* ψ *is bijective.*

(ii) *Conversely, if* ψ *is bijective,* A *is finite, and the* $r_1, ..., r_m$ *lie in the radical of* R, *then* $(r \mid \rho)$ *is regular on* $\operatorname{gr}_F A$.

Things become easier to visualize in the case that $F_1 = 0$, i.e., $\operatorname{gr}_F A = A$. ψ is then the homomorphism

$$\begin{aligned} A/IA[x_1, ..., x_m \mid \xi_1, ..., \xi_n] &\to \operatorname{gr}_I A, \\ [a]\, x^\mu \xi^\nu &\mapsto \sigma_{|\mu|+|\nu|}(ar^\mu \rho^\nu) \end{aligned} \tag{3.1.1}$$

($\operatorname{gr}_I A$ is the module associated to the I-adic filtration on A), and we get:

COROLLARY 3.1.1. *If* A *is a finite module and* $r_1, ..., r_m$ *lie in the radical of* R *then* (3.1.1) *is an isomorphism iff* $(r \mid \rho)$ *is regular on* A.

Proof of the Theorem. First we prove the assertions for the special case $m \mid n = 0 \mid 1$: Put $\rho := \rho_1$. Then it is obvious that:

ψ is injective

$\leftrightarrow$ it holds true that

$$a_0 + a_1\rho \in F_{k+1} + \rho F_k \quad \text{with} \quad a_0 \in F_k,\ a_1 \in F_{k-1} \tag{3.1.2}$$

implies

$$a_0 \in F_{k+1} + \rho F_k \text{ and } a_1 \in F_k + \rho F_{k-1}. \tag{3.1.3}$$

Now if ρ is regular on $\mathrm{gr}_F A$ and (3.1.2) is given, then we have $\rho \cdot \sigma_{k-1}(a_1) = 0$ in $\mathrm{gr}_F(A)$, hence $\sigma_{k-1}(a_1) = \rho \cdot \sigma_{k-1}(a_2)$ with some $a_2 \in F_{k-1}$, hence $a_1 + \rho a_2 \in F_k$ and

$$a_1 \in F_k + \rho F_{k-1}. \tag{3.1.4}$$

From (3.1.2) and (3.1.4) we get

$$a_0 \in F_{k+1} + \rho F_k. \tag{3.1.5}$$

(3.1.4) and (3.1.5) together yield (3.1.3).

Conversely, suppose that (3.1.2) implies (3.1.3), and let $\rho \cdot \sigma_{k-1}(a_1) = 0$; i.e., putting $a_0 := -\rho a_1$, (3.1.2) is fulfilled. Then (3.1.3) yields $a_1 = \rho a_2 + a_3$ with $a_2 \in F_{k-1}$, $a_3 \in F_k$; hence $\sigma_{k-1}(a_1) = \rho \cdot \sigma_{k-1}(a_2)$ and ρ is proved regular on $\mathrm{gr}_F A$.

Now we may prove the theorem by induction on n: In the start of induction $n = 0$ the classical proof given in [6, Proposition 15.1.9] applies. Now let $n > 0$. We put

$$\begin{aligned} J &:= (r_1, ..., r_m \mid \rho_1, ..., \rho_{n-1}) R, \\ F_i'' &:= F_i + JF_{i-1} + \cdots + J^i A. \end{aligned} \tag{3.1.6}$$

Then the F_i'' form another filtration on A, and we have

$$F_i' = F_i'' + \rho_n F_{i-1}''.$$

Consider the map

$$\psi'' \colon \mathrm{gr}_F(A) \otimes_R R/J[x_1, ..., x_m \mid \xi_1, ..., \xi_{n-1}] \to \mathrm{gr}_{F''} A \tag{3.1.7}$$

defined in the same way as ψ. As in the classical case, we factor ψ by

$$\begin{aligned} &\mathrm{gr}_F A \otimes_R R/I[x_1, ..., x_m \mid \xi_1, ..., \xi_n] \\ &\quad = \mathrm{gr}_F A \oplus_R R/J[x_1, ..., x_m \mid \xi_1, ..., \xi_{n-1}] \otimes_R R/\rho_n R[\xi_n] \\ &\quad \xrightarrow{\psi'' \otimes 1_{R/\rho_n R[\xi_n]}} \mathrm{gr}_{F''} A \otimes_R R/\rho_n R[\xi_n] \xrightarrow{\psi'} \mathrm{gr}_{F'} A. \end{aligned}$$

Then we get:

ψ is an isomorphism

$\leftrightarrow \psi'' \otimes 1_{R/\rho_n R[\xi]}$ and ψ' are isomorphisms,

$\leftrightarrow$ (by the case $m \mid n = 0 \mid 1$) $\psi'' \otimes 1_{R/\rho_n R[\xi]}$ is an isomorphism, and ρ_n is regular on $\mathrm{gr}_{F''} A$,

$\leftrightarrow$ (applying the lemma above) ψ'' is an isomorphism, and ρ_n is regular on $\mathrm{gr}_{F''} A$

$\leftrightarrow$ (observing that the l.h.s. of (3.1.7) is a direct sum of copies of $\mathrm{gr}_F A \otimes_R R/J$) ψ'' is an isomorphism, and ρ_n is regular on $\mathrm{gr}_F A \otimes_R R/J$

$\leftrightarrow$ (by hypothesis of induction and (3.1.6)) $(r \mid \rho)$ is A-regular. ▮

In case that $m = 0$ we only need to look at the last piece of (3.1.1):

COROLLARY 3.1.2. *Let $I := (\rho_1, ..., \rho_n) R$, where $\rho_1, ..., \rho_n \in R_1$. Then $(\rho_1, ..., \rho_n)$ is regular on a finite R-module A iff the sequence*

$$\begin{gathered} 0 \to IA \hookrightarrow A \to I^n A \to 0 \\ a \mapsto \rho_1 \cdots \rho_n a \end{gathered} \tag{3.1.7}$$

is exact.

Proof. Exactness of (3.1.7) means that $\psi_n : (A/IA[\xi_1, ..., \xi_n])_n \to \mathrm{gr}_I^n A$ is bijective. Now if for $k < n$ the element $\omega = \sum [a_\mu] \xi^\mu \in (A/IA[\xi_1, ..., \xi_n])_k$ maps under ψ to zero then we get for any μ_0 with $|\mu_0| = k$ $\pm \psi_n([a_{\mu_0}] \xi^{(1)}) = \psi_n(\omega \cdot \xi^{(1) - \mu_0}) = \psi_k(\omega) \rho^{(1) - \mu_0} = 0$ ((1) denotes here the multiindex $(1, ..., 1)$); hence by assumption $[a_{\mu_0}] = 0$ for all μ_0; hence $\omega = 0$. ▮

3.2. Let R be a ring and $I \subseteq R$ be an ideal. Then we have a surjection of R/I-algebras

$$S_{R/I}(I/I^2) \to \mathrm{gr}_I R \tag{3.2.1}$$

which is induced by the identity $I/I^2 = \mathrm{gr}_I^1 R$. Following the terminology of [6], we call I a *regular ideal* if

(a) I/I^2 is a projective R/I-module, and

(b) (3.2.1) is an isomorphism.

PROPOSITION. *Let R be a local ring. An ideal $I \subseteq R$ is regular iff it can be generated by an R-regular sequence. In that case, any minimal base of I is an R-regular sequence.*

Proof. Let $(r \mid \rho) = (r_1, ..., r_m \mid \rho_1, ..., \rho_n)$ be a base of I and $(x_1, ..., x_m \mid \xi_1, ..., \xi_n)$ be a set of even and odd variables. Then we have a commutative diagram of surjections of R/I-algebras

$$\begin{tikzcd} R/I[x \mid \xi] \arrow[rr] \arrow[dr] & & \mathrm{gr}_I R, \\ & S_{R/I}(I/I^2) \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} (x \mid \xi) \arrow[rr, mapsto] \arrow[dr, mapsto] & & ([r] \mid [\xi]) \\ & ([r] \mid [\xi]) \arrow[ur, mapsto] & \end{tikzcd}$$

Here the horizontal line is the Hironaka–Grothendieck map while the ascending line is (3.2.1).

Now if I is regular then I/I^2 is projective and hence free over R/I, and if $(r \mid \xi)$ was chosen to be a minimal base of I it follows that all lines of the diagram are isomorphisms. By Corollary 3.1.1, $(r \mid \xi)$ is a regular sequence.

Conversely, if $(r \mid \xi)$ is a regular sequence then it is obviously an unshortenable base of I. Moreover, the horizontal line and hence all lines of the diagram are isomorphisms, which shows that I is regular. ▮

Remark. Since regularity of an ideal is in an obvious sense a local property, we may loosely say that the regular ideals are just the ideals which are locally generated by regular sequences.

3.3. We call a ring R *oddly regular* if R^1 is a regular ideal of R, and we call R *evenly regular* if $\tilde{R}$ is regular in the classical sense of the word; i.e., all local rings $\tilde{R}_{\tilde{P}}$, $\tilde{P} \in \operatorname{Spec} \tilde{R}$, are (classically) regular local rings. We call R *regular* iff it is both evenly and oddly regular. This terminology will be justified in the theorem below. First we note:

COROLLARY. *For a local ring R the following conditions are equivalent:*

(i) *R is oddly regular;*

(ii) *R^1 can be generated by an odd R-regular sequence;*

(iii) *Every minimal base of R^1 is an odd R-regular sequence.*

Moreover, if this is the case, and if $(r \mid \rho)$ is an R-regular sequence then $R/(r \mid \rho)R$ is oddly regular again.

Proof. By Proposition 3.2 and Theorem 2.4. ▮

By elementary conclusions, one has:

LEMMA. *Let (R, M) be a local ring and let be given a sequence $(r \mid \rho) \in M_0^{m \mid n}$.*

(i) *We have $M = (r \mid \rho)R$ iff $\tilde{M} = (\tilde{r})\,\tilde{R}$ and $R^1 = (\rho)\,R$.*

(ii) *The sequence $(r \mid \rho)$ is a minimal base of M iff $(\tilde{r})$ is a minimal base of $\tilde{M}$ and (ρ) is a minimal base of R^1.*

From this and Corollary 3.3 one has:

THEOREM. *A local ring (R, M) is regular iff M is a regular ideal of R.*

We also note:

PROPOSITION. *For an ideal $I \subseteq M$ in an oddly regular local ring (R, M), the following conditions are equivalent:*

(i) *I is regular;*

(ii) *The canonical map*

$$(I/MI)_1 \to (M/M^2)_1 = \Phi \tag{3.3.2}$$

is injective, and the ideal $\tilde{I} \subseteq \tilde{R}$ is regular.

Proof. If I is regular the regularity of $\tilde{I}$ follows by applying Corollary 3.3(ii) and Theorem 2.4, while the injectivity of (3.3.2) follows from Theorem 2.4 and Theorem 2.6(i). The results quoted yield also the reversed implication (ii) $\to$ (i). ▮

Remark. If R is not oddly regular then regularity of I does not any longer imply regularity of $\tilde{I}$; cf. Remark 3.6(2).

3.4. For later use, we recall the following well-known flatness criterion (cf. [4, Chap. III, Sction 5, No. 2, Theorem 1]; the proof applies to the $\mathbb{Z}_2$-graded case as well):

Let R be a noetherian ring, $I \subseteq R$ an ideal, and A an R-module which is *ideally separated* in the I-adic topology; i.e., for every ideal $J \subseteq R$, $J \otimes_R A$ is I-adically separated.

LEMMA 3.4.1. *A is flat over R iff we have:*

(i) *A/IA is flat over R/I, and*

(ii) *the canonical map $\operatorname{gr}_I R \otimes_R A \to \operatorname{gr}_I A$ is bijective.*

Remark. Any local homomorphism $\varphi \colon R \to R'$ of local rings makes R' ideally separated in the M_R-adic topology (cf. [4, Chap. III, Section 5, No. 4, Proposition 2]).

For technical reasons, we call a local homomorphism $\varphi \colon (R, M, K) \to (R', M', K')$ *special* if the induced map

$$\operatorname{gr}_M R \otimes_K K' \to \operatorname{gr}_{M'} R' \tag{3.4.1}$$

is bijective. This implies that $M' = MR' + (M')^2$, and hence, by the Nakayama lemma, $M' = R'M$. From the lemma above we deduce:

LEMMA 3.4.2. *φ is special iff it is flat and satisfies $M' = R'M$.*

Remarks. (1) A local homomorphism $\varphi \colon R \to R'$ is etale iff it is special and the induced field extension $K \subsetneq K'$ is separable and of finite type.

(2) Let $\varphi \colon R \to R'$ be a local homomorphism of regular local rings of the same dimension, and assume $M' = MR'$. Then φ is special. Indeed,

(3.4.1) is then in every grading a surjection of K'-vector spaces of the same dimension.

We recall (cf. [6, (19.8.5)]) that a classical local ring (W, M_W) is called *Cohen ring* if either W is a field of characteristic zero or W is a complete regular local ring of Krull dimension 1, with $M_W = Wp$, $p := \operatorname{Char} W > 0$. We also recall the following well-known facts:

LEMMA 3.4.3. (i) *For every field K there exists a Cohen ring W which has K as residue field.*

(ii) *If $\varphi: W \to C/J$ is a local homomorphism, W being a Cohen ring, C being any complete local ring and $J \subset C$ being an ideal, then there exists a factorization of φ into $W \to C \to C/J$.*

Remarks. This is trivially true also for non-classical C, since $\varphi(W) = \varphi(W_0) \subseteq (C/J)_0 = C_0/J_0$, and C_0 is a complete local ring again.

We will need the following version of [6, Theorem (19.8.8)(i)]:

LEMMA 3.4.4. *Let (R, M, K) be a complete local ring, and let $(r \mid \rho) \in R_0^{m \mid n}$ be a minimal system of generators of M. Then there exists a surjective local homomorphism*

$$\varphi: W[[x \mid \xi]] = W[[x_1, ..., x_m \mid \xi_1, ..., \xi_n]] \to R, \qquad (3.4.2)$$

where W is a Cohen ring and $(x \mid \xi)$ is a sequence of variables such that φ carries $(x \mid \xi)$ into $(r \mid \rho)$.

Now we may show the super version of Cohen's Structure Theorem.

PROPOSITION. *If R is a complete, equicharacteristic regular local ring then $R \cong K[[x \mid \xi]]$ with a field K and a sequence of variables $(x \mid \xi)$.*

Proof. We choose a surjection $\varphi: W[[x \mid \xi]] \to R$ as in the preceding lemma. Now if $p := \operatorname{Char} K_R$ the hypotheses imply $\varphi(M_W) = \varphi(p \cdot W) = p \cdot R = 0$. Hence φ induces a factor map $\psi: W[[x \mid \xi]]/M_W[[x \mid \xi]] = K_R[[x \mid \xi]] \to R$. This induces a map

$$\operatorname{gr}_{(x \mid \xi) K_R[[x \mid \xi]]} K_R[[x \mid \xi]] \to \operatorname{gr}_{M_R} R. \qquad (3.4.3)$$

The regularity of R implies that this is in each grading a surjection of finite vector spaces of the same dimension; hence (3.6.3) is bijective. By [4, Chap. 3, Section 2, No. 8, Corollary 3] we get that ψ is bijective, too. ∎

COROLLARY. *Let R be a equicharacteristic regular local ring of Krull dimension zero. Then $R = K[\xi]$ with a field K and a sequence (ξ) of odd variables. In other words, R is an ordinary Grassman algebra over a field.*

Proof. The hypotheses imply that $\tilde{R}$ is regular of Krull dimension zero; hence $\tilde{R}$ is a field. It follows $M = R^1$ and hence $(M_R)^k = R^k = 0$ for $k > \dim_{K_R} \Phi_R$, which implies that R is complete. The proposition yields the assertion. ▮

Remark. The condition that R is equicharacteristic cannot be dropped: Consider $R = \mathbb{Z}[\xi, \eta]/(p - \xi\eta)$, where p is a prime. Now $(\xi, \eta \mid p - \xi\eta)$ is obviously a regular sequence in $\mathbb{Z}[\xi, \eta]$. It follows that (ξ, η) is a regular sequence in R. Since $M_R = (\xi, \eta) R$ it follows that R is oddly regular. Moreover, $\tilde{R} \cong \mathbb{Z}/p\mathbb{Z}$, so that $\tilde{R}$ is regular of Krull dimension zero. On the other hand, R is obviously not equicharacteristic. (Strangely enough, we have Char $R = p^2$!)

We will need also the following result from [5, Proposition (10.3.1)]:

LEMMA 3.4.5. *Let (R, M, K) be a local ring and $K \hookrightarrow K'$ be a field extension. Then there exists a special homomorphism $(R, M, K) \to (R', M', K')$ which induces $K \hookrightarrow K'$.*

Proof. Although Grothendieck's original proof goes through in the $\mathbb{Z}_2$-graded case as well, we will give another one since we need its idea later on.

First of all we note that since $R \to \hat{R}$ is special and residually rational, we may assume that R is complete. Let W, W' be Cohen rings with residue fields K, K', respectively. By Lemma 3.4.3, there is an augmentation ψ of the diagram

$$\begin{tikzcd} W \arrow[r, dashed, "\psi"] \arrow[d] & W' \arrow[d] \\ K \arrow[r] & K' \end{tikzcd}$$

so that we may consider W' as W-algebra. Moreover, it is clear by Remark (2) after Lemma 3.4.2 that ψ is special. By Lemma 3.4.3 we may view R as W-algebra, and we claim that $R' := R \otimes_W W'$ solves our task.

By Lemma 3.4.4, we may write $R = W[[x \mid \xi]]/I$ with an ideal I. It then follows $R' = W'[[x \mid \xi]]/IW'$, so that R' is a local ring again. Moreover, the flatness of ψ implies the flatness of $1_R \otimes \psi \colon R \to R'$, and it is also obvious that $M_{R'} = R' M_R = (\text{Char}\, K \cdot 1, x, \xi)\, R'$. Applying Lemma 3.4.2 we get the assertion. ▮

Remark. This proof cannot replace the original proof of [5]: It depends on Lemma 3.4.3(i), and the latter is in [6], a corollary of the lemma above.

3.5. We call an ideal I in a local ring (R, M) *conormally free* iff I/I^2 is a free R/I-module. It is clear from the definitions that any regular ideal is conormally free.

THEOREM. *Let (R, M) be a local ring and $I \subseteq M$ be an ideal.*

(i) *If I is conormally free then the canonical maps of K-vector spaces*

$$(I/MI)_0 \to \tilde{I}/\tilde{M}\tilde{I} \tag{3.5.1}$$

and

$$(I/MI)_1 \to (M/M^2)_1 = \Phi \tag{3.5.2}$$

are injective.

(ii) *If R is regular and equicharacteristic then I is a regular ideal iff it is conormally free.*

Before proving the theorem we will need two lemmata.

LEMMA 3.5.1. *Let R be a local ring and $\omega \in R_0^2$ (cf. 1.2). Let A be a finite nonzero R-module such that $\omega^2 A = 0$. Then the arising sequence*

$$A \xrightarrow{\omega} A \xrightarrow{\omega} A \tag{3.5.3}$$

cannot be exact.

Proof. Suppose the converse. Factoring out the annihilator of A and localizing with respect to some minimal associated prime of A, we may suppose that R is artinian. Thus we have at our disposal the length function $A' \mapsto \ell(A')$, which is defined for any finite R-module A'. We note that if $\omega': A' \to A'$ is a homogeneous endomorphism with $(\omega')^2 = 0$ then $\ell(\omega'(A')) \leqslant \ell(A')/2$, and the equality holds iff $A' \to^{\omega'} A' \to^{\omega'} A'$ is exact.

We may write

$$\omega = \sum_{i=1}^{n} \sigma_i \rho_i$$

with $\sigma_1, ..., \sigma_n,\ \rho_1, ..., \rho_n \in R_1$. Let $(\xi, \eta) = (\xi_1, ..., \xi_n, \eta_1, ..., \eta_n)$ and $(\bar{\xi}, \bar{\eta}) = (\bar{\xi}_1, ..., \bar{\xi}_n, \bar{\eta}_1, ..., \bar{\eta}_n)$ be two sequences of odd variables; put

$$\delta := \sum_{i=1}^{n} (\bar{\xi}_i - \sigma_i)(\bar{\eta}_i - \rho_i) \in \bar{R} := R[\bar{\xi}, \bar{\eta}],$$

$$\bar{A} := A[\bar{\xi}, \bar{\eta}] = A \otimes_R \bar{R},$$

and consider the ideal

$$I := (\bar{\xi}_1 - \sigma_1, ..., \bar{\xi}_n - \sigma_n, \bar{\eta}_1 - \rho_1, ..., \bar{\eta}_n - \rho_n)\,\bar{R}.$$

Set

$$\bar{A}^{(i)} := I^i\bar{A} + \delta^2\bar{A}, \qquad \mathrm{gr}^i\,\bar{A} := \bar{A}^{(i)}/\bar{A}^{(i+1)},$$

$$\mathrm{gr}\,\bar{A} := \bigoplus_{i=0}^{2n} \mathrm{gr}^i\,\bar{A}.$$

We consider the R-linear map

$$A[\xi, \eta] \to \mathrm{gr}\,\bar{A}, \qquad a\xi^\mu\eta^\nu \mapsto [a(\bar{\xi}-\sigma)^\mu\,(\bar{\eta}-\rho)^\nu]$$

(class in $\mathrm{gr}^{|\mu|+|\nu|}\,\bar{A}$) for $a \in A$, $\mu, \nu \in \mathbb{Z}_2^n$.

This should be viewed as a version of the Hironaka–Grothendieck map. Obviously, it is surjective, and because of $\omega^2 A = 0$ it annihilates $\delta^2 A[\xi, \eta]$ with $\delta := \sum_{i=1}^n \xi_i\eta_i$. Hence we get a factor map

$$\psi\colon A[\xi, \eta]/\delta^2 A[\xi, \eta] \to \mathrm{gr}\,\bar{A}.$$

Now $\ell(\mathrm{gr}\,\bar{A}) = \ell(\bar{A}/\delta^2\bar{A})$; by a variable shift we get

$$\ell(\mathrm{gr}\,\bar{A}) = \ell(A[\xi, \eta]/\delta^2 A[\xi, \eta]).$$

Thus ψ is a surjection of modules of equal length and hence it is bijective.

Now the multiplication map $\delta\colon \bar{A} \to \bar{A}$ induces a map $\mathrm{gr}(\delta)\colon \mathrm{gr}\,\bar{A} \to \mathrm{gr}\,\bar{A}$, and the latter acts by

$$\mathrm{gr}(\delta)[b] = [\delta b] = [\omega b] = \omega[b].$$

Thus the diagram

$$\begin{tikzcd} A[\xi, \eta]/\delta^2 A[\xi, \eta] \arrow[r, "\psi"] \arrow[d, "\omega"] & \mathrm{gr}\,\bar{A} \arrow[d, "\mathrm{gr}(\delta)"] \\ A[\xi, \eta]/\delta^2 A[\xi, \eta] \arrow[r, "\psi"] & \mathrm{gr}\,\bar{A} \end{tikzcd}$$

is commutative.

Now we have for each $i \geqslant 0$ a surjection

$$(\delta\bar{A}^{(i)} + \delta^2\bar{A})/(\delta A^{(i+1)} + \delta^2\bar{A}) \to \mathrm{Im}\,\mathrm{gr}^i(\delta).$$

Hence

$$\begin{aligned} \ell(\delta\bar{A}/\delta^2\bar{A}) &= \sum_{i \geqslant 0} \ell((\delta\bar{A}^{(i)} + \delta^2\bar{A})/(\delta\bar{A}^{(i+1)} + \delta^2\bar{A})) \\ &\geqslant \sum_{i \geqslant 0} \ell(\mathrm{Im}\,\mathrm{gr}^i(\delta)) \\ &= \ell(\omega(A[\xi, \eta]/\delta^2 A[\xi, \eta])). \end{aligned}$$

By the exactness of (3.5.3) and our remarks from above we get

$$\ell(\delta\bar{A}/\delta^2\bar{A}) \geqslant \ell(A[\xi, \eta]/\delta^2 A[\xi, \eta])/2. \tag{3.5.4}$$

By a variable shift argument, the l.h.s. of this is equal to $\ell(\delta B)$ with $B := A[\xi, \eta]/\delta^2 A[\xi, \eta]$. Hence (3.5.4) implies (again by the remarks from above) that $B \to^{\delta} B \to^{\delta} B$ is exact. But this is certainly false: If we choose any $a \in A \setminus 0$ the class of $a\eta_1 \cdots \eta_n$ is annihilated by δ without being a multiple of δ. ∎

Remarks. (1) Let (R, M, K) be a local ring and A a finite R-module with $\operatorname{Supp} A \subseteq \{M\}$. Then there exists a composition series

$$A = A^0 \supset A^1 \supset \cdots \supset A^{k-1} \supset A^k = 0, \tag{3.5.5}$$

and every factor A^i/A^{i+1} is isomorphic either to K or to ΠK. Setting $k_0 := \operatorname{Card}\{i: A^i/A^{i+1} \cong K\}$, $k_1 := k - k_0$, it follows from the Jordan–Hölder theorem that these numbers are independent of the choice of (3.5.5). Thus the length is in the $\mathbb{Z}_2$-graded situation actually a pair of numbers $k_0 | k_1$. With some analogy, if A is any module over any ring R we may consider the sets

$$\operatorname{Ass}_i A = \{P \in \operatorname{Spec} R\text{: There exists an injection } R/P \to A \text{ of parity } i\}$$

for $i = 0, 1$. Thus $\operatorname{Ass} A = \operatorname{Ass}_0 A \cup \operatorname{Ass}_1 A$.

(2) With a modification (actually a simplification) of the technique of the proof above, one can show:

Let A be an artinian module over a local ring R and $\omega = \sum_{i=1}^n \rho_i \sigma_i$ with elements $\rho_1, ..., \rho_n, \sigma_1, ..., \sigma_n \in R_1$. Then one has

$$\ell(\omega A) \leqslant \left(1 - \binom{2n+1}{n} 2^{-2n}\right) \ell(A) \tag{3.5.6}$$

This estimate is sharp, at least if $\operatorname{Char} K = 0$: In that case one can show with some combinatorial work that putting $A := R := K[\rho_1, ..., \rho_n, \sigma_1, ..., \sigma_n]$ equality is attained in (3.5.6). Note also that already for $n = 4$ the factor in (3.5.6) becomes $1 - 126/256 < 1/2$. Thus, in the proof of the lemma, the information $\omega^2 A = 0$ had to be fully exploited.

LEMMA 3.5.2. *Let R' be a ring and $R = R'[\xi_1, ..., \xi_k]$ with a sequence $\xi_1, ..., \xi_k$ of odd variables. Let $\varphi: R \to R'$ be the homomorphism given by $\varphi|_{R'} = 1_{R'}$ and $\varphi(\xi_i) = 0$ for all i. Let $I \subseteq R$ be an ideal such that*

$$I/I^2 = (R/I)^{n|0} \tag{3.5.7}$$

for some $n > 0$. Then. with $I' := \varphi(I)$ we have

$$I'/(I')^2 = (R'/I')^{n|0}. \tag{3.5.8}$$

Proof. By induction, we may assume $k = 1$, $\xi := \xi_1$. Let $a_1 + \xi b_1, ..., a_n + \xi b_n$ with $a_1, ..., a_n \in R'_0$, $b_1, ..., b_n \in R'_1$ be elements of I such that their residues mod I^2 form a base of the free R/I-module I/I^2. Let B denote the ideal of R' generated by $b_1, ..., b_n$.

CLAIM. *Given a relation*

$$\sum_i r_i a_i = 0 \tag{3.5.9}$$

with $r_1, ..., r_n \in R'$ there exist elements c_{ij}, $d_{ij} \in R'$ such that

$$r_i = \sum_j (c_{ij} a_j - d_{ij} b_j) \tag{3.5.10}$$

and

$$\sum_j d_{ij} a_j = 0 \tag{3.5.11}$$

for all i.

Proof of the Claim. (3.5.9) can be rewritten to

$$\sum_i \xi r_i (a_i + \xi b_i) = 0.$$

Because of (3.5.7), this implies $\xi r_i \in I$; i.e., we may write

$$\xi r_i = \sum_j (a_j + \xi b_j)(d_{ij} + \xi c_{ij}).$$

Comparison of coefficients yields (3.5.10) and (3.5.11). ∎

CLAIM. *For all $l \geq 0$, the relation* (3.5.9) *implies $r_1, ..., r_n \in I' + B^l$.*

Proof of the Claim. By induction on l, the start $l = 1$ being given by (3.5.10). Now if the claim is true for $l = l_0$, and we are given the relation (3.5.9), we may apply the hypothesis of induction onto the arising relation (3.5.11). We get $d_{ij} \in I' + B^{l_0}$, and (3.5.10) now yields $r_1, ..., r_n \in I' + B(I' + B^{l_0}) = I' + B^{l_0 + 1}$. ∎

Continuation of Proof. Now observe that $B^{n+1} = 0$. Hence the preceding claim says that the images of $a_1, ..., a_n$ in $I'/(I')^2$ form a free base of this R'/I'-module, and the lemma is proved. ∎

Proof of the Theorem. Ad (i). If (3.5.1) was not injective there would be a minimal base $(r \mid \rho) \in I_0^{m|n}$ of I such that $\tilde{r}_1 = 0$, i.e., $r_1 \in R_0^2$. Now, setting $J := (r_1^2, r_2, ..., r_m \mid \rho) R$, we claim that

$$R/J \xrightarrow{r_1} R/J \xrightarrow{r_1} R/J \tag{3.5.12}$$

is exact. Indeed, if $a \in R$ and $r_1[a] = 0$ then $r_1 a \in J$, and the conormal freeness implies $a \in I = J + r_1 R$. Hence $[a] \in r_1(R/J)$, and (3.5.12) is proved exact. Now Lemma 3.5.1 yields a contradiction.

Quite similarly, if (3.5.2) was not injective there would be a minimal base $(r \mid \rho) \in I_0^{m|n}$ such that $\rho_1 \in M^2$. Setting $J' := (r \mid \rho_1, ..., \rho_n) R$, one would get with analogous arguments that ρ_1 is R/J'-regular. But this contradicts Theorem 2.6.

Ad (ii). Let (R, M, K) be equicharacteristic and regular, and let $I \subseteq M$ be conormally free. We have to show that I is regular.

First we note that if $(r \mid \rho)$ is a minimal base of I it follows from assertion (i) and Corollary 3.3 that (ρ) is an R-regular sequence, and that $R/(\rho)R$ is regular again. On the other hand, a direct verification shows that the ideal $I/(\rho) R$ in $R/(\rho) R$ is conormally free again; thus we are reduced to the case that I is generated by even elements. Using Proposition 3.2, we may assume that R is complete, and using Proposition 3.4, we may write $R = \tilde{R}[\xi_1, ..., \xi_n]$, where $\tilde{R}$ is a classical regular local ring. Now, by the preceding lemma, $\tilde{I} \subseteq \tilde{R}$ is conormally free again, and by a classical result of [16] (cf. also [7]) it follows that $\tilde{I}$ is regular. By Proposition 3.3, I is regular. ▮

Remarks. (3) One feels that there should be a proof of assertion (ii) of the theorem which does not make use of Cohen's Structure Theorem (and hence of the axiom of choice) and which also applies in the non-equicharacteristic case. In fact, one expects an adaption of the approach in [16]. A partial result of that kind is the following: First one notes that if $I \subseteq R$ is regular and R is oddly regular then $pd_{\tilde{R}} I \otimes_R \tilde{R} < \infty$. Conversely, if the latter condition is satisfied and if I is conormally free one can show that there exists a minimal base $(r_1, ..., r_m \mid \rho_1, ..., \rho_n)$ of I such that $(r_1, ..., r_{m-1} \mid \rho_1, ..., \rho_n)$ is an R-regular sequence. The proof closely follows that of [16], using the injectivity of (3.5.1) in an essential way (In order to get the needed non-zerodivisor in I, apply [16, Proposition 1.3] onto the kernel of $\tilde{R}^{m|n} \to I \otimes_R \tilde{R} \to 0$).

Curiously enough, there is not guarantee anymore for r_m: Take $R = K[x \mid \xi, \eta]/(x^2 + \xi\eta)$ where K is a field. Then $I := Rx$ is conormaly free, and $I \otimes_R \tilde{R}$ is a free R-module; nevertheless, I is not regular.

3.6. In this section we study some questions on flat local homomorphisms.

PROPOSITION 3.6.1. *Let $\varphi: (R, M, K) \to (R', M', K')$ be a flat local homomorphism.*

(i) *MR' is a conormally free ideal in R'.*

(ii) *The induced $(K \subsetneq K')$-linear map*

$$\Phi_R = (M/M^2)_1 \to (M'/(M')^2)_1 = \Phi_{R'} \tag{3.6.1}$$

is injective.

(iii) *If R' is oddly regular then so is R.*

(iv) *If R' is regular and equicharacteristic then so is R.*

Remark. (4) Of course, the even analogue of (3.6.1), i.e, the map $(M/M^2)_0 \to (M'/(M')^2)_0$, is in general not injective.

Proof. Ad (i). Due to flatness, we have

$$MR'/M^2R' = (M/M^2) \otimes_R R' = (M/M^2) \otimes_{R/M} R'/MR' \tag{3.6.2}$$

and this is, of course, free over R'/MR'.

Ad (ii). The flatness implies that the induced map

$$M/M^2 \to MR'/M^2R' \tag{3.6.3}$$

is injective. The assertion now follows from Theorem 3.5(i).

Ad (iii). If $\rho_1, ..., \rho_n$ is the odd part of a minimal base of M, the assertion (ii) says that the classes of $\varphi(\rho_1), ..., \varphi(\rho_n)$ in $\Phi_{R'}$ are linearly independent. From Theorem 2.5 and Corollary 3.3 we get that $(\varphi(\rho_1), ..., \varphi(\rho_n))$ is an R'-regular sequence. Since "flat + local" implies "faithfully flat" we get that $(\rho_1, ..., \rho_n)$ is an R-regular sequence. By Lemma 3.3 and Corollary 3.3 we get the assertion.

Ad (iv). By assertion (i) and Theorem 3.5(ii), MR' is a regular ideal in R'. Now if $(r \mid \rho) \in M_0^{m \mid n}$ is a minimal base of M, the injectivity of (3.6.3) implies that $(\varphi(r) \mid \varphi(\rho))$ is a mimimal base of MR'. By Proposition 3.2, this is an R'-regular sequence, and the faithful flatness of φ implies that $(r \mid \rho)$ is an R-regular sequence. By Theorem 3.3, R is regular. Moreover, if $p := \mathrm{Char}\, K'$ the nonvanishing of $R \to^p R$ would, again due to faithful flatness, imply the nonvanishing of $R' \to^p R'$, contrary to our assumption. ∎

PROPOSITION 3.6.2. *Let $\varphi: (R, M, K) \to (R', M', K')$ be a local homomorphism, suppose that R is regular.*

(i) *φ is flat iff*

(1) *MR' is a regular ideal in R', and*

(2) *the arising map $M/M^2 \to MR'/MM'$ is injective.*

(ii) *If R' is oddly regular then φ is flat iff*

(3) *the induced homomorphism $\tilde{\varphi}: \tilde{R} \to \tilde{R}'$ is flat, and*

(4) *the induced map $\Phi_R \to \Phi_{R'}$ (cf. (3.6.1) is injective.*

Remark. (5) The flatness of a local homomorphism $\varphi: R \to R'$ does not always imply the flatness of $\tilde{\varphi}: \tilde{R} \to \tilde{R}'$. In order to construct a counterexample, we consider over the polynomial algbra $K[x]$, K being a field, the algebra R freely generated as $K[x]$-module by the symbols 1, r, ξ, η with multiplication table

$$\begin{array}{cccc} 1 & r & \xi & \eta \\ r & 0 & 0 & 0 \\ \xi & 0 & 0 & xr \\ \eta & 0 & -xr & 0. \end{array}$$

Setting $|1| := |r| := 0$, $|\xi| := |\eta| := 1$, R becomes a local ring.

Now x is a non-zero divisor in R. On the other hand, $xr \in R^1$ while $r \notin R^1$; thus $\tilde{x} \in \tilde{R}$ is a zero divisor. By Lemma 3.4, R is flat over $K[x]$ while $\tilde{R}$ is not.

(6) On the other hand, the induced map $\tilde{\varphi}: \tilde{R} \to \tilde{R}'$ carries at least three imprints from the flatness of $\varphi: R \to R'$: First, using Lemma 3.3, Proposition 3.6.1, and Theorem 3.5(i) we get that the induced map $\tilde{M}/\tilde{M}^2 \to \tilde{M}\tilde{R}'/\tilde{M}\tilde{M}'$ is injective; second, we have $Kr\text{-dim}\, \tilde{R}' = Kr\text{-dim}\, \tilde{R} + Kr\text{-dim}\, \tilde{R}'/\tilde{M}\tilde{R}'$ (cf. below), and third, the induced map $\tilde{\varphi}^*$: $\operatorname{Spec} \tilde{R}' \to \operatorname{Spec} \tilde{R}$ is surjective (cf. [4, Chap. II, Section 2, No. 5, Corollary 4] which carries over to the $\mathbb{Z}_2$-graded case).

Proof. Let $(r \mid \rho) \in M_0^{m|n}$ be a fixed minimal base of M.

Ad (i). Suppose φ flat; then (1) is clear from Proposition 3.2. Moreover, due to the isomorphism (3.6.2), the classes of $(\varphi(r) \mid \varphi(\rho))$ in MR'/M^2R' form a minimal base of this R'/MR'-module, which implies the validity of (2).

Conversely, let (1) and (2) hold true, and let $(x \mid \xi)$ be a sequence of $m \mid n$ variables. Then (2) says that $(\varphi(r) \mid \varphi(\rho))$ is a minimal base of MR'. We can write down the diagram

$$\begin{tikzcd} K[x \mid \xi] \otimes_K R'/MR' \arrow[r, "\beta_1"] \arrow[d, "\alpha_1 \otimes 1"'] & R'/MR'[x \mid \xi] \arrow[d, "\alpha_2"] \\ \mathrm{gr}_M R \otimes_K R'/MR' \arrow[r, "\beta_2"] & \mathrm{gr}_{MR'} R' \end{tikzcd}$$

where α_1, α_2 are the Hironaka–Grothendieck maps while β_1, β_2 are induced in an obvious way. By assumption and Corollary 3.1, α_1 and α_2 are bijective, while β_1 is obviously bijective. Hence β_2 is bijective, too, and by Lemma 3.4.1 this means that φ is flat.

Ad (ii). Suppose again φ flat. Then (4) was proved in the preceding proposition. Moreover, assertion (i) says that $(\varphi(r) \mid \varphi(\rho))$ is an R'-regular sequence. Using Theorem 2.4 and Corollary 3.3 we get that $(\widetilde{\varphi(r)})$ is R'-regular. Using assertion (i) again we get (3).

Conversely, if (3), (4) are satisfied then (3) implies by the same arguments as above that the sequence $(\varphi(r))$ is R'-regular. Moreover, (4) implies by Corollary 3.3 and Theorem 2.6 that $(\varphi(\rho))$ is R'-regular. By Theorem 2.4, $(\varphi(r) \mid \varphi(\rho))$ is R-regular. In particular, these elements form a minimal base of MR'. Thus (1) and (2) are satisfied, and an application of assertion (i) completes the proof. ∎

PROPOSITION 3.6.3. *Let* $\varphi\colon (R, M, K) \to (R', M', K')$ *be a local homomorphism.*

(i) *We have*

$$Kr\text{-dim } R' \leqslant Kr\text{-dim } R + Kr\text{-dim } R'/MR' \tag{3.6.4}$$

and

$$\dim_{K'} \Phi_{R'} \leqslant \dim_K \Phi_R + \dim_{K'} \Phi_{R'/MR'} \tag{3.6.5}$$

(ii) *If* φ *is flat we have in* (3.6.4), (3.6.5) *equality.*

(iii) *Conversely, if* R *and* R' *are both regular, and if we have equality in* (3.6.4), (3.6.5) *then* φ *is flat.*

Proof. Ad (i). (3.6.5) follows from the exact sequence

$$\Phi_R \otimes_K K' \to \Phi_{R'} \to \Phi_{R'/MR'} \to 0 \tag{3.6.6}$$

while (3.6.4) is standard (cf. EGA I, (0.16.3.9)).

Ad (ii). By Proposition 3.6.1(ii), the first map of (3.6.6) is injective, which yields the equality in (3.6.5). The equality in (3.6.4) is standard again (cf. [6, Corollary 6.1.2]).

Ad (iii). We first note that $\widetilde{R'/MR'} = \tilde{R}'/\tilde{M}\tilde{R}'$. Therefore we can use [6, Proposition 6.1.5] to conclude that $\tilde{\varphi}\colon \tilde{R} \to \tilde{R}'$ is flat. On the other hand, the equality in (3.6.6) means that $\Phi_R \to \Phi_{R'}$ is injective. Now Proposition 3.6.2(ii) yields the assertion. ∎

4. The Koszul Transform

4.1. We recall that if X is a scheme and Y a subscheme cut out by an ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_X$, the *conormal sheaf* of Y in X is given by $\mathcal{N}^* = \mathcal{N}^*_{X/Y} := \mathscr{I}/\mathscr{I}^2$. We call Y *conormally locally free* in X if $\mathcal{N}^*$ is a locally free $\mathcal{O}_Y$-module. If this is the case we put $\mathcal{N} := \mathscr{H}om(\mathcal{N}^*, \mathcal{O}_Y)$ and

$$\Pi N^* = \Pi N^*_{Y/X} := \operatorname{Spec} S_{\mathcal{O}_Y} \Pi \mathcal{N}.$$

This is the vector bundle over Y determined by the locally free module $\Pi \mathcal{N}^*$; let

$$\mathbb{P}(\Pi N^*) := \operatorname{Proj} S_{\mathcal{O}_Y} \Pi \mathcal{N}$$

denote the corresponding projective bundle. We are going to define covariant functors

$$\begin{array}{c} \operatorname{Qco}(\mathcal{O}_X) \to \mathbb{Z}-\operatorname{Qco}(S_{\mathcal{O}_Y}\Pi\mathcal{N}) \begin{array}{l} \longrightarrow \operatorname{Qco}(\mathcal{O}_{\Pi N^*}) \\ \longrightarrow \operatorname{Qco}(\mathcal{O}_{\mathbb{P}(\Pi N^*)}), \end{array} \\ \mathscr{A} \mapsto \mathscr{C}_{\mathscr{A}} \begin{array}{l} \longmapsto \mathbb{A}\mathscr{C}_{\mathscr{A}} \\ \longmapsto \mathbb{P}\mathscr{C}_{\mathscr{A}} \end{array} \end{array} \tag{4.1.1}$$

We will call each of the objects $\mathscr{C}_{\mathscr{A}}$, $\mathbb{A}\mathscr{C}_{\mathscr{A}}$, $\mathbb{P}\mathscr{C}_{\mathscr{A}}$ the *Koszul transform* of $\mathscr{A}$ along Y, and we will see that they exhibit useful information on the behavior of $\mathscr{A}$ in the neighborhood of Y.

We begin with some observations on the affine level. Therefore we fix for the rest of this section the following situation: R is a neotherian ring and $I \subseteq R$ an ideal such that I/I^2 is a free R/I-module. Let

$$(r_1, ..., r_m \mid \rho_1, ..., \rho_n) \in I_0^{m \mid n}$$

be such that the classes of these elements form a base of I/I^2 over R/I. We fix a set of even and odd variables

$$(\xi \mid x) = (\xi^1, ..., \xi^m \mid x^1, ..., x^n)$$

(note the "opposite" parities) and consider the element

$$\delta = \sum_{i=1}^{m} \xi^i r_i + \sum_{j=1}^{n} x^j \rho_j \in R[\xi \mid x]_1.$$

For any R-module A we set $\mathscr{C}_A^{(r \mid \rho)} := \mathscr{C}(\delta, A[\xi \mid x])$, where $A[\xi \mid x] = A \otimes_R R[\xi \mid x]$, and the r.h.s. is defined as in 2.1.

Lemma. *We have* $I\mathscr{C}_A^{(r \mid \rho)} = 0$.

Proof. It suffices to show $r_i\mathscr{C}_A^{(r|\rho)}=\rho_j\mathscr{C}_A^{(r|\rho)}=0$ for all i, j. Now if $\omega\in A[\xi|x]$ and $\delta\omega=0$ we get $0=(\partial/\partial x_j)(\delta\omega)=\rho_j\omega+\delta(\partial\omega/\partial x_j)$, which shows that $\rho_j\omega$ defines the zero class. Analogously, $0=(\partial/\partial\xi_i)(\delta\omega)=r_i\omega-\delta(\partial\omega/\partial\xi_i)$. ∎

Thus $\mathscr{C}_A^{(r|\rho)}$ is in fact a module over $(R/I)[\xi|x]$.

4.2. Now let X be a locally noetherian scheme and $Y\subseteq X$ be a conormally locally free subscheme which is cut out by the ideal sheaf $\mathscr{I}\subseteq\mathcal{O}_X$. Let $\mathscr{A}$ be a quasicoherent $\mathcal{O}_X$-module. We want to construct a $\mathbb{Z}$-graded quasicoherent module sheaf over $S_{\mathcal{O}_Y}(\Pi(\mathscr{I}/\mathscr{I}^2)^*)$ by constructing it locally as in the previous section and sticking the results together. This can be done straightforwardly: Let $U\subseteq X$ be open such that $\mathscr{I}/\mathscr{I}^2|_U$ is free over $\mathcal{O}_X/\mathscr{I}|_U$, let $(r|\rho)\in H^0(U,\mathscr{I})_0^{m|n}$ be such that the classes of these elements form a base of $\mathscr{I}/\mathscr{I}^2|_U$, and let $(\xi|x)$ be as above a sequence of variables. Then we may fix an isomorphism of $\mathcal{O}_Y$-algebras

$$\psi\colon \mathcal{O}_Y[\xi|x]|_U\to S_{\mathcal{O}_Y}\Pi(\mathscr{I}/\mathscr{I}^2)^*$$

by letting $(\psi(\xi)|\psi(x))\in(\Pi(\mathscr{I}/\mathscr{I}^2)^*)_0^{m|n}$ form the right dual base to the base $(\Pi[r]|\Pi[\rho])$ of $\Pi(\mathscr{I}/\mathscr{I}^2)$. Let (temporarily) $\mathscr{C}_{\mathscr{A}}^{U,(r|\rho)}$ denote the cohomology sheaf of the complex $\mathscr{A}[\xi|x]\to^{\delta}\mathscr{A}[\xi|x]\to^{\delta}\mathscr{A}[\xi|x]$ with $\delta:=(\xi|x)[\frac{r}{\rho}]$ as in 4.1. By Lemma 4.1, it is a quasicoherent $\mathbb{Z}$-graded module over $\mathcal{O}_X[\xi|x]/\mathscr{I}[\xi|x]=\mathcal{O}_Y[\xi|x]$; by means of ψ it becomes an $S_{\mathcal{O}_Y}\Pi(\mathscr{I}/\mathscr{I}^2)^*$-module.

Now let $U'\subseteq X$ be another open set with the indicated property, let $(r'|\rho')$ and $(\xi'|x')$ be as above but with respect to U', and put $U'':=U\cap U'$. We have to construct an isomorphism

$$\gamma_{U,(r|\rho)}^{U',(r'|\rho')}\colon\mathscr{C}_{\mathscr{A}}^{U,(r|\rho)}\to\mathscr{C}_{\mathscr{A}}^{U',(r'|\rho')}. \tag{4.2.1}$$

On U'' we can write

$$\begin{pmatrix} r' \\ \hline \rho' \end{pmatrix}=M\begin{pmatrix} r \\ \hline \rho \end{pmatrix},$$

where M is an even $(m|n)\times(m|n)$ matrix with entries in $H^0(U'',\mathcal{O})$. Considering $\mathcal{O}_Y$ as $\mathcal{O}$-algebra we may fix an isomorphism over $\mathcal{O}_Y$

$$\varphi\colon\mathcal{O}_{U''}[\xi|x]\to\mathcal{O}_{U''}[\xi'|x']$$

by

$$\varphi((\xi|x)):=(\xi'|x')\,M. \tag{4.2.2}$$

This setting ensures that with $\delta' := (\xi' \mid x')(r'/\rho')$ we have $\delta' = \varphi(\delta)$. Indeed,

$$\varphi(\delta) = \varphi((\xi \mid x))\begin{pmatrix} r \\ \rho \end{pmatrix} = (\xi' \mid x')M\begin{pmatrix} r \\ \rho \end{pmatrix} = (\xi' \mid x')\begin{pmatrix} r' \\ \rho' \end{pmatrix}.$$

Moreover, the arising diagram

$$\begin{tikzcd} \mathcal{O}_Y[\xi \mid x] \mid_{U''} \arrow[rr, "\varphi''"] \arrow[dr, "\psi"'] & & \mathcal{O}_Y[\xi' \mid x'] \mid_{U''} \arrow[dl, "\psi'"] \\ & S_{\mathcal{O}_Y}\Pi(\mathcal{I}/\mathcal{I}^2)^* \mid_{U''} & \end{tikzcd} \tag{4.2.3}$$

where ψ' is constructed like ψ and φ'' is the map induced by φ, is easily checked to be commutative. Now (4.2.2) and (4.2.3) yield together the isomorphism (4.2.1) wanted.

It is easy to see that the maps (4.2.1) satisfy the cocycle condition; therefore the $\mathscr{C}^{U,(r \mid \rho)}_{\mathscr{A}}$ can be glued together to a well-defined $\mathbb{Z}$-graded quasicoherent module over the sheaf of $\mathbb{Z}$-graded algebras $S_{\mathcal{O}_Y}\Pi(\mathcal{I}/\mathcal{I}^2)^*$.

PROPOSITION. *Let X be a locally noetherian scheme and Y a conormally locally free subscheme.*

(i) *The assignement $\mathscr{A} \mapsto \mathscr{C}_{\mathscr{A}}$ yields an additive, covariant functor*

$$\mathrm{Qcm}(\mathrm{O}_X) := \{\textit{quasicoherent } \mathcal{O}_X\textit{-modules}\} \to \mathbb{Z} - \mathrm{Qcm}(S_{\mathcal{O}_Y}\Pi(\mathcal{I}/\mathcal{I}^2)^*) := \{\mathbb{Z}\textit{-graded quasicoherent } S_{\mathcal{O}_Y}\Pi(\mathcal{I}/\mathcal{I}^2)^*\textit{-modules}\}.$$

If $\mathscr{A}$ is coherent then so is $\mathscr{C}_{\mathscr{A}}$.

(ii) *Any exact sequence in* $\mathrm{Qcm}(\mathcal{O}_X)$

$$0 \longrightarrow \mathscr{A}' \xrightarrow{\alpha'} \mathscr{A} \xrightarrow{\alpha''} \mathscr{A}'' \longrightarrow 0 \tag{4.2.4}$$

induces an exact triangle

$$\begin{tikzcd} & \mathscr{C}_{\mathscr{A}} \arrow[dr, "\mathscr{C}_{\alpha''}"] & \\ \mathscr{C}_{\mathscr{A}'} \arrow[ur, "\mathscr{C}_{\alpha'}"] & & \mathscr{C}_{\mathscr{A}''} \arrow[ll, "\delta"] \end{tikzcd} \tag{4.2.5}$$

the connection map δ being homogeneous of degree 1 (*so that it can be rewritten as a morphism $\delta\colon \mathscr{C}_{\mathscr{A}''} \to \mathscr{C}_{\mathscr{A}'}(1)$ in* $\mathbb{Z}$-$\mathrm{Qcm}(S_{\mathcal{O}_Y}\Pi(\mathcal{I}/\mathcal{I}^2)^*)$).

Proof. (i) follows from the general nonsense of EGA I, while (ii) is a consequence of Lemma 2.1.1. ∎

In particular, let $I \subseteq R$ be an ideal in a noetherian ring, assume that I/I^2 is projective over R/I. Then any R-module A determines a quasicoherent $\mathcal{O}_X$-module $\mathcal{A} := \mathcal{O}_X \otimes_R A$ on $X := \operatorname{Spec} R$. With respect to $Y := \operatorname{Spec} R/I$ we may form $\mathcal{C}_{\mathcal{A}}$, and we put $\mathcal{C}_A := H^0(X, \mathcal{C}_{\mathcal{A}})$. Thus we get a functor

$$\operatorname{Mod}(R) \to \mathbb{Z}\text{-}\operatorname{Mod}(S_{R/I}\Pi(I/I^2)^*), \qquad A \mapsto \mathcal{C}_A.$$

If I/I^2 is free over R/I with a base formed by the residues of $(r \mid \rho)$, and if we identify $S_{R/I}\Pi(I/I^2)^*$ with $R/I[\xi \mid x]$ as above, then $\mathcal{C}_A$ is just the $\mathcal{C}_A^{(r \mid \rho)}$ considered in 4.1.

4.3. Let X and Y be as the beginning of the previous section. Then $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_Y$-module, and we may associate to it two fibre bundles on Y: first, the vector bundle

$$\Pi N^* = \mathbb{A}_Y := \operatorname{Spec} S_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2)^* \to Y,$$

and, second, the projective bundle

$$\mathbb{P}(\Pi N^*) = \mathbb{P}_Y := \operatorname{Proj} S_{\mathcal{O}_Y}\Pi(\mathcal{I}/\mathcal{I}^2)^* \to Y.$$

If upon writing $m \mid n := \operatorname{rank} \mathcal{I}/\mathcal{I}^2$ we have $n = 0$ then $\mathbb{P}_Y$ is empty.

Now if $\mathcal{A} \in \operatorname{Qcm}(\mathcal{O}_X)$ then, by well-known formalisms, gives rise to a quasicoherent sheaf $\mathbb{A}\mathcal{C}_A$ on $\mathbb{A}_Y$ as well as to a quasicoherent sheaf $\mathbb{P}\mathcal{C}_{\mathcal{A}}$ on $\mathbb{P}_Y$.

4.4. We now study the relations of $\mathcal{C}_{\mathcal{A}}$ with the Koszul complex. Thus we start with a noetherian ring R and an ideal I such that I/I^2 is free over R/I with a basis formed by the classes of $(r \mid \rho) \in I_0^{m \mid n}$. With the notations of 2.3, we recall

$$K(r \mid \rho)_i := R[\eta \mid y]_i, \qquad d = \sum r_i \frac{\partial}{\partial \eta_i} + \sum \rho_j \frac{\partial}{\partial y_j}.$$

Now let $(\xi \mid x)$ be another set of m odd and n even variables which span the $\mathbb{Z}$-graded polynomial ring $R[\xi \mid x] = \bigoplus_{i \geqslant 0} R[\xi \mid x]^i$ over R. We then have the standard dual pairing

$$R[\eta \mid y]_i \times R[\xi \mid x]^i \to R \qquad \text{for} \quad i \geqslant 1,$$

$$(\eta^\alpha y^\beta, \xi^\gamma x^\delta) \mapsto \delta_{\alpha\gamma}\, \delta_{\beta\delta} \cdot \beta!.$$

Here α, β, γ, δ are multiindices, $\delta_{\alpha\gamma}$ and $\delta_{\beta\delta}$ are Kronecker symbols, $\beta! := \beta_1! \cdots \beta_n!$, and, in order to save signs, we used the "backwards notation"

$$\eta^{\alpha} := \eta_m^{\alpha_m} \eta_{m-1}^{\alpha_{m-1}} \cdots \eta_1^{\alpha_1}.$$

It is now easy to check that the transpose to d is multiplication with $\delta := \sum \xi^i r_i + \sum x^j \rho_j$. Thus $\mathrm{Hom}(K(r \mid \rho), R)$ is just the complex

$$0 \longrightarrow R[\xi \mid x]_0 \xrightarrow{\delta} R[\xi \mid x]_1 \xrightarrow{\delta} \cdots .$$

On the other hand, if A is any R-module we have due to the freeness of $R[\eta \mid y]_i$ over R an isomorphism

$$\mathrm{Hom}(K(r \mid \rho), A) = \mathrm{Hom}(K(r \mid \rho), R) \otimes_R A. \tag{4.4.1}$$

The right-hand side of this is the complex

$$0 \longrightarrow A[\xi \mid x]_0 \xrightarrow{\delta} A[\xi \mid x]_1 \xrightarrow{\delta} \cdots .$$

Therefore we have:

PROPOSITION. *In the situation above, we have an isomorphism*

$$\mathscr{C}^i_A = H^i(\mathrm{Hom}(K(r \mid \rho), A) \tag{4.4.2}$$

which is functional in A. Here $\mathscr{C}^i_A$ denotes the ith *graded part of $\mathscr{C}^{\cdot}_A$.*

4.5. Let again X and Y be as in 4.2, and assume that we are given an exact sequence (4.2.4) of quasicoherent sheaves on X. We may "unroll" the exact triangle (4.2.5) into a long exact sequence of quasicoherent $\mathcal{O}_Y$-modules

$$0 \longrightarrow \mathscr{C}^0_{\mathscr{A}'} \longrightarrow \mathscr{C}^0_{\mathscr{A}} \longrightarrow \mathscr{C}^0_{\mathscr{A}''} \xrightarrow{\delta^0} \mathscr{C}^1_{\mathscr{A}'} \longrightarrow \mathscr{C}^1_{\mathscr{A}} \longrightarrow \cdots .$$

Clearly, this varies naturally with (4.2.4). In other words, the system $(\mathscr{C}^{\cdot}_{(.)}, \delta^{\cdot})$ forms a covariant connected functor sequence from $\mathrm{Qcm}(\mathcal{O}_X)$ to $\mathrm{Qcm}(\mathcal{O}_Y)$.

LEMMA. *We have an isomorphism of functors*

$$\mathscr{H}\!om_{\mathcal{O}_Y}(\mathcal{O}_{Y}, .) \to \mathscr{C}^0_{(.)}. \tag{4.5.1}$$

Proof. With the notations of 4.2, we have

$$\mathscr{C}^0_{\mathscr{A}} \mid_U = \mathrm{Ker}\left(\mathscr{A} \mid_U \xrightarrow{\delta} \sum \mathscr{A} \mid_U \xi_i + \sum \mathscr{A} \mid_U x_j\right)$$

$$= \{a \in \mathscr{A} \mid_U : \mathscr{I} a = 0\} = \mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathscr{A}) \mid_U . \quad \blacksquare$$

By well-known formalisms of homological algebra, there arises a unique morphism of connected functor sequences

$$(\mathscr{E}xt^{\cdot}_{\mathcal{O}_X}(\mathcal{O}_Y, .), \delta^{\cdot}) \to (\mathscr{C}^{\cdot}_{(.)}, \delta^{\cdot}) \tag{4.5.2}$$

which extends (4.5.1).

THEOREM. *The following conditions are equivalent:*

(i) (4.5.2) *is an isomorphism;*

(ii) $\mathscr{C}^i_{\mathscr{A}} = 0$ *for all* $i \geqslant 1$ *whenever* $\mathscr{A}$ *is injective;*

(iii) $\mathscr{C}^1_{\mathscr{A}} = 0$ *whenever* $\mathscr{A}$ *is injective;*

(iv) Y *is regularly immersed into* X *(i.e.,* $\mathscr{I}_p$ *is a regular ideal in* $\mathcal{O}_{X,p}$ *for all* $p \in Y$*).*

Proof. We proceed by the logical scheme (i) $\leftrightarrow$ (ii) $\to$ (iii) $\to$ (iv) $\to$ (ii).

(i) $\to$ (ii) follows from known properties of connected functor sequences, while (ii) $\to$ (iii) is trivial. For the proof of (iii) $\to$ (iv) $\to$ (ii) we may obviously assume that $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} R/I$, where (R, M) is a local ring and $I \subseteq M$ is a conormally free ideal with minimal base $(r \mid \rho)$. Now if A is an injective R-module then $\operatorname{Hom}(., A)$ is an exact functor. By Proposition 4.4 we get

$$\mathscr{C}^i_A = H^i(\operatorname{Hom}(K(r \mid \rho)., A)) = \operatorname{Hom}(H_i(K(r \mid \rho).), A).$$

Therefore $\mathscr{C}^1_A = 0$ for all injectives A implies $H_1(K(r \mid \rho).) = 0$, which by Theorem 2.3 implies the R-regularity of the sequence $(r \mid \rho)$. By Proposition 3.2, I is a regular ideal in R. Thus (iii) $\to$ (iv) is proved, while (iv) $\to$ (ii) follows with similar arguments. ∎

Remark. For showing (iv) $\to$ (i) one can argue in the local situation as follows: If $(r \mid \rho)$ is an R-regular sequence then $K(r \mid \rho)$ is a free resolution of R/I with $I := (r \mid \rho) R$; hence it follows from Proposition 4.4 that $\mathscr{C}^i_A$ is just $\operatorname{Ext}^i_R(R/I, A)$.

Now assume that Y is regularly immersed in X, so that (4.5.2) is an isomorphism. Then we should be able to describe the $S_{\mathcal{O}_Y}\Pi(\mathscr{I}/\mathscr{I}^2)^*$-module law on $\mathscr{C}_{\mathscr{A}}$ in terms of the $\mathscr{E}xt$'s, and this we want to do now.

Applying $\mathscr{E}xt^{\cdot}(., \mathscr{A}) := \mathscr{E}xt^{\cdot}_{\mathcal{O}_X}(., \mathscr{A})$ onto

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathcal{O}_X/\mathscr{I}^2 \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

we get a connection homomorphism

$$\delta\colon \mathscr{E}xt^{\cdot}(\mathscr{I}/\mathscr{I}^2, \mathscr{A}) \longrightarrow \mathscr{E}xt^{\cdot}(\mathcal{O}_Y, \mathscr{A})(+1).$$

Because of the local freeness of $\mathscr{I}/\mathscr{I}^2$ over $\mathcal{O}_Y$ we have

$$\mathscr{E}xt^{\cdot}(\mathscr{I}/\mathscr{I}^2, \mathscr{A}) \cong \mathscr{E}xt^{\cdot}(\mathcal{O}_Y, \mathscr{A}) \otimes_{\mathcal{O}_Y} (\mathscr{I}/\mathscr{I}^2)^*.$$

Therefore we may write the diagram

$$\begin{array}{ccc} \Pi(\mathscr{I}/\mathscr{I}^2)^* \otimes \mathscr{E}xt(\mathcal{O}_Y, \mathscr{A}) & \longrightarrow & \mathscr{E}xt^{\cdot}(\mathcal{O}_Y, \mathscr{A})(+1) \\ \downarrow & & \downarrow \\ \Pi(\mathscr{I}/\mathscr{I}^2)^* \otimes \mathscr{C}_{\mathscr{A}} & \longrightarrow & \mathscr{C}_{\mathscr{A}}(+1). \end{array}$$

The upper line is induced by δ, the lower one describes the module law on $\mathscr{C}_{\mathscr{A}}$, and the verticals are induced by (4.5.2). Now our task is solved by:

PROPOSITION. *This diagram is commutative.*

Proof. Of course, it suffices to show this in the affine situation $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} R/I$, I/I^2 free over R/I with base $([r] \mid [\rho])$, where $(r \mid \rho) \in I_0^{m|n}$ is an R-regular sequence.

Consider the Koszul complex $K. = K(r \mid \rho)$. (cf. 2.3) with $K_i = R[y \mid \eta]_i$ and $d = \sum \rho_j(\partial/\partial y_j) + \sum r_i (\partial/\partial \eta_i)$.

The mapping cone of the complex morphism

$$\kappa\colon K. \to K(-1).^{m|n}, \qquad a \mapsto \left(\frac{\partial a}{\partial \eta} \middle| \frac{\partial a}{\partial y}\right)$$

is the complex

$$C. = K. \oplus K(-1).^{m|n}, \qquad d_{C.} = \begin{pmatrix} d_{K.} & \kappa \\ 0 & d_{K.^{m|n}} \end{pmatrix},$$

and we have the standard sequence

$$0 \to K.^{m|n} \to C. \to K. \to 0. \tag{4.5.3}$$

We claim that this is a projective resolution of the sequence

$$0 \to I/I^2 \to R/I^2 \to R/I \to 0. \tag{4.5.4}$$

Indeed, by Theorem 2.3 and the long exact sequence belonging to (4.5.3) we have $H_i(K.) = H_i(C.) = 0$ for $i \geqslant 1$ while in degree zero we find

$$0 \to (R/I)^{m|n} \to H_0(C.) \to R/I \to 0. \tag{4.5.5}$$

Identifying $(R/I)^{m|n}$ with I/I^2 by $([a] \mid [\alpha]) \mapsto [\sum r_i a_i + \sum \rho_j \alpha_j]$, an elementary computation shows that (4.5.5) is the same as (4.5.4), and our claim is proved.

Now (4.5.3) induces for any R-module A the sequence

$$0 \to \mathrm{Hom}(K., A) \to \mathrm{Hom}(C., A) \to \mathrm{Hom}(K.^{m|n}, A) \to 0$$

and therefore a connection map

$$\begin{array}{ccc} H^i(\mathrm{Hom}(K.^{m|n}, A)) & \longrightarrow & H^{i+1}(\mathrm{Hom}(K., A)) \\ \| & & \| \\ \mathrm{Ext}^i_R(I/I^2, A) & \xrightarrow{\delta} & \mathrm{Ext}^{i+1}_R(R/I, A). \end{array} \tag{4.5.6}$$

With all the identifications we made it is now not hard to see that our assertion amounts to saying that (4.5.6) carries the class of $(f|\varphi) \in \mathrm{Hom}(K., A)^{m|n}$ into the class of $\sum \xi_i f_i + \sum x_j \varphi_j = \sum (-1)^{|f_i|} f_i(\partial/\partial\eta_i) + \sum \varphi_j(\partial/\partial y_j)$, and this is easy to check. ∎

4.6. Let the data $Y \subseteq X$, $\mathscr{I}$ be as in 4.2, let $\mathscr{A}$ be a quasicoherent sheaf on X and $\rho \in \mathscr{I}(X)_1$. We call ρ *regular on* $\mathscr{A}$ if the complex $\mathscr{A} \to^{\rho} \mathscr{A} \to^{\rho} \mathscr{A}$ is exact, i.e., its cohomology sheaf $\mathscr{C}(\rho, \mathscr{A})$ vanishes. Analogously, regular sequences are defined.

We note that ρ is $\mathscr{A}$-regular iff $\rho_p \in \mathcal{O}_{X,p}$ is $\mathscr{A}_p$-regular for each $p \in X$. Of course, it suffices to check this only at the closed points of X.

Note that if $\mathscr{A}$ is coherent for any ρ as above the set

$$X \backslash \mathrm{supp}\, \mathscr{C}(\rho, \mathscr{A}) = \{p \in X: \rho_p \in \mathcal{O}_{X,p} \text{ is } \mathscr{A}_p\text{-regular}\}$$

is the maximal open set U such that $\rho\,|\,U$ is $\mathscr{A}\,|_U$-regular. Now ρ defines an even $\mathcal{O}_Y$-homomorphism

$$\Pi\mathscr{N} = \Pi(\mathscr{I}/\mathscr{I}^2)^* \to \mathcal{O}_Y \tag{4.6.1}$$

and hence an $\mathcal{O}_Y$-algebra homomorphism

$$S_{\mathcal{O}_Y} \Pi\mathscr{N} \to \mathcal{O}_Y;$$

i.e., it determines a section $\mathbb{A}\rho$ of the bundle $\mathbb{A}_Y \to X$. Moreover, it is standard that if $\Theta \subseteq \mathbb{A}_Y$ denotes the image of the zero section we have $\mathrm{Im}(\mathbb{A}\rho) \cap \Theta = \varnothing$ iff (4.6.1) is surjective. If this is the case we may form the composite

$$X \xrightarrow{\mathbb{A}\rho} \mathbb{A}_Y \backslash \Theta \xrightarrow{\pi} \mathbb{P}_Y,$$

where π is the standard projection, and we get a section $\mathbb{P}\rho$ of $\mathbb{P}_Y \to X$.

THEOREM. *Let $Y \subseteq X$, $\mathscr{I}$ be as above, let $\rho \in \mathscr{I}(X)_1$, and assume that $\mathscr{A}$ is a coherent $\mathcal{O}_X$-module. Then the following assertions are equivalent.*

(i) *$\rho|_U$ is regular on $\mathscr{A}|_U$, where $U \supseteq Y$ is a sufficiently small neighborhood;*

(ii) *ρ_p is $\mathscr{A}_p$-regular for all $p \in Y$;*

(iii) $\mathrm{Im}(\mathbb{A}_\rho) \cap \mathrm{supp}(\mathbb{A}\mathscr{C}_{\mathscr{A}}) = \varnothing$ *in ΠN^*;*

(iv) *Putting $V := X \backslash (\mathbb{A}_\rho)^{-1}(\Theta)$, we have* $\mathrm{supp}\,\mathscr{A} \subseteq V$ *and* $\mathrm{Im}(\mathbb{P}(\rho|_V)) \cap \mathrm{supp}(\mathbb{P}\mathscr{C}_{\mathscr{A}}) = \varnothing$ *in* $\mathbb{P}(\Pi N^*)$.

Remark. By the remarks of above, $\mathbb{P}(\rho|_V): V \to \mathbb{P}_{Y|V}$ is well defined.

Proof. (i) $\leftrightarrow$ (ii) is clear. For (ii) $\leftrightarrow$ (iii) it obviously suffices to show:

LEMMA. *Assume that $X = \mathrm{Spec}\, R$ with a local ring (R, M), and let $p \in X$ be the closed point. Let $Y = \mathrm{Spec}\, R/I$ with a conormally free ideal $I \subseteq M$. Then $\rho \in I_1$ is regular on a finite R-module A iff $(\mathbb{A}\rho)(p) \notin \mathrm{supp}\, \mathbb{P}\mathscr{C}_A$.*

Proof. Again we choose a sequence $(r|\rho)$ of elements of I which mod I^2 form a base of I/I^2 over R/I, and we identity $S_{R/I}\Pi(I/I^2)^*$ with $R/I[\xi|x]$ as in 4.2. Then, writing

$$\rho = \sum \alpha_i r^i + \sum a^j \rho_j \qquad \text{with} \quad (\alpha|a) \in R_1^{m|n},$$

the morphism $\mathbb{A}\rho: \mathrm{Spec}(R/I) = Y \to \mathbb{A}_Y = \mathrm{Spec}(R/I[\xi|x])$ is given by $(\mathbb{A}\rho)^*(\xi|x) = ([\alpha]|[a])$. Hence the image $q := (\mathbb{A}\rho)(p)$ corresponds to the ideal $Q/I[\xi|x]$ with

$$Q := M[\xi|x] + (\xi - \alpha|x - a)\, R[\xi|x] \subseteq R[\xi|x].$$

Now we have in the localization $R[\xi|x]_Q$,

$$\begin{aligned} \delta/1 - \rho/1 &= \sum (\xi^i - \alpha^i)\, r_i/1 + \sum (x^j - a^j)\, \rho_j/1 \\ &\in (R[\xi|x]_Q)^1 \cdot Q. \end{aligned} \tag{4.6.2}$$

Putting $B := A \otimes_R R[\xi|x]_Q$ we have therefore a chain of equivalences

$q \notin \mathrm{supp}(\mathbb{A}\mathscr{C}_A)$

$\leftrightarrow Q \notin \mathrm{supp}_{R[\xi|x]}(\mathscr{C}_A)$

$\leftrightarrow \delta/1 \in R[\xi|x]_Q$ is B-regular

$\leftrightarrow$ (by (4.6.2) and Theorem 2.6(1); here the finitude of A comes in) $\rho/1 \in R[\xi|x]_Q$ is B-regular.

Now $R \hookrightarrow R[\xi|x]_Q$ is a flat local homomorphism and hence faithfully flat. Therefore ρ is B-regular iff it is A-regular. ∎

Completion of the Proof of the Theorem. The only nontrivial point yet to be done is the implication (i) $\to$ (supp $\mathscr{A} \subseteq V$). Obviously, it suffices to prove that in the situation of the Lemma above, if ρ is A-regular and $A \neq 0$ then $(\mathbb{A}\rho)(p) \notin \Theta$. Now, with the notations of the proof above, $(\mathbb{A}\rho)(p) \in \Theta$ is easily seen to be equivalent with $\{x \mid \xi\} \subseteq Q$; and it is easy to see that this implies $\{a\} \subseteq M$. But in that case it follows from Theorem 2.6.(1) that ρ cannot be A-regular. The theorem is proved. ▌

5. The Singular Scheme

5.1. To prepare the following, we have to do some elementary algebraic geometry over a not necessarily algebraically closed field K; let $\bar{K}$ denote its algebraic closure. Let W be a fixed finite dimensional even vector space over K, and write $\bar{W} := W \times_K \bar{K}$.

We recall that $\mathbb{P}(W) = \operatorname{Proj} S_K W^*$; thus we can identify $\mathbb{P}(\bar{W}) = \mathbb{P}(W) \times_K \bar{K}$ (more correctly written: $\mathbb{P}(W) \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$). We will identify the set $\mathbb{P}(W)(K)$ of the K-points of $\mathbb{P}(W)$ over K with the set of 1-dimensional subspaces of W.

We also recall that the assignment $\bar{X} \to \bar{X}(\bar{K})$ sets up a 1–1 correspondence

$$\{\text{reduced closed subschemes of } \mathbb{P}(\bar{W})\}$$
$$\leftrightarrow \left\{\begin{matrix}\text{subsets of } \mathbb{P}(\bar{W})(\bar{K}) \text{ which are cut out} \\ \text{by homogeneous equations}\end{matrix}\right\}.$$

Now if $\bar{X} \subseteq \mathbb{P}(\bar{W})$ is closed and reduced and if there exists a closed, reduced $X \subseteq \mathbb{P}(W)$ with $\bar{X} = X \times_K \bar{K}$ then X is uniquely determined by $\bar{X}$. In fact, if $\bar{I} \subseteq S_{\bar{K}} \bar{W}^* = (S_K W^*) \otimes_K \bar{K}$ is the $\mathbb{Z}$-graded ideal belonging to $\bar{X}$ (so that $\bar{X} = \operatorname{Proj}(S_{\bar{K}} \bar{W}^*/\bar{I})$) then X exists iff $\bar{I}$ can be generated by elements of $S_K W^*$, i.e., if putting $I := \bar{I} \cap S_K W^*$ we have $\bar{I} = I \cdot S_{\bar{K}} \bar{W}^*$. Of course, in that case we have $X = \operatorname{Proj}(S_K W^*/I)$.

Now let $\bar{V} \subseteq \bar{W}$ be a linear subspace and $\bar{S} \subseteq \mathbb{P}(\bar{W})$ be a closed reduced subscheme. We define the *cone over* $\bar{S}$ *with vertices in* $\mathbb{P}(\bar{V})$, $\bar{C} = \operatorname{Cone}(\mathbb{P}(\bar{V}), \bar{S})$ as the union of all projective lines in $\mathbb{P}(\bar{W})$ which intersect both $\bar{S}$ and $\mathbb{P}(\bar{V})$. That is, $\bar{C}$ is characterized by

$$\bar{C}(\bar{K}) = \pi((\pi^{-1}(\bar{S}(\bar{K})) \cup \{0\} + \bar{V}) \setminus \{0\}), \tag{5.1.1}$$

where π: $\bar{W} \setminus 0 \to \mathbb{P}(\bar{W})(\bar{K})$ is the standard projection. It is easy to write up the defining ideal $I_{\bar{C}}$: if $(x_0, ..., x_k, y_1, ..., y_{n-k})$ are homogenous coordinates in $\mathbb{P}(\bar{W})$ such that $\mathbb{P}(\bar{V})$ is cut out by $y_1 = \cdots = y_{n-k} = 0$ then any

$\varphi \in \bar{K}[x, y]$ vanishes on $\bar{C}$ if upon writing $\varphi = \sum x^{\mu}\varphi_{\mu}(y)$ any $\varphi_{\mu}(y)$ vanishes on $\bar{S}$. Hence

$$I_{\bar{C}} = \bar{K}[x] \cdot (I_{\bar{S}} \cap \bar{K}[y]) \tag{5.1.2}$$

Now if $\bar{S} = S \times_K \bar{K}$ with a subscheme $S \subseteq \mathbb{P}(W)$ and likewise $\bar{V} = V \times_K \bar{K}$ with a linear subspace $V \subseteq W$ it follows easily from (5.1.2) and the remarks from above that we have

$$\bar{C} = C \times_K \bar{K} \tag{5.1.3}$$

with a unique subscheme $C \subseteq \mathbb{P}(W)$. Therefore we may generalize the definition above by putting $\mathrm{Cone}(\mathbb{P}(V), S) := C$ and calling this again the cone over S with vertices in $\mathbb{P}(V)$.

LEMMA. *Let $v_1, ..., v_k \in W$ be linearly independent, put $V_i := Kv_1 + \cdots + Kv_i$ for $i = 0, ..., k$. Let $S \subseteq \mathbb{P}(W)$ be a closed subset.*

(i) *We have*

$$\mathrm{Cone}(\mathbb{P}(V_i), S) = \mathrm{Cone}(\mathbb{P}(Kv_i), \mathrm{Cone}(\mathbb{P}(V_{i-1}), S)). \tag{5.1.4}$$

(ii) *If $\mathbb{P}(V_k) \cap S = \varnothing$ then*

$$\mathbb{P}(Kv_i) \not\subseteq \mathrm{Cone}(\mathbb{P}(V_{i-1}), S) \tag{5.1.5}$$

for all $i = 1, ..., k$, and

$$\dim \mathrm{Cone}(\mathbb{P}(V_k), S) = \dim S + k. \tag{5.1.6}$$

(iii) *If $\mathbb{P}(V_k) \cap S \neq \varnothing$ then there exists an $i \in \{1, ..., k\}$ such that*

$$\mathbb{P}(V_{i-1}) \cap S = \varnothing \tag{5.1.7}$$

and

$$\mathbb{P}(Kv_i) \in \mathrm{Cone}(\mathbb{P}(V_{i-1}), S). \tag{5.1.8}$$

(iv) *If $k \geqslant n - \dim S$ then $\mathbb{P}(V_k) \cap S \neq \varnothing$.*

(v) *Assume that K is infinite. If $k \leqslant n - \dim S - 1$ and $\mathbb{P}(V_k) \cap S = \varnothing$ then there exists a subspace $V' \subseteq W$ with $\bar{V}_k \cap V' = 0$, $\mathbb{P}(V_k + V') \cap S = \varnothing$, and $\dim(V_k + V') = n - 1 - \dim S$.*

Proof. Using the relation (5.1.3) and the remarks from above, assertion (i) is quickly reduced to the case that K is algebraically closed. But in that case it follows immediately from (5.1.1), and (i) is proved. Under the hypothesis of (ii), the relation $\mathbb{P}(Kv_i) \in \mathrm{Cone}(\mathbb{P}(V_{i-1}), S)$ would imply $\mathbb{P}(\bar{K}v_i) \in \mathrm{Cone}(\mathbb{P}(\overline{V_{i-1}}), \bar{S})$ and hence, by (5.1.1), a relation $v_i = s + v'$ with

$s \in \pi^{-1}(S)$ and $v' \in \overline{V_{i-1}}$. Hence $\pi(s) \in \mathbb{P}(\bar{W}) \cap \bar{S}$, contradicting the assumption, and (5.1.5) is proved.

(5.1.6) follows from the well-known relation $\dim \mathrm{Cone}(\mathbb{P}(Kv_1), S) = \dim S + 1$ and (5.1.4) by induction. In the situation of (iii), let i be maximal with (5.1.7). Then $i \leqslant k$, and because of (5.1.1) there is some $s \in \pi^{-1}(\bar{S}) \cap (\overline{V_i} \backslash 0)$. Writing $s = v + cv_i$ with $v \in \overline{V_{i-1}}$ and $c \in \bar{K}$, (5.1.7) implies $c \neq 0$; hence $v_i \in \overline{V_{i-1}} + \pi^{-1}(\bar{S})$. This implies (5.1.8), and (iii) is proved, too.

(iv) is clear from (ii). In order to prove (v), we make an induction on $d := n - \dim S - 1 - k$, the case $d = 0$ being trivial. In the case $d > 0$, the open set $\mathbb{P}(W) \backslash \mathrm{Cone}(\mathbb{P}(V_k), S)$ is not empty, and it follows from elementary arguments that it contains a K-rational point. In other words, there exists a $v \in W \backslash 0$ such that $\mathbb{P}(Kv) \notin \mathrm{Cone}(\mathbb{P}(V_k), S)$. In particular, $v \notin V_k$ and hence $\dim(V_k + Kv) = \dim V_k + 1$. Using (ii) and (iii), we get $\mathbb{P}(Kv + V_k) \cap S = \varnothing$, and the hypothesis of induction now yields some subspace $V'' \subseteq W$ with $V'' \cap (Kv + V_k) = 0$, $\dim(V'' + V_k + Kv) = n - 1 - \dim S$, and $\mathbb{P}(V_k + Kv + V'') \cap S = \varnothing$. Obviously, the subspace $V' := Kv + V''$ now solves our task. ∎

5.2. Let (R, M, K) be a fixed local ring and write $\Phi := (M/M^2)_1$ as in 2.6. Putting

$$Y := \mathrm{Spec}\, K \subsetneq \mathrm{Spec}\, R =: X,$$

Y is conormally free in X, and the Koszul transform yields a functor

$$\mathrm{Mod}(R) \to \mathrm{Qco}(\mathcal{O}_{\mathbb{P}(\Pi(M/M^2))}), \qquad A \mapsto \mathscr{C}_A.$$

If A is a finite R-module we call

$$\mathrm{Sing}\, A = \mathrm{Sing}_R\, A := \mathrm{supp}\, \mathscr{C}_A \subseteq \mathbb{P}(\widetilde{\Pi(M/M^2)}) = \mathbb{P}(\Pi\Phi)$$

the *singular scheme* of A. We will see in 5.4 that it contains all information on the regularity of sequences of odd elements. We call the number

$$\begin{aligned} \mathrm{odpth}\, A &:= \mathrm{codim}_{\mathbb{P}(\Pi\Phi)} \mathrm{Sing}\, A \\ &= \dim_K \Phi - 1 - \dim \mathrm{Sing}\, A \end{aligned}$$

the *odd depth* of A; Theorem 5.4 will justify this name. We stipulate $\dim \varnothing = -1$, so that $\mathrm{Sing}\, A = \varnothing$ iff $\mathrm{odpth}\, A = \dim \Phi$.

We also introduce the notation

$$\begin{aligned} S(A) = S_R(A) := \{ \Pi V \subseteq \Phi &: \dim V = 0 \mid 1, \text{ and} \\ &V \text{ is not } A\text{-regular} \} \end{aligned}$$

(cf. 2.6) for a finite R-module A.

THEOREM. *Let A be a finite module over a local ring R.*

(i) Sing A *is the unique reduced subscheme of* $\mathbb{P}(\Pi\Phi)$ *with the following property*:

If $\varphi\colon (R, M, K) \to (R', M', K')$ *is any special (cf.* 3.4*) homomorphism,* $A' := A \otimes_R R'$, *and if*

$$\mathbb{P}(\Pi\Phi_R) \times_K K' \cong \mathbb{P}(\Pi\Phi_{R'}) \tag{5.2.1}$$

is the isomorphism induced by $\Phi_R \otimes_K K' = \Phi_{R'}$, *we require that* $(\mathrm{Sing}_R A \times_K K')(K')$ *passes under* (5.2.1) *into the set* $S_{R'}(A') \subseteq \mathbb{P}(\Pi\Phi_{R'})(K')$.
In particular, we have

$$(\mathrm{Sing}_R A)(K) = S(A). \tag{5.2.2}$$

(ii) *In the situation of the condition above, we have*

$$\mathrm{Sing}_R A \times_K K' = \mathrm{Sing}_{R'} A'. \tag{5.2.3}$$

Remarks. (1) In the rest of this chapter, we will solely use the property (i) to analyze the behaviour of Sing A.

(2) It seems that $\mathbb{P}(\Pi\Phi)$ in some sense plays the role of the "odd spectrum" of a local ring R. Indeed, while $r \in R_0$ is regular on A iff it is not contained in any $P \in \mathrm{Ass}\, A$, any $\rho \in R_1 \backslash M_1^2$ is A-regular iff its class in $\mathbb{P}(\Pi\Phi)(K)$ is not contained in $(\mathrm{Sing}\, A)(K)$. Thus $\mathrm{Sing}_R A$ seems to be an odd counterpart of $\mathrm{Ass}_R A$. (The picture is somewhat disturbed by the fact that $\mathbb{P}(\Pi\Phi_R)$ is covariant in R while Spec R is contravariant.)

Proof. We first show assertion (ii). For this purpose, we recall the recipe of constructing $\mathscr{C}_A$: Let $(r \mid \rho)$ be a minimal base of M. If $\delta = \sum r_i \xi^i + \sum \rho_j x^j$ is as in 4.1 the $R[\xi \mid x]$-module $\mathscr{C}(\delta, A[\xi \mid x])$ is annihilated by $M[\xi \mid x]$, and therefore it is a $K[\xi \mid x]$-module; identifying $K[\xi \mid x] = S_K \Pi(M/M^2)^*$ by letting $(\xi \mid x)$ be the right dual base to the base $(\Pi[r] \mid \Pi[\rho])$ of $\Pi(M/M^2)$, it becomes the $S_K \Pi(M/M^2)^*$-module $\mathscr{C}_A$.

Now, if φ is as in (i), it is flat (cf. Lemma 3.4.2), and hence $\mathscr{C}(\delta \otimes_R 1, A'[\xi \mid x]) = \mathscr{C}(\delta, A[\xi \mid x]) \otimes_R R'$. The left-hand side of this is nothing but $\mathscr{C}(\delta', A'[\xi \mid x])$ with $\delta' = \sum \varphi(r_i)\, \xi^i + \sum \varphi(\rho_j) x^j$, and $(\varphi(r) \mid \varphi(\rho))$ is due to Lemma 3.4.2 a minimal base of M'. Therefore the relation above implies $\mathscr{C}_{A'} = \mathscr{C}_A \otimes_K K'$ as modules over $S_{K'} \Pi(M'/M'^2)^* = S_K \Pi(M/M^2)^* \otimes_K K'$. Using [4, Chap. II, Section 4, No. 3, Proposition 19] we get (5.2.3). The relation (5.2.2) is a special case of Theorem 4.6. With its help, it is now easy to show that Sing A satisfies the requirements of (i): Under (5.2.1), we have $S(A') = (\mathrm{Sing}\, A')(K') = (\mathrm{Sing}\, A \times_K K')(K')$.

We are left to prove the uniqueness assertion. If the data (R, M, K), A are given, there exists by Lemma 3.4.5 a special homomorphism

$\varphi: (R, M, K) \to (R', M', K')$ such that K' is the algebraic closure of K. Now if $\mathrm{Sing}' A$ is another closed subset of $\mathbb{P}(\Pi\Phi)$ which satisfies the requirements of (i) we will have $(\mathrm{Sing}' A \times_K K')(K') = S(A') = (\mathrm{Sing}\, A \times_K K')(K')$. By the remarks of 5.1, this implies $\mathrm{Sing}' A = \mathrm{Sing}\, A$. ▮

5.3. In the following we will study various transition properties of the singular scheme.

Let (R, M, K) be a local ring and $I \subseteq M$ be an ideal. If we put $\Phi_I := ((I + M^2)/M^2)_1 \subseteq \Phi_R$ we have

$$\Phi_{R/I} = \Phi_R/\Phi_I.$$

Therefore we get a fibre bundle

$$\psi: \mathbb{P}(\Pi\Phi_R) \backslash \mathbb{P}(\Pi\Phi_I) \to \mathbb{P}(\Pi\Phi_{R/I}).$$

PROPOSITION 5.3.1. *In the situation above, let A be a finite non-zero R-module with $IA = 0$. Then*

$$\mathrm{Sing}_R\, A = \mathbb{P}(\Pi\Phi_I) \cup \psi^{-1}(\mathrm{Sing}_{R/I} A). \tag{5.3.1}$$

Proof. We first consider the case that K is algebraically closed. Then the closed subsets of $\mathbb{P}(\Pi\Phi_R)$ are identified by their K-points, and therefore (5.3.1) is equivalent to

$$S_R(A) = \mathbb{P}(\Pi\Phi_I)(K) \cup \psi^{-1}(S_{R/I}(A)),$$

which is a pure tautology.

Now let us consider the general case: Passing to the completions, we get a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/I \arrow[d] \\ \hat{R} \arrow[r] & \hat{R}/\hat{I} = \widehat{R/I}, \end{tikzcd}$$

and the verticals are special. We now follow the line of proof of Lemma 3.4.5: There exists a residually rational local homomorphism $W \to \hat{R}$ with a Cohen ring W, and there is a special homomorphism $W \to \bar{W}$, where $\bar{W}$ is another Cohen ring which has the algebraic closure $\bar{K}$ of K as residue field. Setting $\bar{R} := \hat{R} \otimes_W \bar{W}$ and $\bar{I} := \hat{I} \otimes_W \bar{W}$, we get the diagram

$$\begin{tikzcd} \hat{R} \arrow[r, dash] \arrow[d] & \hat{R}/\hat{I} \arrow[d] \\ \bar{R} \arrow[r, dash] & \bar{R}/\bar{I} \end{tikzcd}$$

Both verticals are special again. Setting $\bar{A} := A \otimes_R \bar{R}$ we get from Theorem 5.2 and the residually algebraically closed case, $\mathrm{Sing}_R A \times_K \bar{K} = \mathrm{Sing}_{\bar{R}} \bar{A} = \mathbb{P}(\Pi\Phi_{\bar{I}}) \cup \psi^{-1}(\mathrm{Sing}_{\bar{R}/\bar{I}} \bar{A}) = (\mathbb{P}(\Pi\Phi_I) \cup \psi^{-1}(\mathrm{Sing}_{R/I} A)) \times_K \bar{K}$. By the remarks of 5.1 we get the assertion. ▮

Now let (R, M, K) be a local ring and

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

be an exact sequence of finite R-modules. By Proposition 4.2(ii) we get an exact triangle of $\mathbb{Z}$-graded $S_K \Pi(M/M^2)^*$-modules

$$\begin{tikzcd} & \mathscr{C}_{A_2} \arrow[dr] & \\ \mathscr{C}_{A_1} \arrow[ur] & & \mathscr{C}_{A_3} \arrow[ll, "\delta"] \end{tikzcd} \tag{5.3.2}$$

with δ being of degree $+1$. This implies

$$\mathrm{Sing}\, A_{\pi(1)} \subseteq \mathrm{Sing}\, A_{\pi(2)} \cup \mathrm{Sing}\, A_{\pi(3)} \tag{5.3.3}$$

for every permutation $\pi \in S_3$. By induction we get

PROPOSITION 5.3.2. *Let*

$$0 \to A \to B_1 \to \cdots \to B_k \to C \to 0$$

be an exact sequence of finite R-modules, put

$$U := \mathbb{P}(\Pi\Phi) \Big\backslash \bigcup_{i=1}^{k} \mathrm{Sing}\, B_i.$$

Then

$$\mathrm{Sing}\, A \cap U = \mathrm{Sing}\, C \cap U.$$

Let again (R, M, K) be a local ring. If $r \in M_0$ is regular on a finite R-module A then the exact triangle (5.3.2) associated with the exact sequence $0 \to A \to^{r} A \to A/rA \to 0$ splits up since the induced map $\mathscr{C}_r : \mathscr{C}_A \to \mathscr{C}_A$ is easily seen to be zero. We get an exact sequence of $\mathbb{Z}$-graded modules over $S_K \Pi(M/M^2)^*$

$$0 \longrightarrow \mathscr{C}_A \longrightarrow \mathscr{C}_{A/rA} \xrightarrow{\delta} \mathscr{C}_A(+1) \longrightarrow 0.$$

This implies $\mathrm{Sing}\, A/rA = \mathrm{Sing}\, A$. By induction we get:

PROPOSITION 5.3.3. *If $(r_1, ..., r_k)$ is an even regular sequence on a finite R-module A we have*

$$\mathrm{Sing}\, A/(r_1, ..., r_k)\, A = \mathrm{Sing}\, A.$$

Now let $\varphi: (R, M, K) \to (R', M', K')$ be a flat local homomorphism. By Proposition 3.6.1, the induced linear map $\Phi := \Phi_R \to \Phi_{R'} =: \Phi'$ is injective. We get an imbedding

$$\mathbb{P}(\Pi\Phi) \times_K K' \hookrightarrow \mathbb{P}(\Pi\Phi'). \tag{5.3.4}$$

PROPOSITION 5.3.4. *Let A be a finite R-module, put $A' := A \otimes_R R'$. Suppose that K' is separable over K. Then, viewing* (5.3.4) *as inclusion, we have*

$$(\mathrm{Sing}_R\, A) \times_K K' = (\mathrm{Sing}_{R'}\, A') \cap (\mathbb{P}(\Pi\Phi) \times_K K'). \tag{5.3.5}$$

Remark. It is not clear whether (5.3.5) can fail if the separability assumption is dropped.

Proof. First we consider the residually rational case, i.e., $K = K'$. Using Theorem 5.2, we may assume that R and R' are both complete. We can now follow the line of proof of Proposition 5.3.1; with its notations we get a diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R \otimes_W \bar{W} & \longrightarrow & R' \otimes_W \bar{W}. \end{array}$$

The lower line is a flat homomorphism of local rings again, and the verticals are special. Using Theorem 5.2 we get

$$\begin{aligned}(\mathrm{Sing}_R\, A) \times_K \bar{K} &= \mathrm{Sing}_{R \otimes_W \bar{W}}(A \otimes_W \bar{W}) \\ &= \mathrm{Sing}_{R' \otimes_W \bar{W}}(A \otimes_R (R' \otimes_W \bar{W})) \cap \mathbb{P}(\Pi\Phi_{R \otimes_W \bar{W}}) \\ &= (\mathrm{Sing}_{R'}(A \otimes_R R') \cap \mathbb{P}(\Pi\Phi_R)) \times_K \bar{K},\end{aligned}$$

which implies (5.3.5) for $K = K'$.

Turning now to the general case, we can again suppose that R and R' are complete. By the arguments in the proof of Proposition 5.3.1, we can write up a diagram

$$\begin{array}{ccc} W & \longrightarrow & W' \\ \downarrow & & \downarrow \\ K & \hookrightarrow & K' \end{array}$$

where $W \to W'$ is a special homomorphism of Cohen rings and the verticals are the projections onto the residue fields. Moreover, we may view R as W-algebra. Now the separability assumption implies by [6, Theorems (19.7.1) and (19.6.1)] that W' is formally smooth over W. Hence we can augment the diagram

$$\begin{tikzcd}
R \arrow[rrr] \arrow[dr] & & & R' \arrow[dl] \\
& K \arrow[r] & K' & \\
& W \arrow[uul] \arrow[u] \arrow[r] & W' \arrow[u] \arrow[uur, dashed] &
\end{tikzcd}$$

(cf. [6, Proposition (19.3.10)]; it is applicable without problems since $(R')_0$ is a classical complete local ring). Thus we may factorize φ into

$$R = R \otimes_W W \xrightarrow{\varphi_1} R \otimes_W W' \xrightarrow{\varphi_2} R'.$$

Now φ_1 is special again. On the other hand, applying [5, (10.2.5)] we learn that φ_2 is flat. Moreover, it is residually rational. Now we get (5.3.5) by linking Theorem 5.2(ii) with the residually rational case. ∎

5.4. Let (R, M, K) be a local ring and $A \neq 0$ a finite R-module. We call a subspace $V \subseteq \Phi$ *maximally A-regular* if it is A-regular (cf. 2.6) and there is no larger subspace $V' \supset V$ of Φ which is A-regular, too.

The following is our central result on odd regular sequences.

THEOREM. (i) *Let $(\rho_1, \ldots, \rho_k)$, $k \geqslant 1$, be an odd A-regular sequence, let $V \subseteq \Phi$ be the subspace generated by the residue classes of the ρ_i in Φ. Then we have*

$$\operatorname{Sing}(\rho_1 \cdots \rho_k A) = \operatorname{Cone}(\mathbb{P}(\Pi V), \operatorname{Sing} A) \tag{5.4.1}$$

and

$$\operatorname{odpth}(\rho_1 \cdots \rho_k A) = \operatorname{odpth} A - k. \tag{5.4.2}$$

(ii) *Let $V \subseteq \Phi$ be any linear subspace. Then V is A-regular iff $\mathbb{P}(\Pi V) \cap \operatorname{Sing} A = \varnothing$. If this is the case then* $\dim_K \Pi V \leqslant \operatorname{odpth} A$.

(iii) *Suppose that the field K is infinite. A subspace $V \subseteq \Phi$ is maximally A-regular iff we have $\mathbb{P}(\Pi V) \cap \operatorname{Sing} A = \varnothing$ and* $\dim_K \Pi V = \operatorname{odpth} A$.

Remark. It follows in particular that the following assertions are equivalent:

(1) $\operatorname{Sing} A = \varnothing$, i.e., $\operatorname{odpth} A = \dim_{K} \Pi\Phi$;

(2) Φ is A-regular, i.e., any minimal base of R^1 is an A-regular sequence.

However, it may happen that although all $\rho \in R_1 \backslash M_1^2$ are A-regular, i.e., $\operatorname{Sing} A$ has no K-points, it is nevertheless not empty. As an example, consider $A = \mathbb{C}[\xi, \eta]/(\xi + \sqrt{-1}\,\eta)\,\mathbb{C}[\xi, \eta]$ over $R = \mathbb{R}[\xi, \eta]$.

LEMMA 5.4.1. *Let Φ be an odd finite vector space over a field K, $R := S_K\Phi$, A a finite R-module, and assume that $\rho \in \Phi \backslash 0$ is A-regular. Then*

$$\operatorname{Sing} \rho A = \operatorname{Cone}(\mathbb{P}(\Pi K_\rho), \operatorname{Sing} A).$$

Proof. By tensoring over K with the algebraic closure of K and using Theorem 5.2 we can reduce the question onto the case that K is algebraically closed. In that case, we have to inspect the K-rational points only.

We therefore introduce the auxiliary set

$$S_{A'} := \{\rho \in \Phi : \rho \text{ is not } A\text{-regular}\}.$$

The assertion is now equivalent to

$$S'_{\rho A} = S'_A + K\rho. \tag{5.4.3}$$

By Lemma 2.1.1, $\lambda \in S'_A$ implies $\lambda \in S'_{\rho A}$; and since for any $k \in K$ the element $\lambda + k\rho$ acts on ρA in the same way as λ, it follows that $\lambda + k\rho \in S'_{\rho A}$. This proves the inclusion $\supseteq$ in (5.4.3).

We are left to show $\subseteq$. Let $\lambda \in S'_{\rho A}$. If $\lambda \in S'_A$ we are done. Else we have by Lemma 2.1.1 an isomorphism

$$\mathscr{C}(\lambda, \rho A) \circlearrowleft, \qquad [\rho a] \mapsto [\lambda a]$$

(classes taken modulo $\lambda\rho A$).

Now this is an automorphism of a finite vector space over K; hence it has a non-zero eigenvalue c. Substituting λ by $c\lambda$ if necessary we may assume $c = 1$; i.e., there exists an $a \in A$ with $[\rho a] = [\lambda a]$, i.e.,

$$(\rho - \lambda)a = \rho\lambda a' \tag{5.4.4}$$

with suitable $a' \in A$ and

$$\rho a \notin \rho\lambda A. \tag{5.4.5}$$

Now (5.4.4) can be rewritten to $(\rho-\lambda)(a-\rho a')=0$. On the other hand, if there existed an $a''\in A$ with $a-\rho a'=(\rho-\lambda)\,a''$ we would get a contradiction to (5.4.5). Hence $\rho-\lambda\in S'_A$; i.e., λ is contained in the r.h.s. of (5.4.3). ∎

LEMMA 5.4.2. *Let (R, M, K) and A be as in the theorem, let $\rho\in R_1$ be A-regular. Then*

$$\operatorname{Sing}(\rho A)=\operatorname{Cone}(\mathbb{P}(\Pi K\rho), \operatorname{Sing} A).$$

Proof. We will reduce the situation to that of the preceding lemma. Using Theorem 5.2, we first may assume that R is complete. Using Lemma 3.4.4, we may write $R=\tilde{S}[\xi]/I$, where $\tilde{S}$ is a classical regular local ring, (ξ) is a sequence of n odd variables, and I is an ideal with $\Phi_I=0$ (cf. 5.3). Now, by Proposition 5.3, we are reduced to the case $R=\tilde{S}[\xi]$.

We may choose an exact sequence

$$0\to B\to F_n\to\cdots\to F_0\to A\to 0$$

with free R-modules $F_0, ..., F_n$ and $n := Kr\text{-dim } S$. Because of Theorem 2.6 we have $\rho\notin(M_R)^2$; hence ρ is regular on all F_i, and Proposition 2.2 yields that the sequence

$$0\to\rho B\to\rho F_n\to\cdots\to\rho F_0\to\rho A\to 0$$

is exact again. Now Proposition 5.3.2 yields

$$\operatorname{Sing} A=\operatorname{Sing} B \qquad \text{and} \qquad \operatorname{Sing}(\rho A)=\operatorname{Sing}(\rho B).$$

On the other hand, since $\tilde{S}$ is classically regular, B is now a free $\tilde{S}$-module. In particular, any minimal base of the maximal ideal of $\tilde{S}$ will be a B-regular sequence, and Proposition 5.3.3 therefore yields

$$\operatorname{Sing} B=\operatorname{Sing} B/M_{\tilde{S}}B \qquad \text{and} \qquad \operatorname{Sing}(\rho B)=\operatorname{Sing} B/(\rho, M_{\tilde{S}})\,B.$$

Now $C := B/M_{\tilde{S}}B$ is a module over $R/M_{\tilde{S}}R=K[\xi]$, and by Proposition 5.3.1 we have

$$\operatorname{Sing}_R C=\operatorname{Sing}_{K[\xi]} C \qquad \text{and} \qquad \operatorname{Sing}_R C/\rho C=\operatorname{Sing}_{K[\xi]} C/\rho C.$$

Altogether, we are reduced into the situation of the previous lemma, and the assertion follows. ∎

Proof of the Theorem. The relation (5.4.1) follows from the previous lemma and lemma 5.1(i) by induction. Now assume that $V\subseteq\Phi$ is A-regular. Then $\mathbb{P}(\Pi V)\cap\operatorname{Sing} A=\varnothing$ follows from (5.4.1) and Lemma 5.1(i) and (iii). Applying Lemma 5.1(ii) we then get (5.1.2).

Conversely, if $\mathbb{P}(\Pi V) \cap \operatorname{Sing} A = \varnothing$ then Lemma 5.1(ii) and induction show that V is A-regular. The estimate $\dim \Pi V \leqslant \operatorname{odpth} A$ follows from Lemma 5.1(iv). Thus the assertions (i), (ii) of the theorem are proved, while (iii) follows from Lemma 5.1(v) and the preceding assertions. ∎

5.5. We now return to the Koszul transform in the general setting of Chap 4. We will give a complete characterization of $\operatorname{supp} \mathbb{P}\mathscr{C}_{\mathscr{A}}$ in terms of the singular schemes $\operatorname{Sing}_{\mathcal{O}_p} \mathscr{A}_p$, where p runs through the points of Y.

Let $Y \subseteq X$ be a conormally locally free subscheme of a locally noetherian scheme X. Let us compute the fibre of the projection $\psi\colon \mathbb{P}(\Pi N^*_{Y/X}) \to Y$ over $p \in Y$. Let $(\mathcal{O}_p, \mathscr{M}_p, \mathscr{K}_p)$ be the stalk of $\mathcal{O}_X$ at p as local ring, let $\mathscr{I}$ be the ideal sheaf of Y in X. Then

$$\begin{aligned}\psi^{-1}(\{p\}) &= \operatorname{Proj}(S_{\mathcal{O}_Y} \Pi(\mathscr{I}/\mathscr{I}^2)^* \otimes_{\mathcal{O}_Y} \mathscr{K}_p)\\ &= \operatorname{Proj}(S_{\mathscr{K}_p} \Pi(\mathscr{I}_p/\mathscr{M}_p \mathscr{I}_p)^*)\\ &= \mathbb{P}(\Pi(\mathscr{I}_p/\mathscr{M}_p \mathscr{I}_p)).\end{aligned}$$

Writing as usual, $\Phi_p := (\mathscr{M}_p/\mathscr{M}_p^2)_1$, we have by Proposition 3.4 an injection $(\mathscr{I}_p/\mathscr{M}_p\mathscr{I}_p)_1 \hookrightarrow \Phi_p$ and therefore a closed embedding

$$\widetilde{\psi^{-1}(\{p\})} \subseteq \mathbb{P}(\Pi \Phi_p).$$

THEOREM. *Let $\mathscr{A}$ be a coherent sheaf on X. Then*

$$\operatorname{supp} \mathbb{P}\mathscr{C}_{\mathscr{A}} \cap \psi^{-1}(\{p\}) = \operatorname{Sing}_{\mathcal{O}_p} \mathscr{A}_p \cap \psi^{-1}(\{p\})$$

for all $p \in Y$.

Proof. It obviously suffices to prove: Let I be a conormally free ideal in a local ring (R, M, K), let A be a finite R-module and $\mathscr{C}_A$ be the Koszul transform of A with respect to $Y = \operatorname{Spec}(R/I) \hookrightarrow \operatorname{Spec} R = X$. Let p be the closed point of X. With the notations from above, we have

$$\operatorname{supp} \mathbb{P}\mathscr{C}_A \cap \psi^{-1}(\{p\}) = \operatorname{Sing} A \cap \psi^{-1}(\{p\}). \tag{5.5.1}$$

As usual, we first consider the case that K is algebraically closed. Both sides of (5.5.1) are closed subsets of $\mathbb{P}(\Pi\Phi)$, and hence it suffices to compare their K-rational points. Now Theorem 4.6 yields the answer.

Turning to the general case, we can choose again a special homomorphism $R \to (R', M', K')$ with K' being algebraically closed. Setting $I' := IR'$ we have $I'/(I')^2 = I/I^2 \otimes_{R/I} R'/I'$; hence I' is conormally free in R'. Moreover, setting $A' := A \otimes_R R'$ we have as in the proof of Theorem 5.2 that

$$\mathscr{C}^{I'}_{A'} = \mathscr{C}^I_A \otimes_{S_{R/I}\Pi(I/I^2)^*} S_{R'/I'} \Pi(I'/(I')^2)^*.$$

From this the assertion easily follows. ∎

5.6. Let $X \to S$ be a scheme which is defined over a classical scheme S (i.e., $\mathcal{O}_{S,1} = 0$). As usual, the embedding $i\colon \tilde{X} \hookrightarrow X$ determines an exact sequence

$$\Psi \xrightarrow{\mu} i^*(\Omega^1_{X/S}) \xrightarrow{\nu} \Omega^1_{\tilde{X}/S} \to 0 \tag{5.6.1}$$

with $\Psi := \mathcal{O}^1/\mathcal{O}^2$ and $\mu\colon \rho \mapsto i^*(d\rho)$.

PROPOSITION. *μ is injective, and the resulting short exact sequence splits in a natural way, so that*

$$i^*(\Omega^1_{X/S}) = \Pi\Psi \oplus \Omega^1_{\tilde{X}/S}.$$

This is a decomposition into even and odd parts.

Proof. We define a derivation

$$\mathcal{O}_{\tilde{X}} = \mathcal{O}_X/\mathcal{O}^1_X \to i^*(\Omega^1_{X/S}) \tag{5.6.2}$$

by $[a] \mapsto i^*(da)$ for $a \in \mathcal{O}_{X,0}$. This is well defined, since $[a] = 0$ implies $a \in \mathcal{O}^1_X \cap \mathcal{O}_{X,0} = \mathcal{O}^2_{X,0}$ and hence $da \in \mathcal{O}^1\Omega^1_{X/S}$.

Now (5.6.2) determines an $\mathcal{O}_{\tilde{X}}$-linear map $\Omega^1_{\tilde{X}/S} \to i^*(\Omega^1_{X/S})$, and it is easy to see that this is a right inverse to Ψ.

On the other hand, we have a derivation $\mathcal{O} \to \Psi$, $a \mapsto [a_1]$, for $a = a_0 \oplus a_1 \in \mathcal{O} = \mathcal{O}_0 \oplus \mathcal{O}_1$. It induces an $\mathcal{O}$-linear map $\Omega^1_{X/S} \to \Psi$ which obviously vanishes on $\mathcal{O}^1\Omega^1_{X/S}$ and therefore gives rise to $i^*(\Omega^1_{X/S}) \to \Psi$. This map is the left inverse to μ. The assertions follow. ∎

Now assume that X is a scheme such that the subscheme $\tilde{X} \hookrightarrow X$ is co-normally locally free, i.e., Ψ is a locally free $\mathcal{O}_{\tilde{X}}$-module. Then $\mathbb{P}(\Pi\Psi) \to \tilde{X}$ is a classical projective bundle, and the Koszul transform yields a functor $\mathrm{Qco}(\mathcal{O}_X) \to \mathrm{Qco}(\mathcal{O}_{\mathbb{P}(\Pi\Psi)})$, $\mathcal{A} \mapsto \mathbb{P}\mathcal{C}_{\mathcal{A}}$. Now $\mathbb{P}(\Pi\Psi)$ is just the disjoint union of all $\mathbb{P}(\Pi\Phi_{\mathcal{O}_p})$ with $p \in X$; and from Theorem 5.5 we get:

COROLLARY. *For any coherent $\mathcal{O}_X$-module $\mathcal{A}$,* $\operatorname{supp} \mathbb{P}\mathcal{C}_{\mathcal{A}}$ *is exactly the union of all* $\mathrm{Sing}_{\mathcal{O}_p} \mathcal{A}_p$ *with $p \in X$.*

6. PROJECTIVE AND INJECTIVE RESOLUTIONS OVER LOCAL RINGS

6.1. PROPOSITION. *Let R be an oddly regular local ring and A a finite R-module with* $\mathrm{Sing}\, A = \varnothing$.

(i) *If $0 \to A \to Q^{\cdot}$ is an injective resolution of A then $0 \to A/R^1A \to Q^{\cdot}/R^1Q^{\cdot}$ is an injective resolution of A/R^1A over $\tilde{R} = R/R^1$* (cf. 1.2).

(ii) *The injective resolution* $0 \to A \to Q^{\cdot}$ *is minimal* (i.e., Q^i *is the injective envelope of* $\mathrm{Ker}(Q^i \to Q^{i+1})$ *for all* i) *iff* $0 \to A/R^1A \to Q^{\cdot}/R^1Q^{\cdot}$ *is minimal.*

(iii) *If* $F. \to A \to 0$ *is a projective resolution of* A *then* $F./R^1F. \to A/R^1A \to 0$ *is a projective resolution of* A/R^1A *over* $\tilde{R}$.

(iv) *The projective resolution* $F. \to A \to 0$ *is minimal* (*i.e.,* $\mathrm{Im}(F_{i+1} \to F_i) \subseteq MF_i$ *for all* i) *iff* $F./R^1F. \to A/R^1A \to 0$ *is minimal.*

(v) *For the projective and injective dimensions we have*

$$\mathrm{pd}_R\, A = \mathrm{pd}_{\tilde{R}}\, A/R^1A$$

and

$$\mathrm{id}_R\, A = \mathrm{id}_{\tilde{R}}\, A/R^1A.$$

For the proof, we first note:

LEMMA 6.1.1. *Let* Q *be an injective module over a ring* R, *and let* $I \subseteq R$ *be an ideal. Then the* R/I*-module*

$$0 :_Q I = \{q \in Q : Iq = 0\}$$

is injective. Moreover, if Q *is an essential extension of a submodule* A *then* $0 :_Q I$ *is an essential extension of* $0 :_A I$.

Proof. Given a diagram of R/I-modules

$$\begin{array}{ccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C_2 \\ & & \downarrow & & \\ & & 0 :_Q I & & \end{array}$$

we may view it as diagram of R-modules and complete it to

$$\begin{array}{ccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C_2 \\ & & \downarrow & & \downarrow \\ & & 0 :_Q I & \subsetneq & Q. \end{array}$$

On the other hand, since $IC_2 = 0$, the image of $C_2 \to Q$ lies automatically in $0 :_Q I$. This proves the first assertion. Now if $B \subseteq 0 :_Q I$ is a submodule then $IB = 0$ implies $A \cap B \subseteq (0 :_A I) \cap B$, from which the second assertion follows. ∎

LEMMA 6.1.2. *Let R be a ring, Q an injective R-module and $(\rho_1, ..., \rho_n)$ an R-regular sequence. Then $(\rho_1, ..., \rho_n)$ is also Q-regular, and $Q/(\rho_1, ..., \rho_n)Q$ is injective over $R/(\rho_1, ..., \rho_n) R$.*

Proof. By induction, we may assume $n = 1$. Applying $\mathrm{Hom}_R(., Q)$ onto $R \to^{\rho_1} R \to^{\rho_1} R$ we get that ρ_1 is Q-regular; hence $\mathrm{Ker}(Q \to^{\rho_1} Q) = Q/\rho_1 Q$ over $R/\rho_1 R$. Therefore the preceding lemma proves that $Q/\rho_1 Q$ is injective over $R/\rho_1 R$. ∎

Proof of the Proposition. Given the injective resolution $0 \to A \to Q^{\cdot}$, we can break it up at the ith place, getting an exact sequence

$$0 \to A \to Q^0 \to Q^1 \to \cdots \to Q^i \to B^i \to 0.$$

Using Corollary 3.3, the previous lemma and Proposition 2.2 we get that

$$0 \to A/R^1 A \to Q^0/R^1 Q^0 \to \cdots \to Q^i/R^1 Q^i \to B^i/R^1 B^i \to 0$$

is exact and that all $Q^j/R^1 Q^j$ are injective. Assertion (i) follows, while (iii) is proved quite analogous. The assertion (ii) is easily proved with the help of Lemma 6.1.1 and Proposition 2.2, while (iv) is obvious. Finally, (v) is a consequence of the previous assertions and the fact that, due to the nilpotency of R^1, $C/R^1 C = 0$ for any R-module C implies $C = 0$ (cf. [4, Chap. II, Section 3, No. 2, Proposition 4]). ∎

COROLLARY. *Let R be an oddly regular local ring. Then for any finite R-module A we have*

$$\mathrm{pd}_R A < \infty \leftrightarrow (\mathrm{Sing}\, A = \emptyset \text{ and } \mathrm{pd}_{\bar{R}} A/R^1 A < \infty)$$

and

$$\mathrm{id}_R A < \infty \leftrightarrow (\mathrm{Sing}\, A = \emptyset \text{ and } \mathrm{id}_{\bar{R}} A/R^1 A < \infty).$$

Proof. The only point yet to be proved is that finite injective or projective dimension implies $\mathrm{Sing}\, A = \emptyset$. Now if $0 \to A \to Q^0 \to \cdots \to Q^n \to 0$ is an exact sequence with all Q^i injective then Lemma 6.1.2 and Proposition 2.2 yield that any minimal base of R^1 is an A-regular sequence; hence $\mathrm{Sing}\, A = \emptyset$. The same argument holds for projective resolutions. ∎

6.2. Let (R, M, K) be a regular local ring (cf. 3.3) with $\dim_K M/M^2 = m|n$, let E denote the injective envelope of K.

THEOREM. *Let A be a finite R-module, put $B := \mathrm{Hom}_R(A, E)$.*

(i) *For $i \geqslant 0$ we have natural isomorphisms of K-vector spaces*

$$\mathscr{C}^i_A = \mathrm{Ext}^i_R(K, A) \tag{6.2.1}$$

and

$$\mathscr{C}^i_B = \mathrm{Ext}^i_R(K, B) = \mathrm{Ext}^i_R(A, K). \tag{6.2.2}$$

(ii) $\mathbb{P}\mathscr{C}_B$ *is a coherent sheaf on* $\mathbb{P}(\Pi(M/M^2)^*)$.

(iii) *We have*

$$\mathrm{supp}\ \mathbb{P}\mathscr{C}_B = \mathrm{supp}\ \mathbb{P}\mathscr{C}_A = \mathrm{Sing}\ A. \tag{6.2.3}$$

(iv) *There exist polynomials* $P_j(X)$, $Q_j(X) \in \mathbb{Q}[X]$ $(j = 0, 1)$ *such that for* $i \gg 0$ *we have*

$$P_0(i) \mid P_1(i) = \dim_K \mathrm{Ext}^i_R(K, A),$$

$$Q_0(i) \mid Q_1(i) = \dim_K \mathrm{Ext}^i_R(A, K).$$

Moreover,

$$\begin{aligned}\max(\deg P_0, \deg P_1) &= \max(\deg Q_0, \deg Q_1)\\ &= \dim \mathrm{Sing}\ A = n - 1 - \mathrm{odpth}\ A.\end{aligned}$$

Proof. Ad(i). Let $\alpha\colon K \subsetneq E$ denote the standard embedding. First we note

$$\dim_K \mathrm{Hom}_R(K, E) = 1. \tag{6.2.4}$$

Indeed, if there was another $\beta\colon K \to E$ which is not a multiple of α then $\beta(K)$ would be a nonzero submodule of E which does not cut $\alpha(K)$; but this contradicts the essentiality of the extension $\alpha\colon K \subsetneq E$.

Now we have the standard isomorphism of functors

$$\mathrm{Hom}_R(K, \mathrm{Hom}_R(.\,, E)) = \mathrm{Hom}_R(K \otimes_R .\,, E);$$

taking the derived functors, we get an isomorphism

$$\mathrm{Ext}^i_R(K, \mathrm{Hom}_R(A, E)) = \mathrm{Hom}_R(\mathrm{Tor}^R_i(K, A), E). \tag{6.2.5}$$

On the other hand, it is well known (and easy to see by using a minimal resolution of A) that

$$\mathrm{Ext}^i_R(A, K) = \mathrm{Hom}_R(\mathrm{Tor}^R_i(K, A), K).$$

Comparing this with (6.2.5) and using (6.2.4) we get

$$\mathrm{Ext}^i_R(A, K) = \mathrm{Ext}^i_R(K, B).$$

Using Theorem 4.5 we now get the isomorphisms (6.2.1) and (6.2.2).

Ad (ii). Since E is never a finite R-module (unless $n = 0$, in which case one has $E = R$) it is not quite obvious that $\mathbb{P}\mathscr{C}_B$ is coherent.

LEMMA 6.2.1. *If $R^1A = 0$ then $\mathbb{P}\mathscr{C}_B$ is coherent.*

Proof. Let $(r\,|\,\rho)$ be a minimal base of M. In the notations of 4.1, we may write $\mathscr{C}_B = \mathscr{C}(\delta, B[\xi\,|\,x]) = \mathscr{C}(\delta', B[\xi]) \otimes_K K[x]$ with $\delta' := \sum \xi^j r_j$. This is a product of $\mathbb{Z}$-graded modules, and the coherence of $\mathbb{P}\mathscr{C}_B$ will be proved once we show that $\mathscr{C}(\delta', B[\xi])$ is finite over $K[\xi]$.

Otherwise we have for at least one graded part $(\cdots)^{i_0}$ that

$$\dim_K \mathscr{C}(\delta', B[\xi])^{i_0} = \infty. \tag{6.2.6}$$

Now

$$\mathscr{C}_B^{i_0} \supseteq \mathscr{C}(\delta', B[\xi])^{i_0} \otimes_K (K[x])^0 = \mathscr{C}(\delta', B[\xi]),$$

hence

$$\dim_K \mathscr{C}_B^{i_0} = \infty. \tag{6.2.7}$$

On the other hand, we know already that $\mathscr{C}_B^{i_0} = \operatorname{Ext}_R^{i_0}(A, K)$, and since A is finite and R is noetherian, this is certainly finite. Thus (6.2.7) and hence (6.2.6) cannot hold, and $\mathbb{P}\mathscr{C}_B$ is proved coherent. ▌

Continuation of the Proof. Turning now to the general case, the exact sequences

$$0 \to R^{i+1}A \to R^iA \to \operatorname{gr}^i_{R^1} A \to 0$$

yield exact triangles

$$\begin{tikzcd}[column sep=small]
 & \mathbb{P}\mathscr{C}_{\operatorname{Hom}_R(R^iA,E)} \arrow[dl] & \\
\mathbb{P}\mathscr{C}_{\operatorname{Hom}_R(R^{i+1}A,E)} \arrow[rr] & & \mathbb{P}\mathscr{C}_{\operatorname{Hom}_R(\operatorname{gr}^i_{R^1} A,E)} \arrow[ul]
\end{tikzcd},$$

and by the lemma above the right lower term is coherent.

Since $R^{n+1}A = 0$ it now follows by descending induction on i that $\mathbb{P}\mathscr{C}_{\operatorname{Hom}(R^iA,E)}$ is coherent for all i. In particular, for $i = 0$ we get assertion (ii).

Ad (iii). We first note:

LEMMA 6.2.2. *An element $\rho \in R_1$ is A-regular iff it is B-regular.*

Proof. If C is any non-zero R-module then C/MC is a vector space over K. Hence there exists always a non-zero composite $C \to C/MC \to K \subsetneq E$, i.e., $\operatorname{Hom}_R(C, E) \neq 0$.

It follows that the functor $\operatorname{Hom}_R(\,.\,, E)$ both preserves and detects exactness. In particular, $A \to^{\rho} A \to^{\rho} A$ is exact iff $B \to^{\rho} B \to^{\rho} B$ is so. ▌

Continuation of the Proof. This lemma already suggests the relation (6.2.3); but, unfortunately, B is in general not finite over R, and thus Theorem 4.6 or Theorem 5.5 is not applicable. Instead of this, we do induction on $m = Kr\text{-dim } R$.

Start of induction. If $m = 0$ it follows from Proposition 6.1 that R is an injective R-module. Moreover, if we embed $K = R/MR$ into R by $[1] \mapsto \rho_1 \cdots \rho_n$ then R is easily seen to be an essential extension of K, and thus $E = R$.

It follows that B is finite over R, and Sing B is well defined. We are to show Sing A = Sing B. Again, we use our standard technique: The case that K is algebraically closed is done with Theorem 4.6 and the lemma above, while the general case is reduced to it with the help of Theorem 5.2(ii).

Step of induction. Choose some $r \in M_0 \backslash M_0^2$. Then, by Corollary 3.3 and known facts on classically regular local rings, $R' := R/rR$ is regular again of the embedding dimension $(m-1)|n$, and we want to play back the problem onto R'.

First we note that, without loss of generality, we may assume that r is A-regular. Indeed, we may choose an exact sequence

$$0 \to A'' \to R^{k|l} \to A \to 0. \tag{6.2.8}$$

Setting $B'' := \operatorname{Hom}_R(A'', E)$ we get

$$0 \to B \to E^{k|l} \to B'' \to 0. \tag{6.2.9}$$

Now we have $\mathbb{P}\mathscr{C}_E = 0$ from Theorem 4.5 and $\mathbb{P}\mathscr{C}_R = 0$ because of Remark 5.4. Therefore the exact triangles belonging to (6.2.8) and (6.2.9) yield isomorphisms $\mathbb{P}\mathscr{C}_A = \mathbb{P}\mathscr{C}_{A''}(+1)$ and $\mathbb{P}\mathscr{C}_{B''} = \mathbb{P}\mathscr{C}_B(+1)$. On the other hand, r is certainly regular on A'', so that we can replace A by A'' if necessary.

Now, setting $A' := A/rA$, the exact sequence

$$0 \longrightarrow A \xrightarrow{r} A \longrightarrow A' \longrightarrow 0 \tag{6.2.10}$$

yields with $B' := \operatorname{Hom}_R(A', E)$,

$$0 \longrightarrow B' \longrightarrow B \xrightarrow{r} B \longrightarrow 0. \tag{6.2.11}$$

On the other hand, since $r \in M$ the induced maps $\mathscr{C}_r : \mathscr{C}_B \circlearrowleft$ and $\mathscr{C}_r : \mathscr{C}_A \circlearrowleft$ are zero, so that the exact triangles belonging to (6.2.10) and (6.2.11) yield exact sequences

$$0 \to \mathbb{P}\mathscr{C}_A \to \mathbb{P}\mathscr{C}_{A'} \to \mathbb{P}\mathscr{C}_A(+1) \to 0$$

and

$$0 \to \mathbb{P}\mathscr{C}_B \to \mathbb{P}\mathscr{C}_{B'}(+1) \to \mathbb{P}\mathscr{C}_B(+1) \to 0.$$

Thus

$$\operatorname{supp} \mathbb{P}\mathscr{C}_A = \operatorname{supp} \mathbb{P}\mathscr{C}_{A'} \tag{6.2.12}$$

and

$$\operatorname{supp} \mathbb{P}\mathscr{C}_B = \operatorname{supp} \mathbb{P}\mathscr{C}_{B'}. \tag{6.2.13}$$

Now let $E' := \operatorname{Ker}(E \to^{r} E)$. Then $K \subseteq E'$ in E, and it follows from Lemma 6.1.1 that E' is just the injective envelope of K considered as R'-module.

On the other hand, we have $rA' = 0$ and therefore

$$B' = \operatorname{Hom}_R(A', E) = \operatorname{Hom}_{R'}(A', E').$$

With $M' := M_{R'} = M/rR$, the hypothesis of induction now yields

$$\operatorname{supp} \mathbb{P}\mathscr{C}_{A'}^{R', M'} = \operatorname{supp} \mathbb{P}\mathscr{C}_{B'}^{R', M'} \tag{6.1.14}$$

(in obvious notation). On the other hand, we claim that under the natural identification $\Phi_R = \Phi_{R'}$ we have

$$\operatorname{supp} \mathbb{P}\mathscr{C}_{A'}^{R, M} = \operatorname{supp} \mathbb{P}\mathscr{C}_{A'}^{R', M'} \tag{6.2.15}$$

and

$$\operatorname{supp} \mathbb{P}\mathscr{C}_{B'}^{R, M} = \operatorname{supp} \mathbb{P}\mathscr{C}_{B'}^{R', M'}. \tag{6.2.16}$$

Indeed, the natural surjection $M/M^2 \to M'/(M')^2$ induces an embedding $S_K\Pi(M'/(M')^2)^* \subsetneq S_K\Pi(M/M^2)^*$. Analyzing the recipe of constructing $\mathscr{C}_{B'}^{R, M}$ and using the fact that r can be included into a minimal base of M, one finds

$$\mathscr{C}_{B'}^{R, M} = \mathscr{C}_{B'}^{R', M'} \otimes_{S_K\Pi(M'/(M')^2)^*} S_K\Pi(M/M^2)^*,$$

from which (6.2.16) and analogously (6.2.15) easily follow. (For (6.1.15), one could also use Proposition 5.3.1.) Combining (6.2.14), (6.2.15), (6.2.16), (6.2.12), and (6.2.13), we get the statement of induction.

Thus assertion (iii) of the theorem is proved while (iii) follows from the (obvious) supervariant of the Hilbert–Serre theorem. ∎

REFERENCES

1. M. AUSLANDER AND D. A. BUCHSBAUM, Codimension and multiplicity, *Ann. of Math.* **68** (1958), 625–657.
2. Ф. А. БЕРЕЗИН, Введение в алгебру и анализ с антикоммутирующими переменными, Изд. Моск. Ун-та 1983.
3. И. Н. БЕРНШТЕЙН, И. М. ГЕЛЬФАНД, и С. И. ГЕЛЬФАНД, Алгебраические расслоения на P^n и задачи линейной алгебры, Функц. анализ и его прил. т. **12** в. 3 (1978), 66–67.
4. N. BOURBAKI, "Algèbre Commutative." Éléments de Math. Vols. 27 and 28, Hermann, Paris, 1961.
5. A. GROTHENDIECK AND J. DIEUDONNÉ, Éléments de Géométrie Algébrique, EGA III, Étude cohomologique des faisceaux cohérents, *Inst. Hautes Études Sci. Publ. Math.* **11** (1961); **17** (1963).
6. A. GROTHENDIECK, AND J. DIEUDONNÉ, Éléments de Géométrie Algébrique, EGA IV, *Inst. Hautes Études Sci. Publ. Math.* **20** (1964); **32** (1967).
7. M. HERMANN, R. SCHMIDT, AND W. VOGEL, "Theorie der normalen Flachheit," Teubner, Leipzig, 1977.
8. Д. А. ЛЕЙТЕС, Введение в теорию супермногообразий, Успехи матем. наук т. **35**, в. I (1980), 3–57.
9. Д. А. ЛЕЙТЕС, "Теория супермногообразий," Петрозаводск, 1983.
10. Д. А. ЛЕЙТЕС, Спектры градуированно-коммутативных колец, Успехи матем. наук т. **29**, в. 3 (1974), 209–210.
11. Ю. И. МАНИН, Новые размерности в геометрии, Успехи матем. наук т. **39**, в. 6 (1984), 47–74.
12. Ю. И. МАНИН, "Калибровочные поля и комплексная геометрия," Наука, Москва, 1984.
13. H. MATSUMARA, "Commutative Algebra," Benjamin, New York, 1970.
14. О. В. ОГИЕВЕЦКИЙ и И. Б. ПЕНКОВ, Двойственность Серра для проективных супермногообразий, Функц. анализ и его прил. т. **18**, в. I (1984), 78–79.
15. T. SCHMITT, "Super Differential Geometry," Report 05/84 des IMath, Berlin, 1984.
16. W. V. VASCONSELOS, Ideals generated by *R*-sequences, *J. Algebra* **6** (1967), 309–316.