MATHEMATICS

ALGEBRAIC DEPENDENCE OF ARITHMETIC FUNCTIONS

BY

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1. *Introduction*

In recent years several authors have studied rings of arithmetic functions $f(n)$ in which the multiplicative operation is introduced by means of the convolution product $(f_1 \star f_2)(n)$ and not by means of the ordinary product $f_1(n)f_2(n)$. Here the function $h(n) = (f_1 \star f_2)(n)$ is defined by

$$h(n) = \sum_{n_1 n_2 = n} f_1(n_1) f_2(n_2),$$

where the summation is extended over all ordered pairs of positive integers (n_1, n_2) so that $n_1 n_2 = n$.

In the sequel we take the multiplication of arithmetic functions always in the convolution sense. It is then well-known that there exists a large number of simple algebraic relations between functions known from elementary number-theory. For instance, the Möbius inversion formula can be written

$$\mu \star I_0 = e,$$

where μ stands for $\mu(n)$, I_0 for the function $I_0(n) = 1$ for all positive integers n, and e for the "unit-function".

$$e(n) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n \geqslant 2. \end{cases}$$

There exist many more formulae of the same kind, but on the other hand no such relations are known for other sets of functions. One might suspect that such a system forms an algebraic independent set. In fact, CARLITZ [1] has given several examples of sets of arithmetic functions which are algebraically independent in the above sense over the field of complex numbers. His main result is that for any integer $r \geqslant 0$ the functions $|\mu(n)|$, $I_0, I_1, \ldots, I_r$ are algebraically independent. Here I_ϱ denotes the arithmetic function $I_\varrho(n) = n^\varrho$ $(\varrho = 0, 1, \ldots, r)$.

In this paper we study the algebraic relations between arithmetic functions from a more general standpoint, but we also arrive in this way at certain theorems which enable us to prove the algebraic independence of many sets of functions.

In this theory it proved to be useful to extend the notion of arithmetic functions in a natural way. Ordinary arithmetic functions $f(n)$ are defined on the set of positive integers n only. Here we extend the domain of the functions to an arbitrary commutative semi-group in which a unique factorization condition holds. In this way the theory actually becomes a branch of commutative algebra.

The ring of ordinary arithmetic functions $f(n)$ with functional values in the field of complex numbers is isomorphic to the ring of formal Dirichlet series

$$\sum_{n=1}^{\infty} f(n)\, n^{-s},$$

and it is therefore possible to translate our results into the theory of Dirichlet series. In this way our study is linked to OSTROWSKI's theory [3] of Dirichlet series satisfying algebraic differential equations.

In a second paper we shall study more closely multiplicative functions.

2. *Definitions*

Let S be a commutative semi-group with an identity element i which moreover satisfies the cancellation law. Let i have no proper divisors in S and let every element of S, except the identity, be a finite product of irreducible factors. Moreover, let such a factorization be unique to within the order of the factors. We shall call such a set a "unique factorization semi-group" (U.F.S.). For example, S may be the set of positive integers, the set of all ideals in a finite algebraic number-field and so on. Elements of S in general shall be denoted by x, y; irreducible elements (or prime-elements) by p, π. x^0 always means the identity i.

Further, let R be an arbitrary commutative ring with unity e. We consider here single valued functions $f(x)$ with domain the U.F.S. S and with their values entirely in the ring R. We shall call these functions "arithmetic functions". If in particular S is the set of positive integers, then we get the "ordinary arithmetic functions" $f(n)$ with their range in an arbitrary commutative ring R with unity [1]).

For two arithmetic functions f_1 and f_2 we define the sum $g = f_1 + f_2$ in the usual way $g(x) = f_1(x) + f_2(x)$ for every $x \in S$; we write this sum also: $(f_1 + f_2)(x)$. We further introduce the convolution product $h = f_1 \star f_2$ by means of the finite sum

$$h(x) = \sum_{x_1 x_2 = x} f_1(x_1)\, f_2(x_2),$$

where (x_1, x_2) runs through all ordered pairs of elements in S such that $x_1 x_2 = x$. We usually write $(f_1 \star f_2)(x)$ instead of $h(x)$. The symbol $\star$ is used

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1) CASHWELL and EVERETT [2] have shown that the ring of ordinary arithmetic functions can be made into a U.F.S. Hence we can even consider general arithmetic functions with as their domain the set of ordinary arithmetic functions (more precisely the set of classes of associate functions).

in order to avoid confusion with the multiplication in the ring R. However, we always shall write f^2, f^3 instead of $f \star f$, $f \star (f \star f)$ a.s.o..

It is easy to see that the set of arithmetic functions thus becomes a commutative ring R^*. The verification of the associative law leads to the following useful expression for $h = f_1 \star f_2 \star f_3$, namely

$$h(x) = \sum_{x_1 x_2 x_3 = x} f_1(x_1)\, f_2(x_2)\, f_3(x_3),$$

where the summation is extended over all ordered triples (x_1, x_2, x_3) of elements of S such that $x_1 x_2 x_3 = x$. There are similar formulae for products of four and more factors. The zero-element in R^* clearly is the null-function 0, defined by $0(x) = 0$ for all $x \in S$ [2]).

Instead of "elements of R^*" we usually shall speak of "arithmetic functions". The elements a of R correspond one–one with the particular arithmetic functions a' defined by

$$a'(x) = \begin{cases} a & \text{if } x = i \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $(a' \star f)(x) = af(x)$. The functions a' form a sub-ring R' of R^*, which is isomorphic with R. Therefore, instead of a' we usually shall write simply a. Since R has the unity e, R^* has unity $e'(x)$. By a simple reasoning we can show: if R is an integral domain, then equally R^* is an integral domain. The proof has been omitted since we do not need this result here. In the sequel for any arithmetic function f we always put $f^0 = e'$.

3. *The fundamental formulae*

Let $f_1, f_2, \ldots, f_r$ be given arithmetic functions. If

$$q(u_1, u_2, \ldots, u_r) = \sum_{(i)} a_{(i)}\, u_1^{i_1}\, u_2^{i_2} \ldots u_r^{i_r}$$

denotes an arbitrary polynomial in the variables $u_1, u_2, \ldots, u_r$ with coefficients $a_{(i)}$ from the ring R', then

$$q(f_1, f_2, \ldots, f_r) = \sum_{(i)} a_{(i)} \star f_1^{i_1} \star f_2^{i_2} \star \ldots \star f_r^{i_r}$$

represents also an arithmetic function. If $f_1, f_2, \ldots, f_r$ are fixed then we shall denote this function by $q(x)$ or simply by q.

In this section we shall derive certain formulae for expressions like $q(px)$, $q(p^2x)$, $q(pp'x)$, where x is an arbitrary element of S while p and p' represent irreducible elements relatively prime to x. Our main tool shall be Taylor's formula for a polynomial $q(u_1, u_2, \ldots, u_r)$

$$(1)\quad \begin{cases} q(u_1+v_1, \ldots, u_r+v_r) = q(u_1, \ldots, u_r) + \sum\limits_{\varrho=1}^{r} v_\varrho q_\varrho(u_1, \ldots, u_r) \\ \quad + \sum\limits_{1 \leqslant \varrho \leqslant \sigma \leqslant r} v_\varrho v_\sigma q_{\varrho\sigma}(u_1, \ldots, u_r) + \sum\limits_{1 \leqslant \varrho \leqslant \sigma \leqslant \tau \leqslant r} v_\varrho v_\sigma v_\tau q_{\varrho\sigma\tau}(u_1, \ldots, u_r) + \ldots \end{cases}$$

[2]) We also shall write in the sequel $0(x) \equiv 0$ on S.

where

$$(1^*)\quad \begin{cases} q_\varrho(u_1, \ldots, u_r) = \sum\limits_{(i)} \binom{i_\varrho}{1} a_{(i)}\, u_1^{i_1} \ldots u_\varrho^{i_\varrho - 1} \ldots u_r^{i_r} \\ q_{\varrho\varrho}(u_1, \ldots, u_r) = \sum\limits_{(i)} \binom{i_\varrho}{2} a_{(i)}\, u_1^{i_1} \ldots u_\varrho^{i_\varrho - 2} \ldots u_r^{i_r} \\ q_{\varrho\sigma}(u_1, \ldots, u_r) = \sum\limits_{(i)} \binom{i_\varrho}{1}\binom{i_\sigma}{1} a_{(i)}\, u_1^{i_1} \ldots u_\varrho^{i_\varrho - 1} \ldots u_\sigma^{i_\sigma - 1} \ldots u_r^{i_r} \quad (\varrho < \sigma), \end{cases}$$

a.s.o.

First we prove three simple lemma's.

Let $p_1, p_2, \ldots, p_t$ be a given set of prime-elements in S. They generate a semi-group K with identity i of elements

$$x = p_1^{n_1}\, p_2^{n_2} \ldots p_t^{n_t},$$

where $n_1, n_2, \ldots, n_t$ run through the integers $\geqslant 0$.

Let S/K denote the complementary set in S. Clearly S/K is also closed under multiplication.

Lemma 1. The arithmetic functions φ with the property

$$(2)\qquad \varphi(x) = 0 \text{ for all } x \in S/K$$

form a sub-ring of R^*.

Proof. We only have to show: if two functions φ_1, φ_2 have the property (2) then the same applies to $\varphi = \varphi_1 \star \varphi_2$. Now

$$(3)\qquad \varphi(x) = \sum_{x_1 x_2 = x} \varphi_1(x_1)\, \varphi_2(x_2).$$

If $x = x_1 x_2 \in S/K$, then at least one of the factors x_1, x_2 is in S/K and therefore every term in the right-hand member of (3) vanishes [3]).

Clearly every element a' of R' has the property (2); hence:

Corollary. If $\varphi_1, \varphi_2, \ldots, \varphi_r$ satisfy (2) and if $q(u_1, u_2, \ldots, u_r)$ is an arbitrary polynomial over R', then the arithmetic function

$$\bar{q} = q(\varphi_1, \varphi_2, \ldots, \varphi_r)$$

vanishes for every element x from S/K.

Lemma 2. The arithmetic functions ψ with

$$(4)\qquad \psi(x) = 0 \text{ for all } x \in K$$

form an ideal in R^*. More generally: let $\psi_1, \psi_2, \ldots, \psi_k$ denote k arithmetic

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[3]) The ring of arithmetic functions $\varphi(x)$ with the property (2) is clearly isomorphic to the ring of formal power series in the variables $u_1, u_2, \ldots, u_t$

$$\sum_{(n)} a_{n_1, \ldots, n_t}\, u_1^{n_1} u_2^{n_2} \ldots u_t^{n_t}$$

over the ring R.

functions with the property (4), let f be an arbitrary element from R^*. Then the function

$$g = f \star \psi_1 \star \psi_2 \star \dots \star \psi_k$$

vanishes for any element of S which has less than k prime-factors different from $p_1, p_2, \dots, p_t$.

Proof. We have

$$g(x) = \sum_{x_0 x_1 \dots x_k = x} f(x_0)\, \psi_1(x_1) \dots \psi_k(x_k). \tag{5}$$

If $x = x_0 x_1 \dots x_k$ has at most $k-1$ prime-factors outside the sequence $p_1, p_2, \dots, p_t$, then at least one of the factors $x_1, x_2, \dots, x_k$ has none and therefore belongs to K. Hence in every term of (5) at least one of the factors $\psi_1(x_1), \dots, \psi_k(x_k)$ vanishes. This proves our assertion.

Our method used in the sequel depends on the principle that we split each of the given functions f_ϱ into two terms φ_ϱ and ψ_ϱ, the first with the property (2), the other with (4). Put for $\varrho = 1, 2, \dots, r$

$$\varphi_\varrho(x) = \begin{cases} f_\varrho(x) & \text{if } x \in K \\ 0 & \text{if } x \in S/K \end{cases}, \quad \psi_\varrho(x) = \begin{cases} 0 & \text{if } x \in K \\ f_\varrho(x) & \text{if } x \in S/K \end{cases},$$

so that $f_\varrho = \varphi_\varrho + \psi_\varrho$.

Lemma 3. *Let $q(u_1, \dots, u_r)$ be an arbitrary polynomial over R'. In the sequel we shall write $\bar{q}$ instead of $q(\varphi_1, \varphi_2, \dots, \varphi_r)$, q instead of $q(f_1, f_2, \dots, f_r)$. Then*

$$\bar{q}(x) = q(x) \ \textit{for any } x \in K.$$

Proof. On account of $f_\varrho = \varphi_\varrho + \psi_\varrho$ we have by Taylor's formula in the form (1)

$$\begin{cases} q(f_1, \dots, f_r) = q(\varphi_1, \dots, \dots, \varphi_r) + \sum_{\varrho=1}^{r} \psi_\varrho \star q_\varrho(\varphi_1, \dots, \varphi_r) \\ \qquad + \sum_{1 \leqslant \varrho \leqslant \sigma \leqslant r} \psi_\varrho \star \psi_\sigma \star q_{\varrho\sigma}(\varphi_1, \dots, \varphi_r) + \dots . \end{cases} \tag{6}$$

By their definition each of the ψ_ϱ vanishes identically on the set K; however, all functions ψ with this property form an ideal (lemma 2). Hence in the right-hand member of (6) every term, except the first one $q(\varphi_1, \dots, \varphi_r)$, vanishes for $x \in K$. This proves our lemma.

Definition 1. Let f denote a certain arithmetic function, x_0 a given element from S, then we introduce the "translated function" $f^{(x_0)}$ by means of

$$f^{(x_0)}(x) \overset{\text{def}}{=} f(x_0 x) \ \textit{for all } x \in S.$$

Theorem 1. *Let $f_1, f_2, \dots, f_r$ be given arithmetic functions, $q(u_1, u_2, \dots, u_r)$ an arbitrary polynomial over R' with first partial derivatives $q_\varrho(u_1, u_2, \dots u_r)$*

$(\varrho = 1, 2, \ldots, r)$. Then (in the notations of lemma 3 and definition 1)

$$q(py) = \sum_{\varrho=1}^{r} (f_\varrho^{(p)} \star q_\varrho)(y),$$

for any $y \in S$ and for any prime-element p of S, relatively prime to y.

Proof. We apply the foregoing lemma's by choosing for $p_1, p_2, \ldots, p_t$ the different prime-factors of y. Hence $y \in K$ and $p, py \in S/K$. Applying Taylor's formula just as before (see formula (6)) we get

$$q = \bar{q} + \sum_{\varrho=1}^{r} \psi_\varrho \star \bar{q}_\varrho + \sum_{1 \leqslant \varrho \leqslant \sigma \leqslant r} \psi_\varrho \star \psi_\sigma \star \bar{q}_{\varrho\sigma} + \ldots. \tag{7}$$

In order to evaluate $q(py)$ we calculate every term in the right-hand member of (7) after substituting $x = py$. Note that py contains exactly *one* prime-factor not in the sequence $p_1, p_2, \ldots, p_t$.

By the corollary of lemma 1

$$\bar{q}(py) = 0.$$

On the other hand lemma 2 finishes the second- and higher-order terms: In fact any term $\psi_\varrho \star \psi_\sigma \star \bar{q}_{\varrho\sigma}$ contains two factors ψ which vanish on the set K, but $x = py$ contains only one prime-factor different from $p_1, p_2, \ldots, p_t$. Therefore by lemma 2 (with $k = 2$) every second-order term $\psi_\varrho \star \psi_\sigma \star \bar{q}_{\varrho\sigma}$ must vanish for $x = py$. A similar reasoning shows that also the higher-order terms in (7) must vanish for $x = py$. It follows

$$q(py) = \sum_{\varrho=1}^{r} (\psi_\varrho \star \bar{q}_\varrho)(py).$$

Now every divisor of y belongs to K; therefore

$$\begin{aligned}(\psi_\varrho \star \bar{q}_\varrho)(py) &= \sum_{x_1 x_2 = y} \psi_\varrho(px_1)\, \bar{q}_\varrho(x_2) \\ &= \sum_{x_1 x_2 = y} f_\varrho(px_1)\, q_\varrho(x_2) \quad \text{(using lemma 3)} \\ &= (f_\varrho^{(p)} \star q_\varrho)(y) \quad \text{(by the definition of } f_\varrho^{(p)}).\end{aligned}$$

From this the assertion of our theorem follows immediately.

Using the same kind of reasoning we find:

Theorem 1*. Let the conditions of theorem 1 be satisfied and let $q_{\varrho\sigma}(u_1, \ldots, u_r)$ denote the second-order "partial derivatives" of the polynomial $q(u_1, \ldots, u_r)$ (defined by (1*)). Then

$$q(p^2 y) = \sum_{\varrho=1}^{r} (f_\varrho^{(p^2)} \star q_\varrho)(y) + \sum_{1 \leqslant \varrho \leqslant \sigma \leqslant r} (f_\varrho^{(p)} \star f_\sigma^{(p)} \star q_{\varrho\sigma})(y)$$

and

$$q(pp'y) = \sum_{\varrho=1}^{r} (f_\varrho^{(pp')} \star q_\varrho)(y) + \sum_{1 \leqslant \varrho \leqslant \sigma \leqslant r} \{(f_\varrho^{(p)} \star f_\sigma^{(p')} + f_\varrho^{(p')} \star f_\sigma^{(p)}) \star q_{\varrho\sigma}\}\,(y),$$

for any $y \in S$ and for different irreducible elements p and p', both relatively prime to y.

Moreover it is clear how one can proceed to find similar formulae for $q(p^3y)$ a.s.o. However, the results tend to become more and more complicated.

4. *Algebraic dependence*

A set $f_1, f_2, \ldots, f_r$ of arithmetic functions is called *algebraically dependent over R'* if there exists a non-trivial polynomial $q(u_1, u_2, \ldots, u_r)$ over R' such that the arithmetic function

$$q = q(f_1, f_2, \ldots, f_r)$$

vanishes identically on S.

In the sequel let the ring R have the property, that for any non-zero element a also every multiple ma $(m=1, 2, \ldots)$ does not vanish. Such a ring (e.g. an integral domain of characteristic zero) shall be called *torsion-free.*

Theorem 2. Let R be torsion-free. Let $f_1, f_2, \ldots, f_r$ be algebraically dependent over R'. Then there exist elements $a_1, a_2, \ldots, a_r$ in R, not all zero, such that the linear relation

$$\sum_{\varrho=1}^{r} a_\varrho f_\varrho(p) = 0$$

is satisfied by all prime-elements p in S except perhaps by finitely many.

Proof. Let $q(u_1, u_2, \ldots, u_r)$ denote a non-trivial polynomial over R', such that $q = q(f_1, f_2, \ldots, f_r) \equiv 0$ on S and that moreover $q(u_1, \ldots, u_r)$ is of minimal total degree. Since R is torsion-free at least one of the partial derivatives $q_\varrho(u_1, u_2, \ldots, u_r) \not\equiv 0$, hence $q_\varrho = q_\varrho(f_1, f_2, \ldots, f_r) \not\equiv 0$ on S. Consider all elements x of S with the property

$$q_\varrho(x) \neq 0 \text{ for some } \varrho \text{ in } 1 \leqslant \varrho \leqslant r.$$

Select from these particular elements one with a minimal number of prime-factors [4]). Let us denote such an element by y. We put

$$a_\varrho \overset{\text{def}}{=} q_\varrho(y) \qquad (\varrho = 1, 2, \ldots, r);$$

then it is clear that not all a_ϱ vanish in R. Moreover by the minimal condition for y

$$\text{(8)} \qquad \left\{ \begin{array}{l} q_\varrho(x) = 0 \text{ for any } x \neq y \text{ which divides } y \\ \qquad\qquad \text{and for } \varrho = 1, 2, \ldots, r. \end{array} \right.$$

[4]) The multiplicity of the factors is taken into account.

Finely, let p be a prime-element in S, not a divisor of y but otherwise arbitrarily chosen. Then by the formula of theorem 1

$$0 = q(py) = \sum_{\varrho=1}^{r} (f_\varrho^{(p)} \star q_\varrho)(y),$$

where

$$\begin{aligned}(f_\varrho^{(p)} \star q_\varrho)(y) &= \sum_{x_1 x_2 = y} f_\varrho(px_1)\, q_\varrho(x_2) \\ &= f_\varrho(p)\, q_\varrho(y) \quad \text{(by (8))} \\ &= a_\varrho f_\varrho(p).\end{aligned}$$

It follows

$$\sum_{\varrho=1}^{r} a_\varrho f_\varrho(p) = 0$$

for any prime p not a divisor of y. This proves our theorem.

Corollary. Let $f_1(n), f_2(n), \ldots, f_r(n)$ denote ordinary arithmetic functions with values in a commutative torsion-free ring R with unity. Let for any sequence $\{a_1, a_2, \ldots, a_r\}$ of elements from R, not all zero, there exist an infinity of natural primes p, such that

$$\sum_{\varrho=1}^{r} a_\varrho f_\varrho(p) \neq 0. \tag{9}$$

Then $f_1, f_2, \ldots, f_r$ are algebraically independent over R'.

As an example take for R say the field of complex numbers and for $f_\varrho(n)$ the functions $I_{\varrho-1}(n) = n^{\varrho-1}$ ($\varrho = 1, 2, \ldots, r$) mentioned in the introduction. Clearly, if $a_1, a_2, \ldots, a_r$ are arbitrarily chosen complex numbers, not all zero, then (9) is satisfied for all sufficiently large primes p and it follows that $I_0, I_1, \ldots, I_{r-1}$ are algebraically independent.

Translated into the theory of ordinary Dirichlet series this last result becomes: the series

$$\sum_{n=1}^{\infty} n^{-s}, \quad \sum_{n=1}^{\infty} n \cdot n^{-s}, \quad \ldots, \quad \sum_{n=1}^{\infty} n^{r-1} \cdot n^{-s}$$

are algebraically independent over the complex numbers. It is not difficult to show that this is equivalent to the assertion that the Riemann zêta-function $\zeta(s)$ does not satisfy any algebraic difference equation of the type $F(s, \zeta(s), \zeta(s-1), \ldots, \zeta(s-r+1)) = 0$, where $F(u_0, u_1, u_2, \ldots, u_r)$ denotes a non-trivial polynomial with complex coefficients. Ostrowski [3] has shown that more generally the zêta-function does not satisfy an algebraic differential-difference equation. This more general result also follows from our corollary quite easily. Compare also the application of theorem 3 at the end of this paper.

The same kind of reasoning which we used in the proof of theorem 2 leads with the help of theorem 1* to:

Theorem 2*. Let R be torsion-free. Let $f_1, f_2, \ldots, f_r$ be algebraically dependent over R'. Then there exist elements $a_\varrho, a_{\varrho\sigma} (1 \leqslant \varrho \leqslant \sigma \leqslant r)$ of R, not all zero, such that the three linear relations

$$\sum_{\varrho=1}^{r} a_\varrho f_\varrho(p) = 0$$

$$\sum_{\varrho=1}^{r} a_\varrho f_\varrho(p^2) + \sum_{1 \leqslant \varrho \leqslant \sigma \leqslant r} a_{\varrho\sigma} f_\varrho(p)\, f_\sigma(p) = 0$$

$$\sum_{\varrho=1}^{r} a_\varrho f_\varrho(pp') + \sum_{1 \leqslant \varrho \leqslant \sigma \leqslant r} a_{\varrho\sigma} \{f_\varrho(p)\, f_\sigma(p') + f_\varrho(p')\, f_\sigma(p)\} = 0$$

are simultaneously satisfied for all prime elements $p, p' \neq p$ in S, except for those of a certain finite set.

5. *Conjugated functions*

The next theorem has a more special character, but the result lies much deeper. We therefore need the following definition:

Definition 2. We shall say that the arithmetic functions $f_1, f_2, \ldots, f_r$ are *conjugated* if for arbitrarily chosen $a_1, a_2, \ldots, a_r$ from R, but not all zero, there exists in S a semi-group F with a finite number of generators, such that the condition $\sum_{\varrho=1}^{r} a_\varrho f_\varrho(x_0) = 0$ for any $x_0 \in S/F$ implies

$$f_1(x_0) = f_2(x_0) = \ldots = f_r(x_0) = 0.$$

This is a very strong condition imposed on $f_1, \ldots, f_r$. If R contains at least one regular element it implies that if one of the functions vanishes, say for $x_0 \in S$, then every function of the set vanishes for $x = x_0$, except if x_0 is chosen in a certain semi-group F_0 generated by a finite number of prime elements.

If R is an integral domain, then one may obtain conjugated functions as follows. First take arithmetic functions $g_1, g_2, \ldots, g_r$ which are linearly independent in the following strong sense: For arbitrarily chosen $a_1, a_2, \ldots, a_r$ from R, not all zero, there are in S only a finite number of roots of the equation

$$\sum_{\varrho=1}^{r} a_\varrho g_\varrho(x) = 0.$$

If now f represents an arbitrary arithmetic function, then the f_ϱ defined by

$$f_\varrho(x) = g_\varrho(x) \cdot f(x) \ \textit{for all} \ x \in S \qquad (\varrho = 1, 2, \ldots, r)$$

are conjugated.

For example, if R is the field of complex numbers, let

$$\gamma_0 < \gamma_1 < \ldots < \gamma_r; \quad \delta_0 < \delta_1 < \ldots < \delta_t$$

be two increasing sequences of real numbers. Then for any ordinary arithmetic function $f(n)$ the $(r+1)(t+1)$ functions

$$(10) \qquad f_{\varrho\tau}(n) = n^{\gamma_\varrho} (\log n)^{\delta_\tau} f(n) \qquad (\varrho=0, 1, \ldots, r;\ \tau=0, 1, \ldots, t)$$

are conjugated. [5])

Theorem 3. Let R be torsion-free. Let the conjugated arithmetic functions $f_1, f_2, \ldots, f_r$ be algebraically dependent over R'. Then there exists in S a semi-group K with only a finite number of generators, so that every $f_\varrho(x)$ vanishes identically outside K $(\varrho=1, 2, \ldots, r)$.

Proof. Let $q(u_1, \ldots, u_r)$ be a non-trivial polynomial over R' of minimal degree such that $q=q(f_1, \ldots, f_r) \equiv 0$ on S. Let $q_\sigma(u_1, \ldots, u_r)$ denote a partial derivative not vanishing identically in $u_1, \ldots, u_r$. Then there exists some $\bar{x} \in S$, so that $q_\sigma(\bar{x}) \neq 0$ (q_σ denoting as usual $q_\sigma(f_1, \ldots, f_r)$). Let the different prime-divisors $p_1, p_2, \ldots, p_h$ of $\bar{x}$ generate the semi-group H. The elements of H are ordered lexicographically, so that

$$p_1 < p_2 < \ldots < p_h$$

form an increasing sequence. Here the lexicographic order will always be denoted by the symbol $<$. Now a new order is imposed on the elements of H (we use the symbol $\prec$ to indicate this new order):

If x_1, x_2 are two different elements of H, then put $x_1 \prec x_2$

a) if the total number of prime-factors (taking the multiplicity into account) of x_1 is less than that of x_2,

b) if the numbers of prime-factors tally, but $x_1 < x_2$ in the lexicographic order.

Then we are able to write all elements of H in an ascending sequence

$$i;\ p_1, p_2, \ldots, p_h;\ p_1^2, p_1 p_2, \ldots, p_{h-1}p_h, p_h^2;\ p_1^3, \ldots \text{ [6])}.$$

This sequence will be denoted in the sequel by

$$(11) \qquad y_0 = i,\ y_1 = p_1,\ y_2, \ldots, y_t, \ldots.$$

From $x_1 \prec x_2$ it follows clearly $x_1 x \prec x_2 x$ for any $x \in H$. Applying this rule twice one obtains for x_1, x_2, x_3, x_4 from H:

$$(12) \qquad \textit{If } x_1 \prec x_2,\ x_3 \preceq x_4, \textit{ then } x_1 x_3 \prec x_2 x_4.$$

Consider all $x \in H$ with $q_\varrho(x) \neq 0$ for some ϱ in the sequence $\varrho=1, 2, \ldots, r$. This set is not vacuous since $q_\sigma(\bar{x}) \neq 0$. Let η denote the element of this set which is smallest with respect to the order just defined. Because of

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[5]) The functional values of $f_{\varrho\tau}$ for $n=1$ are not defined by (10) if $\delta_\tau \leqslant 0$. In these cases one can take arbitrary values for $f_{\varrho\tau}(1)$.

[6]) If $\bar{x} = i$, then this sequence reduces to only one element i. The proof in this special case is a simplified version of the proof for the more general case given below.

the well-ordering η exists. Put $a_\varrho \overset{\text{def}}{=} q_\varrho(\eta)$ ($\varrho=1, 2, \dots, r$), so that the a_ϱ do not vanish simultaneously; moreover

(13) $$q_\varrho(x) = 0 \ (\varrho = 1, 2, \dots, r) \ \textit{for any } x \prec \eta \textit{ in } H.$$

In the sequel we shall need the function $f(t)$ defined for $t=0, 1, 2, \dots$ by the relation

(14) $$\eta y_t = y_{f(t)},$$

where the y_t are the elements of the sequence (11). Clearly $f(t)$ is an increasing function of t; for $\eta=i$ we have $f(t)=t$.

The functions $f_1, \dots, f_r$ are conjugated. Let F denote the semi-group of definition 2 belonging to $a_1, \dots, a_r$, so that $\sum_{\varrho=1}^{r} a_\varrho f_\varrho(x_0)=0$ for any $x_0 \in S/F$ implies $f_1(x_0)=f_2(x_0)= \dots =f_r(x_0)=0$. F has a finite number of generators, so has H. Let K be generated by the prime-factors in F and n H, so that K encloses both F and H. We shall show that our assertion holds for this semi-group K. Therefore we put

(15) $$\varphi_\varrho(x) = \begin{cases} f_\varrho(x) & \textit{if } x \in K \\ 0 & \textit{if } x \in S/K \end{cases}, \quad \psi_\varrho(x) = \begin{cases} 0 & \textit{if } x \in K \\ f_\varrho(x) & \textit{if } x \in S/K \end{cases},$$

so that we have to prove that every $\psi_\varrho(x)$ ($\varrho=1, 2, \dots, r$) vanishes identically on S.

Let

$$x_0 = p_1^{n_1} \dots p_h^{n_h} \pi_1^{m_1} \dots \pi_l^{m_l}$$

be the factorization of an arbitrary element of S; $\pi_1, \dots, \pi_l$ *denoting prime-elements outside* H. We shall call

(16) $$x_0' = p_1^{n_1} \dots p_h^{n_h}, \ x_0'' = \pi_1^{m_1} \dots \pi_l^{m_l}$$

the *components* of x_0, respectively in H and in S/H. Using the sequence (11) one can also write

$$x_0' = y_{t_0}, \quad x_0 = y_{t_0} \pi_1^{m_1} \dots \pi_l^{m_l},$$

where t_0 is some integer $\geqslant 0$.

Put $M_0 \overset{\text{def}}{=} m_1 + m_2 + \dots + m_h$, the number of prime-factors of x_0 outside H.

Clearly, $\psi_\varrho(x_0)=0$ for all x_0 with $M_0=0$ and arbitrary $t_0=0, 1, 2, \dots$, since then $x_0 \in H$. We shall prove $\psi_\varrho(x_0)=0$ for all other x_0 by means of induction.

Suppose $M_0 \geqslant 1$ and suppose that we already have proved $\psi_\varrho(x)=0$ ($\varrho=1, 2, \dots, r$) for all

$$x = y_t \pi_1^{\mu_1} \dots \pi_l^{\mu_l}$$

with

(17) $$\mu_1 + \dots + \mu_l < M_0, \ t \leqslant f(t_0)$$

and for all x with

$$\mu_1 + \ldots + \mu_l = M_0, \ t < t_0 \ \text{[7]}). \tag{18}$$

Then we shall show that these hypotheses imply $\psi_\varrho(x_0)=0$. (In a $t-M$ diagram this means that if our assertion is true for all lattice points (t, M) in the rectangle $0 \leqslant t \leqslant f(t_0)$, $0 \leqslant M \leqslant M_0 - 1$, and also for lattice points with $0 \leqslant t < t_0$, $M = M_0$, then the assertion is also true for the point (t_0, M_0).) In this way we can prove our assertion for an arbitrarily chosen element x_0 in a finite number of steps. For the usual inductive procedure see the remark at the end of this proof.

By Taylor's formula in the form (7)

$$0 = q(x_0\eta) = \bar{q}(x_0\eta) + \sum_{\varrho=1}^{r} (\psi_\varrho \star \bar{q}_\varrho)(x_0\eta) + \sum_{1 \leqslant \varrho \leqslant \sigma \leqslant r} (\psi_\varrho \star \psi_\sigma \star \bar{q}_{\varrho\sigma})(x_0\eta) + \ldots,$$

where $\bar{q} = q(\varphi_1, \ldots, \varphi_r)$, a.s.o.

We treat the terms of different orders separately. If $x_0 \in K$, then $\psi_\varrho(x_0) = 0$ $(\varrho = 1, 2, \ldots, r)$ by the definition of the ψ_ϱ; hence in the sequel let $x_0 \in S/K$. Then $\bar{q}(x_0\eta) = 0$ by the corollary of lemma 1.

Now consider an arbitrary second-order term in Taylor's formula:

$$\sum_{x_1x_2x_3 = x_0\eta} \psi_\varrho(x_1)\, \psi_\sigma(x_2)\, \bar{q}_{\varrho\sigma}(x_3).$$

Taking the components in H and in S/H of $x_1x_2x_3$ separately (compare (16)) we get

$$x_1'x_2'x_3' = x_0'\eta, \quad x_1''x_2''x_3'' = x_0''.$$

If x_1'' should have the same number of prime-factors as x_0'', then $x_2'' = i$, $x_2 \in H$ and therefore $\psi_\sigma(x_2) = 0$. Therefore we may suppose that x_1'' (and similarly x_2'') have less prime-factors than x_0''. Then x_1 has the form

$$x_1 = y_u \pi_1^{\mu_1} \ldots \pi_l^{\mu_l}, \quad y_u \in H,$$

with $\mu_1 + \ldots + \mu_l < M_0$; moreover $y_u = x_1'$ is a divisor of $x_0'\eta = \eta y_{t_0} = y_{f(t_0)}$ (by (14)); it follows that $u \leqslant f(t_0)$. Hence by the induction hypothesis (17) we have $\psi_\varrho(x_1) = 0$ for $\varrho = 1, 2, \ldots, r$. Thus we see that every second-order term in Taylor's formula vanishes.

By the same method we prove that also the terms of the third and higher order cancel. Therefore Taylor's formula reduces to

$$\sum_{\varrho=1}^{r} (\psi_\varrho \star \bar{q}_\varrho)\,(x_0\eta) = 0. \tag{19}$$

Any term can be written

$$\sum_{x_1x_2 = x_0\eta} \psi_\varrho(x_1)\, \bar{q}_\varrho(x_2).$$

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[7]) In the case $t_0 = 0$, this hypothesis has to be cancelled.

We may suppose that $x_2 \in H$; for otherwise x_2 should contain at least one prime-factor from the sequence $\pi_1, \pi_2, \ldots, \pi_l$ at the expense of x_1, so that x_1 should have less prime-factors of this type than x_0. By a similar reasoning as above we then find $\psi_\varrho(x_1)=0$, so that the term vanishes. Hence $x_2 \in H$, $x_2 \in K$ and thus $\bar{q}_\varrho(x_2)=q_\varrho(x_2)$ by lemma 3.

From $x_1x_2=x_0\eta$ we have, taking the components in H and in S/H,

$$x_1'x_2 = x_0'\eta, \; x_1'' = x_0'', \tag{20}$$

so that x_1 must have the form

$$x_1 = y_t \pi_1^{m_1} \ldots \pi_l^{m_l}, \quad y_t \in H.$$

We distinguish three cases:

1) $x_2 \prec \eta$, then $q_\varrho(x_2)=0$ by (13),

2) $\eta \prec x_2$, then $x_1' \prec x_0'$, for otherwise $x_0' \preceq x_1'$ and therefore $\eta x_0' \prec x_1'x_2$ by (12), but this contradicts (20). Hence $x_1' \prec x_0'$ or $y_t \prec y_{t_0}$, so that $t<t_0$. From the induction hypothesis (18) it follows $\psi_\varrho(x_1)=0$.

Hence in the cases 1) and 2) the corresponding terms vanish and we only have to consider

3) $x_2=\eta$, then $x_1=x_0$, which gives the contribution $\psi_\varrho(x_0)\, q_\varrho(\eta)=a_\varrho\psi_\varrho(x_0)$ to (19). It follows

$$\sum_{\varrho=1}^{r} a_\varrho \psi_\varrho(x_0) = 0.$$

Now $x_0 \in S/K$ and therefore

$$\sum_{\varrho=1}^{r} a_\varrho f_\varrho(x_0) = 0.$$

Since $x_0 \in S/F$ and $f_1, f_2, \ldots, f_r$ are conjugated, we find $f_\varrho(x_0)=0$, hence $\psi_\varrho(x_0)=0$ for $\varrho=1, 2, \ldots, r$. The proof is complete.

Remark. We could have used the usual induction procedure by introducing the "height" $h(x)=h(t, M)$ for any element

$$x = y_t \pi_1^{m_1} \pi_2^{m_2} \ldots \pi_l^{m_l} \text{ with } m_1+m_2+\ldots+m_l = M$$

as follows:

Let $f(t)$ denote the function defined in (14), put $f^{(0)}(t)=t$, $f^{(1)}(t)=f(t)$, $f^{(2)}(t)$ the iterated function $f\{f(t)\}, \ldots, f^{(M)}(t)=f^{(M-1)}\{f(t)\}, \ldots$. Then define $h(x)=h(t, M)$ for $t=0, 1, \ldots,$ $M=0, 1, \ldots$ by

$$h(t, M) = \begin{cases} 0 & \text{if } M=0,\; t=0, 1, \ldots \\ M + \sum\limits_{k=0}^{M-1} f^{(k)}(t) & \text{if } M \geqslant 1,\; t=0, 1, \ldots \end{cases}.$$

For $h(x_0)=0$ the assertion $\psi_\varrho(x_0)=0 (\varrho=1, 2, \ldots, r)$ is trivial. For $h(x_0) \geqslant 1$

we prove the assertion by induction with respect to $h(x_0)$, using the formulae

$$h(t-1, M) < h(t, M) \quad \text{for } t=1, 2, \ldots; \ M=1, 2, \ldots$$
$$h(f(t), M-1) < h(t, M) \quad \text{for } t=0, 1, \ldots; \ M=1, 2, \ldots .$$

In this way, however, the proof becomes somewhat more artificial.

As an application of theorem 3 take for R the field of complex numbers and instead of $f_1, \ldots, f_r$ the conjugated ordinary arithmetic functions

$$f_{\varrho\tau}(n) = n^{\gamma_\varrho} (\log n)^{\delta_\tau} f(n) \quad (\varrho=0, 1, \ldots, r, \ \tau=0, 1, \ldots, t)$$

from (10). If these $(r+1)$ $(t+1)$ functions are algebraically dependent over the complex field, then from our theorem it follows that $f(n)$ must vanish for all n if we exclude a set of positive integers, generated by finitely many primes.

If we translate this result into the theory of Dirichlet series, then we can easily deduce from this the following well-known result of Ostrowski [3, p. 242]:

If the Dirichlet series

$$\sum_{n=1}^{\infty} f(n)\, n^{-s} \tag{21}$$

converges in some half-plane and represents there a regular analytic function $y(s)$ satisfying a non-trivial algebraic differential-difference equation

$$F(s, y^{(\tau)}(s-\gamma_\varrho)) = 0 \text{ [8]},$$

then $f(n)$ must vanish for all n except for the n generated by a finite number of certain primes.

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[8]) Here the left-hand member represents a polynomial with real or complex coefficients in s and $y^{(\tau)}(s - \gamma_\varrho)$ $(\varrho = 0, 1, \ldots, r, \ \tau = 0, 1, \ldots, t)$. One has to use a lemma of Ostrowski that if $y(s)$ satisfies an algebraic equation of this kind it also satisfies such an equation in which s does not occur explicitly.

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