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Discrete Mathematics 306 (2006) 452–455
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Communication

A note on minimal matching covered graphs

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Received 12 June 2005; accepted 7 December 2005

Communicated by L. Lovasz

Dedicated to Anush

Abstract

A graph is called matching covered if for its every edge there is a maximum matching containing it. It is shown that minimal matching covered graphs without isolated vertices contain a perfect matching.
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Keywords: Maximum matching; Matching covered graph; Minimal matching covered graph; Perfect matching

Let Z^+ denote the set of nonnegative integers. We consider finite undirected graphs $G=(V(G), E(G))$ without multiple edges or loops [2], where $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. For a vertex $u \in V(G)$ define the set $N_G(u)$ as follows:

$$N_G(u) \equiv \{e \in E(G)/e \text{ is incident with } u\}.$$

In a connected graph G the length of the shortest $u-v$ path [2] is denoted by $\rho(u, v)$, where u, v are vertices of the graph G. For a vertex $w \in V(G)$ and $U \subseteq V(G)$ set

$$\rho(w, U) \equiv \min_{u \in U} \rho(w, u).$$

The set of all maximum matchings [2,4] of a graph G is denoted by $M(G)$, and for $e \in E(G)$ define the set $M(e)$ as follows:

$$M(e) \equiv \{F \in M(G)/e \in F\}.$$

A vertex $u \in V(G)$ is said to be covered (missed) by a matching $F \in M(G)$ if $N_G(u) \cap F \neq \emptyset$ ($N_G(u) \cap F = \emptyset$). A matching $F \in M(G)$ is called perfect if it covers every vertex $v \in V(G)$.

For a graph G define the subgraph $C(G)$ as follows:

$$C(G) \equiv G \backslash \{e \in E(G)/\text{for every } F \in M(G) e \notin F\}.$$

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doi:10.1016/j.disc.2005.12.006

The graph G is said to be matching covered if $G = C(G)$, and is said to be minimal matching covered if it satisfies the following condition, too:

$$G - e \neq C(G - e) \quad \text{for every } e \in E(G).$$

In this paper it is proved that every minimal matching covered graph without isolated vertices contains a perfect matching.

The idea of the subgraph $C(G)$ of a graph G is not new in graph theory. It stems from the idea of the core of a graph introduced in [1,3]. Roughly speaking, the core of a graph G is the subgraph $C(G)$ if the cardinality of a maximum matching of G equals that of minimum point cover for G, and is the empty graph otherwise.

The same is true for the idea of matching covered graph. In the "bible" of matching theory [4] one can find a detailed analysis of the structure of 1-extendable graphs (connected matching covered graphs containing a perfect matching) and all necessary references of its development. In terms of [4] the main result of the present paper can be reformulated in the following way: connected minimal matching covered graphs are 1-extendable.

Non-defined terms and conceptions can be found in [2,4,5].

Lemma. *If G is a connected, matching covered graph, which does not contain a perfect matching, then*

(1) *for every edge* $e = (u, v) \in E(G)$ *there is a* $F \in M(G)$ *such that F misses either u or v*;
(2) *if for edges* $e, e' \in E(G)$ $M(e) = M(e')$ *then* $e = e'$.

Proof. (1) For every $F \in M(G)$ consider the sets $A(F)$ and $B(F)$ defined in the following way:

$$A(F) \equiv \{w \in V(G)/F \text{ covers } w\},$$
$$B(F) \equiv \{w \in V(G)/F \text{ misses } w\}.$$

Clearly, for each $F \in M(G)$ the following holds:

$$V(G) = A(F) \cup B(F), \quad A(F) \cap B(F) = \emptyset, \quad A(F) \neq \emptyset, \quad B(F) \neq \emptyset.$$

For an edge $e = (u, v) \in E(G)$ define a mapping $\mu_e : M(G) \to Z^+$ as follows:

$$\mu_e(F) \equiv \min\{\rho(u, B(F)), \ \rho(v, B(F))\} \quad \text{where } F \in M(G).$$

Choose $F_0 \in M(G)$ satisfying the condition:

$$\mu_e(F_0) \equiv \min_{F \in M(G)} \mu_e(F).$$

Let us show that F_0 misses either u or v. For the sake of contradiction assume F_0 to cover both u and v. Let $w_0, (w_0, w_1), w_1, \ldots, w_{k-1}, (w_{k-1}, w_k), w_k$ be a simple path of the graph G satisfying the conditions:

$$w_0 \in B(F), \quad \{w_1, \ldots, w_k\} \subseteq A(F_0), \quad \{w_{k-1}, w_k\} = \{u, v\}, \quad k = 1 + \mu_e(F_0),$$
$$k \geqslant 2.$$

Set $e' \equiv (w_1, w_2)$. Let us prove that $e' \notin F_0$. If $e' \in F_0$ then consider the matching $F_1 \in M(G)$ defined as follows:

$$F_1 \equiv (F_0 \backslash \{e'\}) \cup \{(w_0, w_1)\}.$$

It is clear that $\mu_e(F_1) < \mu_e(F_0)$, which contradicts the choice of F_0, therefore $e' \notin F_0$. Take a maximum matching $F_0' \in M(e')$ satisfying the condition

$$|F_0 \cap F_0'| = \max_{F' \in M(e')} |F_0 \cap F'|.$$

Let us show that $w_0 \in A(F_0')$. If $w_0 \notin A(F_0')$ then assume

$$F_0'' \equiv (F_0' \backslash \{e'\}) \cup \{(w_0, w_1)\}.$$

Note that $F_0'' \in M(G)$ and $\mu_e(F_0'') < \mu_e(F_0)$, which is impossible, therefore $w_0 \in A(F_0')$. It is not hard to see that the choice of F_0' implies that there is a simple path $v_0, (v_0, v_1), v_1, \ldots, v_{2l-1}, (v_{2l-1}, v_{2l}), v_{2l}$ $(l \geqslant 1)$ of the graph G satisfying the conditions

$$\{(v_0, v_1), \ldots, (v_{2l-2}, v_{2l-1})\} \subseteq F_0', \{(v_1, v_2), \ldots, (v_{2l-1}, v_{2l})\} \subseteq F_0,$$
$$e' \notin \{(v_0, v_1), \ldots, (v_{2l-2}, v_{2l-1})\}, \quad v_0 = w_0, v_{2l} \in \{w_1, w_2\}.$$

Set

$$\tilde{F}_0 \equiv (F_0 \backslash \{(v_1, v_2), \ldots, (v_{2l-1}, v_{2l})\}) \cup \{(v_0, v_1), \ldots, (v_{2l-2}, v_{2l-1})\}.$$

Clearly, $\tilde{F}_0 \in M(G)$ and $\mu_e(\tilde{F}_0) < \mu_e(F_0)$, which contradicts the choice of F_0, therefore F_0 misses either u or v.

(2) Suppose $e, e' \in E(G)$, $e = (u, v)$ and $e \neq e'$. Let us show that $M(e) \neq M(e')$. Take a matching $F_1 \in M(G)$ missing either u or v. For the sake of definiteness let us assume that F_1 covers u and misses v. If $e' \in F_1$ then $M(e) \neq M(e')$, therefore without loss of generality we may assume that $e' \notin F_1$. As F_1 covers u, then there is a $w \in V(G)$ such that $(u, w) \in F_1$. Set

$$F_2 \equiv (F_1 \backslash \{(u, w)\}) \cup \{(u, v)\}.$$

Clearly, $F_2 \in M(G)$, $e \in F_2$ and $e' \notin F_2$, therefore $M(e) \neq M(e')$. The proof of the lemma is complete. □

From the results [1,3] and lemma we have the following interesting result:

Corollary. *Let G be a connected, bipartite, matching covered graph and let (U, W) be the bipartition of the set $V(G)$. If there is a $w_0 \in W$ and a $F_0 \in M(G)$ such that F_0 misses w_0, then for every $w \in W$ there is a $F \in M(G)$ such that F misses w.*

Theorem. *Suppose that the graph G which does not contain an isolated vertex satisfies the following two properties:*

(1) *G is a matching covered graph,*
(2) *$G - e$ is not a matching covered graph for every edge $e \in E(G)$.*

Then the graph G has a perfect matching.

Proof. Without loss of generality we may assume G to be connected. Let us show that there are two distinct edges e and e' such that $M(e) = M(e')$.

Take an arbitrary edge $e_0 \in E(G)$. Suppose that the edges $e_0, \ldots, e_k$ $(k \geqslant 0)$ are already defined, and consider the graph $G - e_k$. As it is not a matching covered graph, then there exists an edge $\tilde{e} \neq e_k$ such that $M(e_k) \supseteq M(\tilde{e})$. Set $e_{k+1} \equiv \tilde{e}$.

Consider the infinite sequence $\{e_k\}_{k=0}^{\infty}$ of edges of the graph G. Clearly, there are numbers $i, j \in Z^+$, $i < j$ such that $e_i = e_j$. The construction of the sequence $\{e_k\}_{k=0}^{\infty}$ implies that

$$M(e_i) \supseteq M(e_{j-1}) \supseteq M(e_j) = M(e_i) \quad \text{and} \quad e_{j-1} \neq e_j,$$

therefore

$$M(e_{j-1}) = M(e_j).$$

The lemma implies that G has a perfect matching. The proof of the theorem is complete. □

I would like to thank my Supervisor Rafayel R. Kamalian for his constant attention and support during all those years that we worked together.

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