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On the Non-sequential Nature of Domain Models of Real-number Computation

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Abstract

Escardó, Hofmann and Streicher showed that real-number computations in the interval-domain environment are inherently parallel, in the sense that they imply the presence of weak parallel-or. Part of the argument involves showing that the addition operation is not Vuillemin sequential. We generalize this to all continuous domain environments for the real line. The key property of the real line that leads to this phenomenon is its connectedness. We show that any continuous domain environment for any connected topological space exhibits a similar parallel effect.

Keywords: domain theory, real number computation, sequentiality, connectedness

1 Introduction

Escardó, Hofmann and Streicher [4] investigated the possibility of sequential computation on the real line via its well known interval-domain environment, considered by e.g. Edalat [2] and Escardó [3]. The main result of [4] is that sequential computation on the reals via the interval domain is extremely restrictive, to the extent that not even a basic operation such as addition is sequential. The argument in [4] has two main steps: (1) no extension of the addition operation on the real numbers to the interval domain is Vuillemin sequential (see Definition 2.1 below), and (2) under natural assumptions for a sequential programming language, the weak parallel-or operator is definable from any function that fails to be Vuillemin sequential. Escardó, Hofmann and Streicher asked whether this would hold for any domain environment of the real line, among a class of domains of interest, or this would be a limitation of the interval domain. A main result of the present paper is that the first step generalizes to any continuous domain environment for the real

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line (Theorem 3.4 and its corollary). Thus, we can say that this is not a limitation of the interval domain, but rather an intrinsic property of the real line. A second contribution of the present paper is to identify this property. We show that any continuous domain environment for any *connected* topological space exhibits the parallel effect. More precisely, we prove that only very restricted binary operations on such a space can be extended to Vuillemin sequential operations on a continuous domain environment (Theorem 3.3).

2 Preliminaries

We assume some familiarity with topology and domain theory [1], including the notions of dcpo (directed complete poset), continuous domain, Scott topology and (Scott) continuous function. The upper set of an element x of a poset is denoted by $\uparrow x$. Given a function $f\colon D \times E \to F$ and elements $d_0 \in D$, $e_0 \in E$, we denote by $f(-, e_0)$ the function $D \to F$ that maps any d to $f(d, e_0)$, and by $f(d_0, -)$ the function $E \to F$ that maps any e to $f(d_0, e)$. By the real line we mean the set of real numbers endowed with its usual topology, generated by the open intervals.

The following definition is taken from [4]. The intuition is that if a binary function is sequential, then it must look at one of its arguments first, and if there is no progress at this argument, then the value of the function itself cannot make any progress either.

Definition 2.1 A continuous function $f\colon D \times E \to F$ of domains is called *Vuillemin sequential* if, for any d in D and e in E,

(i) $f(d, e') = f(d, e)$ for all $e' \sqsupseteq e$, or

(ii) $f(d', e) = f(d, e)$ for all $d' \sqsupseteq d$.

Equivalently, the function $f(-, e)$ is constant at $\uparrow d$ or the function $f(d, -)$ is constant at $\uparrow e$.

Definition 2.2 A *domain environment* for a topological space X is a continuous domain E containing X as a subspace in the relative Scott topology.

Notice that we don't require a domain environment to be the subspace of maximal elements, thus being less restrictive than in [5].

Definition 2.3 If E is a domain environment for a topological space X, we say that an operation $X \times X \to X$ is *Vuillemin sequential* if it has at least one Vuillemin sequential extension $E \times E \to E$.

3 Vuillemin sequentiality on connected spaces

We show that for any continuous domain environment of a connected topological space, only very restricted binary operations on the space can be Vuillemin sequential. We split the argument in two steps. Lemma 3.1 doesn't assume connectedness, but item (i) of its conclusion is closely related to connectedness, as it becomes apparent in Lemma 3.2.

Lemma 3.1 *If $f\colon D \times E \to F$ is a Vuillemin sequential function on continuous domains and R is a subspace of E, then for every d in D,*

(i) *$f(d,-)$ is constant at each open set of some open cover of R, or*

(ii) *$f(-,r)$ is constant at $\uparrow d$ for some r in R.*

Proof. Assume that (ii) doesn't hold. To prove (i), let r be in R and Y the set of elements way below r. Because $f(-,r)$ is not constant at $\uparrow d$, there are d_0 and d_1 in $\uparrow d$ such that

$$f(d_0,r) \neq f(d_1,r). \tag{1}$$

Since f is Scott continuous and r is the supremum of the directed set Y, we have both

$$f(d_0,r) = \bigsqcup_{y\in Y} f(d_0,y) \text{ and } f(d_1,r) = \bigsqcup_{y\in Y} f(d_1,y). \tag{2}$$

From assertions (1) and (2), one concludes that, for some y_0 in Y,

$$f(d_0,y_0) \neq f(d_1,y_0)$$

as, otherwise, one would have $\bigsqcup_{y\in Y} f(d_0,y) = \bigsqcup_{y\in Y} f(d_1,y)$ by continuity of f, and hence $f(d_0,r) = f(d_1,r)$, contradicting (1). In other words, the function $f(-,y_0)$ is not constant at $\uparrow d$. So, because f is Vuillemin sequential, the function $f(d,-)$ is constant at $\uparrow y_0$. And because $\uparrow y_0$ is a neighbourhood of r, there is an open subset U_r of $\uparrow y_0$ such that U_r has r as a member. Then the collection $\{U_r\}_{r\in R}$ is an open cover of R such that $f(d,-)$ is constant at each U_r, as required. □

Recall that a topological space R is called *connected* if whenever two open sets V and W are such that $V \cap W = \emptyset$ and $V \cup W = R$, one of the sets V and W is empty. Equivalently, R is connected if R and $\emptyset$ are the only clopens (both closed and open). Typical examples of connected spaces are the real intervals, including $\mathbb{R}$ itself. We will use the characterization given by Lemma 3.2(iii) below, which is a reformulation of the well known characterization (ii).

Lemma 3.2 *The following are equivalent for any non-empty space R:*

(i) *R is connected.*

(ii) *Any continuous function from R to a discrete space is constant.*

(iii) *Any set-theoretical function defined on R which is constant at each open set of some open cover of R is constant.*

Although this result is standard, we include a proof for the sake of completeness.

Proof. (i) $\Longrightarrow$ (ii). Suppose X is a discrete space and $f\colon R \to X$ is continuous. Let $r \in R$. The set $f^{-1}(\{f(r)\})$ is not empty (because it contains r) and is a clopen (because $\{f(r)\}$ is a clopen and f is continuous), hence it is the whole space R (because R is connected), which means that f is constant.

(ii) $\implies$ (iii). Let $f : R \to X$ be a set-theoretical function defined on R and constant at each open set of some open cover $\{U_i\}$ of R. Endow X with the discrete topology. Then f is continuous, because the singletons form a base of X, and, for any element $\{x\}$ of that base, the set $f^{-1}(\{x\})$ is open as it is a union of open sets, namely those U_i whose direct image is $\{x\}$. So, by (ii), f is constant.

(iii) $\implies$ (i). If U is a clopen in R, then U and its complement form an open cover of R and, by (iii), the characteristic function of U has to be constant, so U has to be $\emptyset$ or R.

□

The following is an immediate consequence of Lemmas 3.1 and 3.2.

Lemma 3.3 *If $f: D \times E \to F$ is a Vuillemin sequential function on continuous domains and R is a connected subspace of E, then for every $d \in D$,*

(i) *the function $f(d, -)$ is constant at R , or*

(ii) *the function $f(-, r)$ is constant at $\uparrow d$ for some $r \in R$.*

We say that a function is *locally constant at a point* if it is constant at some neighbourhood of the point. Given property p and a point y, we say that $p(y')$ *holds for some y' as close to y as one wishes* if for every neighbourhood V of y there is y' such that $p(y')$ holds. Recall that a space is locally connected if every point has a neighbourhood base of connected sets.

Theorem 3.4 *If X is a locally connected space and a function $g: X \times X \to X$ is Vuillemin sequential with respect to some continuous domain environment, then for any x and y in X,*

(i) *the function $g(x, -)$ is locally constant at y, or*

(ii) *the function $g(-, y')$ is locally constant at x, for some y' as close to y as one wishes.*

Proof. Let E be a domain environment for X and $f: E \times E \to E$ a Vuillemin sequential extension of g. Let x and y be in X and assume (ii) doesn't hold. This means that for some neighbourhood V of y and all $y' \in V$, the function $g(-, y')$ fails to be locally constant at x, and so, for every d way below x, its extension $f(-, y')$ is not constant at $\uparrow d$. By local connectedness, we may assume that V is connected, and, by Lemma 3.3 with $R = V$, we conclude that for every d way below x, the function $f(d, -)$ is constant at V. Hence, by continuity of f, the function $f(x, -)$, and hence $g(x, -)$ is constant at V, i.e. (i) holds. □

As discussed in the introduction, it was shown in [4] that when E is the interval domain and X is the Euclidean real line embedded into E by the singleton map $x \mapsto \{x\}$, there is no Vuillemin sequential map $E \times E \to E$ extending the addition operation. Our main result is that this holds for all domain environments and a vast class of operations.

Corollary 3.5 *Addition, multiplication, maximum and minimum fail to be Vuillemin sequential, for any continuous domain environment for the real line.*

Note that condition (ii) in Theorem 3.4 doesn't imply that $g(-, y)$ is locally constant at x. For example, let X be the real line and define g as $g(x, y) = \max(|x|, y)$. For all y except 0, the function $g(-, y)$ is locally constant at 0. So, for $x = 0$ and $y = 0$, the function g satisfies condition (ii). But the function $g(-, 0)$ is not locally constant at 0. However, this example fails to be Vuillemin sequential (consider $x = y = 1$ in the theorem).

4 Possible generalizations

By just skipping the definition of Y in the proof of Lemma 3.1, one obtains a proof of the following, slightly more general lemma.

Lemma 4.1 *Let $f: D \times E \to F$ be a Vuillemin sequential function on dcpos and R a subset of E. Suppose that, for every r in R, there exists a directed subset Y of E such that r is the supremum of Y and, for each y in Y, the upper set $\uparrow y$ is a neighbourhood of r. Then, for every d in D,*

(i) *$f(d, -)$ is constant at each open set of some open cover of R, or*

(ii) *$f(-, r)$ is constant at $\uparrow d$ for some r in R.*

One example where this generalization is of interest is a particular quotient of the domain 3^∞, where 3^∞ is the set of finite and infinite sequences of digits -1, 0 or 1, with the prefix ordering. The maximal elements in 3^∞, that is the infinite sequences, are identified with real numbers in the interval $[-1, 1]$ via the map $s \mapsto \sum_{i \geq 0} \frac{s_i}{2^{i+1}}$. For any s in 3^∞, the set of real numbers above s is an interval. We define an equivalence relation $\sim$ on 3^∞ by stipulating that $s \sim s'$ if and only if s and s' define the same interval. The quotient $3^\infty/\sim$ is a non-continuous dcpo. In fact, it is easy to verify that, although every maximal element is the directed join of elements way below it, a non-maximal element doesn't have any element way below it other than bottom. Nevertheless, Lemma 4.1 allows us to immediately extend Theorem 3.3 and Theorem 3.4, with the same proofs, to the case where $\mathbb{R}$ is embedded into $3^\infty/\sim$. It is not too hard to check that this dcpo is quasicontinuous [5, p. 226], and it is natural to ask whether our results generalize to such dcpos. Another class of domains one may want to consider is the category of topological domains [6]. This is left for future work.

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