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Entanglement Entropy in 2D non-abelian pure gauge theory

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ABSTRACT

We compute the Entanglement Entropy (EE) of a bipartition in 2D pure non-abelian $U(N)$ gauge theory. We obtain a general expression for EE on an arbitrary Riemann surface. We find that due to area-preserving diffeomorphism symmetry EE does not depend on the size of the subsystem, but only on the number of disjoint intervals defining the bipartition.

In the strong coupling limit on a torus we find that the scaling of the EE at small temperature is given by $S(T) - S(0) = O\left(\frac{m_{\text{gap}}}{T} e^{-\frac{m_{\text{gap}}}{T}}\right)$, which is similar to the scaling for the matter fields recently derived in literature. In the large N limit we compute all of the Renyi entropies and identify the Douglas–Kazakov phase transition.

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1. Introduction

Entanglement has become a useful tool in the study of the properties of the states of matter [1]. A particularly useful measure of entanglement in quantum systems is the Entanglement Entropy (EE). In a pure state, EE measures the entanglement present between a subsystem and its complement. When the subsystem is a subset of the configuration space (real space partition) EE can be computed using the replica trick [2,3].

It was suggested on the basis of the AdS/CFT arguments [4] that in confining gauge theories with N_c colors the EE has a non-analyticity in the large N_c limit: dependence of EE on N_c suddenly jumps from N_c^2 to N_c^0 as the size of the subsystem crosses a critical value. This suggestion was confirmed numerically [5] for a \mathbb{Z}_2 gauge theory, where the authors found an intrinsic ambiguity in the definition of the EE in the presence of gauge fields. It was shown that it is impossible to separate the Hilbert space into a tensor product of two Hilbert spaces without violating the Gauss law on the boundary of the bi-partition. In [5] it was suggested to use a “minimal way” of violating the Gauss law just in the border of the partition to compute the EE in a lattice gauge theory. This construction was formalized [6] for the spin-network states and arbitrary gauge groups.

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In the physics of black holes the EE of the matter fields in a gravitational background coincides with the entropy of the black hole for scalars and fermions [7]. In the same reference it was found that in the presence of gauge fields there is an additional contact term that makes the EE negative for spatial dimension $D < 8$ and thus cannot be interpreted as EE of any quantum field theory. The term was written as sum over trajectories starting and ending on the horizon. This observation carries a close resemblance with the construction of [5,8]. In two dimensions, the apparent negative contribution of the gauge fields to the entropy is an artifact which disappears as the zero modes are properly considered [9].

In condensed matter physics the (topological) EE can be used to classify the topological phases [10–12] of gapped systems. There is an intriguing possibility that EE can help to understand the gapless topological phases [13].

The issue of ambiguity of the EE was recently addressed in [14] where it was found that different extensions of the physical Hilbert space lead to different values of the (topological) EE. Nevertheless, it was concluded that the ambiguity becomes irrelevant in the continuum limit and the usual replica method gives the correct answer.

In this letter we study the $U(N)$ Yang–Mills theory in $1+1$ dimensions (YM_2). This theory is exactly solvable and superrenormalizable. The simplicity of the model is explained by the large symmetry group of area-preserving diffeomorphisms that prevents the existence of any local degree of freedom. The model can be interpreted as a closed string theory [15] and in the large N limit the partition function can be written as a sum over the

Riemann surfaces. It was used as a toy model to test the relationship between large- N QCD and the (free) string theory. We take advantage of the simplicity of the model and derive expressions for the EE of YM_2 with $U(N)$ gauge group on a Riemann surface of genus g for arbitrary bi-partition. We derive the entanglement spectrum and the large N limit for the EE.

2. Definitions

2.1. Entanglement Entropy of a partition

The system under consideration is defined on one spatial and one (euclidean) time dimensions. We will take the subsystem A to be a union of l disjoint intervals in the spatial dimension. In the text we will refer to it as l cuts. We denote the complement of A by \bar{A} .

The Entanglement Entropy (EE) is defined as the von Neumann entropy of the reduced density matrix $\rho_A = \text{tr}_{\bar{A}} \rho$ by

$$S = -\text{tr} \rho_A \ln \rho_A. \quad (1)$$

More generally, we define Renyi entropy for any integer n .

$$S_n = \frac{1}{1-n} \ln \text{tr} \rho_A^n. \quad (2)$$

Then the EE is obtained from the Renyi entropy by

$$S = -\lim_{n \rightarrow 1+0} \frac{\partial}{\partial n} \text{tr} \rho_A^n. \quad (3)$$

The subtle point in (3) is the definition of the derivative with respect to n . In principle, if there are no divergences then every eigenvalue of ρ_A is smaller than 1 and $\text{tr} \rho_A^n$ is absolutely convergent making the analytic continuation to any real n easy. In reality, there are divergences and we have to deal with the analytic continuation to the real n on the case by case basis.

2.2. Replica method

A standard way to compute EE is the replica method [3,16]. In order to compute $\text{tr} \rho_A^n$ one replaces the base manifold of the quantum field theory Σ by its n -sheeted ramified covering Σ_n , with ramification points being the end points of the l cuts. We denote the partition functions of the QFT on Σ and on Σ_n as Z and Z_n correspondingly. It was shown [3] that

$$\text{tr} \rho_A^n = \frac{Z_n}{Z^n}. \quad (4)$$

Thus the computation of the Entanglement Entropy is reduced to the computation of the partition function on Σ_n . EE is given by

$$S = -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{Z_n}{Z^n} = \ln Z - \frac{1}{Z} \lim_{n \rightarrow 1} \frac{\partial}{\partial n} Z_n. \quad (5)$$

2.3. Pure Yang–Mills theory in 2 dimensions

Two-dimensional Yang Mills (YM_2) is an exactly solvable, super-normalizable model and has been studied extensively [17,18]. The model is fully specified by the choice of a gauge group G and a Riemann surface Σ . The action is given by

$$S[A] = \frac{1}{4e^2} \int_{\Sigma} d^2x \sqrt{g} g^{\mu\lambda} g^{\rho\nu} \text{tr} F_{\mu\nu} F_{\rho\lambda}, \quad (6)$$

where $F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + f^{abc} A_{\mu}^b A_{\nu}^c$ is the field strength tensor associated with the gauge group G and $g_{\mu\nu}$ is the metric on

the Riemann surface Σ . The action (6) is invariant with respect to area-preserving diffeomorphisms [18]. This large symmetry group is responsible for the simplicity of the theory.

The partition function is given by

$$Z = \int \mathcal{D}A e^{-S[A]}. \quad (7)$$

Notice that both the action and the partition function are invariant with respect to the transformation

$$e \rightarrow \sqrt{r}e \quad g_{\mu\nu} \rightarrow r g_{\mu\nu}. \quad (8)$$

This implies that the coupling constant e and the area A will always enter together, so from now on we absorb e into the definition of the area.

2.4. Partition function

Let us fix G to be $U(N)$ or $SU(N)$. The partition function can be computed explicitly [17,19]

$$Z(A, g) = \sum_R (d_R)^{\chi(\Sigma)} e^{-\frac{A}{2N} C_2(R)}, \quad (9)$$

where the sum runs over all irreducible representations of the gauge group, including the trivial one [17]. Here, $C_2(R)$ is the quadratic Casimir, d_R is the dimension of the representation R , N is the t'Hooft coupling constant and $\chi(\Sigma)$ is the Euler characteristic of the Riemann surface Σ .

In general, one encounters divergences when computing the partition function (7) in the presence of the external gravitational field. There are only two local counter terms needed to cancel them [20].

$$I_1 = u \int d^2x \sqrt{g} \quad (10)$$

$$I_2 = -\frac{v}{2\pi} \int d^2x \sqrt{g} R \quad (11)$$

So the partition function has to be modified accordingly

$$Z'(A, \Sigma, u, v) = e^{uA - v\chi(\Sigma)} Z(A, \Sigma). \quad (12)$$

Thus taking the regularization scheme into account gives a 2-parametric family of theories with u and v being parameters.

3. Entanglement Entropy in YM_2 theory

We are going to merge the results reviewed in the previous section. Since the partition function depends only on the total area and the Euler characteristic of the Riemann surface, thus application of the replica method is straightforward.

In order to compute Z_n we need to know the Euler characteristic $\chi(\Sigma_n)$ of the n -sheeted ramified covering Σ_n . This is given by Riemann–Hurwitz theorem. We have $2l$ ramification points of degree n .

$$\chi(\Sigma_n) = n\chi(\Sigma) - 2l(n-1). \quad (13)$$

Direct application of the replica (5) gives

$$\text{tr} \rho_A^n = e^{-v l (2-2n)} \frac{\sum_R d_R^{n\chi(\Sigma) - 2l(n-1)} e^{-AnC_2(R)/2N}}{(\sum_R d_R^{\chi(\Sigma)} e^{-AC_2(R)/2N})^n}. \quad (14)$$

Already at this point we see dependence on u canceled as it always happens in using the replica trick. Combining (3) and (14) we get the EE

$$S = 2lv + \ln \left(\sum_R d_R^{\chi(\Sigma)} e^{-\frac{A}{2N} C_2(R)} \right) - \frac{\sum_R d_R^{\chi(\Sigma)} e^{-\frac{A}{2N} C_2(R)} \ln(d_R^{\chi(\Sigma)-2l} e^{-\frac{A}{2N} C_2(R)})}{\sum_R d_R^{\chi(\Sigma)} e^{-\frac{A}{2N} C_2(R)}}. \quad (15)$$

This formula is the first major result of this letter. A special case of this expression for $l=1$ and $v=0$ was obtained in [21].

This can be written in compact form as follows. For any operator X , that is diagonal in the character basis we introduce the notation

$$\langle X \rangle = \frac{\sum_R d_R^{\chi(\Sigma)} e^{-\frac{A}{2N} C_2(R)} X_R}{\sum_R d_R^{\chi(\Sigma)} e^{-\frac{A}{2N} C_2(R)}}, \quad (16)$$

where $X_R = \langle R|X|R \rangle$ is the eigenvalue of X on the state labeled by the irrep R . Then we can rewrite (15) as

$$S = 2lv + \ln Z - \langle \ln(d_R^{\chi(\Sigma)-2l} e^{-\frac{A}{2N} C_2(R)}) \rangle. \quad (17)$$

3.1. Torus

In order to lighten up the notations and to give the area A a thermal interpretation we choose Σ to be a torus. This corresponds to a gauge theory with periodic boundary conditions in a thermal bath. In this case $\chi(\Sigma) = 0$. We take one of the radii to be $\frac{1}{T}$ and the other one to be 1.

Thus the Entanglement Entropy is given by

$$S = 2lv + \ln \left(\sum_R e^{-\frac{1}{2TN} C_2(R)} \right) - \frac{\sum_R e^{-\frac{1}{2TN} C_2(R)} \ln(d_R^{-2l} e^{-\frac{1}{2TN} C_2(R)})}{\sum_R e^{-\frac{1}{2TN} C_2(R)}}, \quad (18)$$

where we also replaced A by $\frac{1}{T}$.

3.2. Strong coupling

The sum over irreps is essentially a strong coupling (low temperature) expansion. In the strong coupling limit we keep contributions from the trivial and the fundamental representations. Quadratic Casimir is normalized such that $C_2(\square) = N$ and $d_{\square} = N$. We have

$$S(T) = 2lv + e^{-\frac{1}{2T}} \left(1 + 2l \ln N + \frac{1}{2T} \right) + \dots \quad (19)$$

At this point we make two observations.

First, at zero temperature the Entanglement Entropy is completely determined by the regulator

$$S(0) = 2lv. \quad (20)$$

Second, since the mass gap in the problem is set by the eigenvalues of the quadratic Casimir $m_{\text{gap}} = \frac{C_2(\square)}{2N}$ the EE scales as

$$S(T) - S(0) = O\left(\frac{m_{\text{gap}}}{T} e^{-\frac{m_{\text{gap}}}{T}}\right). \quad (21)$$

This scaling is similar to the scaling found for the matter fields [22–24], but with additional factor of $\frac{1}{T}$. This factor is natural since the entropy is essentially an average of the logarithm of the density matrix and it goes as the Boltzmann weight times $\frac{1}{T}$. Eq. (21) is another new result of this paper.

3.3. Weak coupling

Weak coupling limit is more tricky. In this limit the sum over representations is not well behaved. Of course, we expect the Entanglement Entropy to be divergent at high temperature, as it should approach the scaling of the thermal entropy. To observe this scaling, we take a simple gauge group. Let $G = SU(2)$. Representations of $SU(2)$ are labeled by an integer m . We have

$$d_R = m, \quad C_2(R) = \frac{m^2 - 1}{2}. \quad (22)$$

The partition function is given by

$$Z = \sum_{m=1}^{\infty} e^{-\frac{m^2-1}{8T}} = \frac{e^{\frac{1}{8T}}}{2} (\vartheta_3(0, e^{-\frac{1}{8T}}) - 1), \quad (23)$$

where ϑ_3 is a Jacobi theta function. In the weak coupling limit ($T \gg 1$) EE scales as

$$S(T) = \frac{3}{2} \ln T + \frac{1}{2} \ln 8\pi - \gamma + O\left(\frac{\ln T}{T^{1/2}}\right), \quad (24)$$

where γ is the Euler–Mascheroni constant.

3.4. Higher genus

The situation is a little different for genus $g > 1$. First of all, the quantity (5) does not immediately have the EE meaning, but we will retain the same terminology.

In this case Eq. (14) still holds, with one important difference: d_R enters the sum with a negative power, thus making the sum over R absolutely convergent. The low temperature (large area) behavior is found in the same way as before.

In the zero-area limit YM₂ theory becomes a TQFT. The Entanglement Entropy is given by

$$S_l^{\text{top}} = 2lv + \ln Z^{\text{top}} + \frac{2(l+g-1)}{Z^{\text{top}}} \sum_R d_R^{2-2g} \ln d_R \quad (25)$$

where we have denoted

$$Z^{\text{top}} = \sum_R d_R^{2-2g}. \quad (26)$$

Note, that sum in (25) is convergent, but this is not surprising as the area does not have a thermal interpretation anymore.

3.5. Entanglement spectrum on the torus

Using the previous results we can also study the eigenvalues of the logarithm of the partial density matrix, the so-called entanglement spectrum [25]. It is introduced as follows. Given a reduced density matrix ρ_A we can define an operator.

$$\rho_A = e^{-He} \quad (27)$$

If we denote the eigenvalues of ρ_A by Λ_R and their degeneracies by g_R then we have

$$\text{tr} \rho_A^n = \sum_R g_R \Lambda_R^n, \quad (28)$$

comparing with (14) we obtain ($\lambda = A/2N$)

$$\Lambda_R = \frac{\exp(-\lambda C_2(R) - 2l \ln d_R)}{\sum_R e^{-\lambda C_2(R)}}, \quad (29)$$

$$g_R = d_R^{2l} \quad (30)$$

we find then that the eigenvalues of H_e (the entanglement spectrum) are

$$\xi_R = \lambda C_2(R) + (2l - \chi(\Sigma)) \ln d_R, \quad (31)$$

each of them with degeneracy g_R .

4. Large N limit

In two dimensions the large N Yang–Mills theory on a sphere has the celebrated 3-rd order Douglas–Kazakov phase transition [26]. For the other base manifolds the large N limit is either not well behaved (for $\chi(\Sigma) < 0$) or trivial (for $\chi(\Sigma) = 0$). In this section we will study the EE in the large N limit on a sphere.

4.1. Free energy in large N

In order to take the large N limit we make group theory explicit. This procedure is discussed in great detail in [17,18,26,27] so we will be very brief.

The irreps of $U(N)$ are labeled by non-decreasing integers $n_1 \leq n_2 \leq \dots \leq n_N$. Quadratic Casimir and the dimensions of the irreps are parametrized as follows.

$$C_2(R) = \sum_{i=1}^N n_i(n_i - 2i + N + 1), \quad (32)$$

$$d_R = \prod_{i < j} \left(1 - \frac{n_i - n_j}{i - j}\right). \quad (33)$$

Then introducing $x = \frac{i}{N}$, $n(x) = \frac{n_i}{N}$, $h(x) = -n(x) + x - \frac{1}{2}$, and

$$\rho(x) = \frac{dx}{dh}, \quad (34)$$

we write the partition function on arbitrary Riemann surface of Euler characteristic χ as

$$Z(A, \chi) = e^{N^2 F(A)} = \int D\rho e^{-N^2 S_{eff}^\chi}, \quad (35)$$

where

$$S_{eff}^\chi = \frac{A}{2} \int_0^1 dh \rho(h) h^2 - \chi \int dh \int ds \rho(h) \rho(s) \ln |h - s| - \frac{A}{24} - \frac{3}{2} \chi. \quad (36)$$

We are keeping χ arbitrary because we want to apply the expressions to the partition function on Σ_n . Integral in (35) can be evaluated via the saddle point approximation. Since n_i 's are ordered $\rho(x) \leq 1$. This restriction is responsible for the DK phase transition.

4.2. Sphere

We take $\chi > 0$ and solve the saddle point equation [26,27].

$$\rho(h) = \frac{A}{\pi} \frac{1}{\chi} \sqrt{\frac{2\chi}{A} - h^2}. \quad (37)$$

Evaluating the action on this solution we find

$$Z(A, \chi) = \exp N^2 \left[\frac{\chi}{2} \ln \frac{\chi}{2A} + \chi + \frac{A}{24} \right]. \quad (38)$$

Notice that the last two terms in this expression are precisely of the form (12).

We first compute the Renyi entropy S_n for $l = 1$. Combining (5), (13) and (38) and setting $\chi(\Sigma) = 2$ we have

$$S_n = N^2 \left(\ln \frac{1}{A} + 1 - \frac{\ln n}{1-n} \right), \quad (39)$$

and the EE is given by

$$S = N^2 \ln \frac{1}{A}. \quad (40)$$

These expressions are another new result of this letter.

For general l we encounter an issue. The Euler characteristic doesn't stay positive for arbitrary n , that is, $\chi(\Sigma_n) > 0$ implies $n < \frac{l}{l-1}$. So if we proceed as before, our expression will be valid only for n slightly bigger than 1. This raises a question about the validity of the naive analytic continuation. Nonetheless, a sensible answer is obtained

$$S = l N^2 \ln \frac{1}{A}. \quad (41)$$

The end points of the cuts simply add additional degrees of freedom and the EE gets a contribution.

We point out that a similar issue arises on a torus in the large N limit with the difference that for any $n > 1$ the Euler characteristic changes sign and the analytic continuation is impossible. This leads to a divergence of the limit (3). Even though all Renyi entropies are well defined, the analytic continuation to EE is impossible.

The expressions (40) and (41) are valid only for weak coupling (small area) since at some value of area A_{cr} the condition $\rho(x) \leq 1$ is violated.

4.3. Douglas–Kazakov phase transition

The EE is sensitive to the DK phase transition. After some gymnastics with elliptic functions (in line with [26]) we find that near the critical point $x_c = \pi^2$ we have for $\Delta S = S_{strong}(x) - S_{weak}(x)$ (with $x = A/\chi$)

$$\frac{\Delta S(x)}{N^2} = -2l \left(\frac{x - x_c}{\pi^2} \right)^2 \left[\frac{2x + x_c}{3\pi^2} \right] + \dots \quad (42)$$

thus the Entanglement Entropy also has a critical point at $x = x_c$. In this point although the EE is continuous, but its 2nd derivative is discontinuous. This is a signal of the DK phase transition. This result coincides with the large N expansion discussed in [28] for the geometric entropy.

5. Conclusions

We have computed the Entanglement Entropy of 2D pure Yang–Mills theory on a Riemann surface of arbitrary genus with any number of disjoint intervals in the bipartition. We found an exponential scaling at low temperatures which is similar to the ones found for the matter fields [22,23]. We have investigated the behavior of the EE near the DK phase transition and found that it has a critical point at the phase transition. For the higher genus Riemann surfaces the small area (topological) limit is convergent and the EE well defined.

We did not find the non-analytic behavior predicted in [4], but it was expected as YM_2 does not know about any length scales except the size of the system. Another way to say it: all of the dependence of the EE on the length of the cut L has the form of the Heaviside function $S \sim N^2 \theta(L)$, so that $\frac{\partial S}{\partial L} \sim N^2 \delta(L)$, thus the critical value of L is pushed to 0 by the symmetries of the model.

It would be interesting to study the EE with additional matter fields and take advantage of the integrability of the Schwinger model. We leave this for the future work.

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