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Is every nonsingular matrix diagonally equivalent to a matrix with all distinct eigenvalues?

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ABSTRACT

It is shown that a 2×2 complex matrix A is diagonally equivalent to a matrix with two distinct eigenvalues iff A is not strictly triangular. It is established in this paper that every 3×3 nonsingular matrix is diagonally equivalent to a matrix with 3 distinct eigenvalues. More precisely, a 3×3 matrix A is not diagonally equivalent to any matrix with 3 distinct eigenvalues iff $\det A = 0$ and each principal minor of A of order 2 is zero. It is conjectured that for all $n \geq 2$, an $n \times n$ complex matrix is not diagonally equivalent to any matrix with n distinct eigenvalues iff $\det A = 0$ and every principal minor of A of order $n - 1$ is zero.

1. 2 × 2 matrices diagonally equivalent to matrices with two distinct eigenvalues

Two $n \times n$ matrices A and B over $\mathbb{C}$ are said to be *diagonally equivalent* if there are invertible diagonal matrices D_1 and D_2 such that $B = D_1 A D_2$.

Matrices all of whose eigenvalues are distinct have many desirable properties, such as diagonalizability (see [1]). Considerable research has been done on matrices with all distinct eigenvalues, see for example [2,4–6,8]. In this paper, we aim to identify matrices that are diagonally equivalent to matrices with no multiple eigenvalues.

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Note that if D_1 and D_2 are invertible diagonal matrices of order n and A is any matrix of order n, then D_1AD_2 is similar to D_2D_1A. Thus in order to investigate the eigenvalues of D_1AD_2, it suffices to consider matrices of the form D_1A for invertible diagonal matrices D_1.

We denote the resultant (see [3,7] or [9]) of two polynomials $f(x)$ and $g(x)$ by $\mathrm{Res}(f(x), g(x))$. It is well known that $f(x)$ and $g(x)$ have no common zero (in an extension field that contains all the zeros of $f(x)g(x)$) iff $\mathrm{Res}(f(x), g(x)) \neq 0$. The discriminant (see [7] or [9]) of a polynomial $f(x)$, denoted $\mathrm{discr}(f(x))$, is defined as the product of the squares of the pairwise differences of the roots of $f(x)$. It is also well known that a polynomial $f(x)$ has no multiple root iff $\mathrm{Res}(f(x), f'(x)) \neq 0$, iff $\mathrm{discr}(f(x)) \neq 0$. In fact, the discriminant of a monic polynomial $f(x)$ of degree n is given by

$$\mathrm{discr}(f(x)) = (-1)^{\frac{n(n-1)}{2}} \mathrm{Res}(f(x), f'(x)).$$

Horn and Lopatin [2] gave an alternative method for finding the discriminant of the characteristic polynomial of a matrix A by computing the determinant of the moment matrix, whose (i, j) entry is the trace of A^{i+j-2}.

Theorem 1.1. *A 2×2 matrix A is diagonally equivalent to a matrix with two distinct eigenvalues iff A is not strictly upper triangular or strictly lower triangular.*

Proof. We prove the equivalent statement that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not diagonally equivalent to any matrix with two distinct eigenvalues iff A is strictly upper triangular or strictly lower triangular.

Clearly, if A is strictly upper (or lower) triangular, then every matrix diagonally equivalent to A is also strictly upper (or lower) triangular and thus would have 0 as the eigenvalue of multiplicity 2. Thus A is not diagonally equivalent to any matrix with two distinct eigenvalues.

We now prove the converse. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not diagonally equivalent to any matrix with two distinct eigenvalues. Since a scalar multiple of a matrix preserves the property of the eigenvalues being distinct or otherwise, without loss of generality, we may restrict our attention to matrices of the form D_2A, where D_2 is a diagonal matrix of the form $D_2 = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$, $x \neq 0$. The characteristic polynomial (in t) of D_2A is

$$t^2 - (a + dx)t + xD,$$

where $D = ad - bc$ denotes the determinant of A. Since D_2A has a multiple eigenvalue for every nonzero x, the discriminant of the above characteristic polynomial is equal to 0, namely,

$$(a + dx)^2 - 4xD = 0, \quad \text{or} \quad d^2x^2 + (2ad - 4D)x + a^2 = 0$$

for all nonzero values of x. For the polynomial $d^2x^2 + (2ad - 4D)x + a^2$ to have infinitely many roots, all the coefficients must be zero. Thus, $d^2 = 0$, $a^2 = 0$, and $2ad - 4D = 0$. It follows that $a = 0$, $d = 0$ and $D = 0$. Consequently, $bc = ad - D = 0$. We now have $a = d = 0$ and ($c = 0$ or $b = 0$), namely, A is strictly upper triangular or A is strictly lower triangular. □

The above theorem can be rephrased in terms of the principal minors as follows.

Corollary 1.2. *A 2×2 matrix A is not diagonally equivalent to any matrix with two distinct eigenvalues iff* $\det A = 0$ *and each diagonal entry of A is zero, iff each principal minor of A is zero.*

The following result follows immediately from the above theorem.

Corollary 1.3. *Every* 2×2 *nonsingular matrix is diagonally equivalent a matrix with two distinct eigenvalues.*

2. 3 × 3 matrices diagonally equivalent to matrices with 3 distinct eigenvalues

The main result of this section is the following.

Theorem 2.1. *A* 3×3 *complex matrix* A *is not diagonally equivalent to any matrix with 3 distinct eigenvalues iff* $\det A = 0$ *and every principal minor of* A *of order 2 is zero.*

In order to prove Theorem 2.1, we need the following lemma, which can be proved by induction on the number of variables following the Proof of Theorem 2.19 in Section 2.12 of [3]. For convenience, we define the degree of the zero polynomial to be $-\infty$, which is naturally considered to be less than any integer.

Lemma 2.2. *Let* $f(x_1, x_2, \ldots, x_n)$ *be a polynomial over a field* $\mathbb{F}$ *with degree at most* m *in each variable* x_i. *Let* S_i $(1 \leqslant i \leqslant n)$ *be a subset of* $\mathbb{F}$ *with at least* $m+1$ *elements. Suppose that* $f(c_1, c_2, \ldots, c_n) = 0$ *for all* $(c_1, c_2, \ldots, c_n) \in S_1 \times S_2 \times \cdots \times S_n$. *Then* $f(x_1, x_2, \ldots, x_n)$ *is the zero polynomial, namely, all of the coefficients of* $f(x_1, x_2, \ldots, x_n)$ *are zeros.*

Proof of Theorem 2.1. Consider a 3×3 complex matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}.$$

For an invertible diagonal matrix $D_3 = \mathrm{diag}(x, y, z)$, the characteristic polynomial of D_3A is

$$p(t) = t^3 - (ax + ey + kz)t^2 + (xyM_1 + yzM_2 + xzM_3)t - xyzD,$$

where $D = \det A = aek + bfg + cdh - afh - bdk - ceg$, while $M_1 = (ae - bd)$, $M_2 = (ek - fh)$, and $M_3 = (ak - cg)$ are the principal minors of A of order 2. Hence,

$$p'(t) = 3t^2 - 2(ax + ey + kz)t + (xyM_1 + yzM_2 + xzM_3).$$

With the help of Maple, we find that

$\mathrm{Res}(p(t), p'(t)) = -a^2M_1^2x^4y^2 + (-2a^2M_3M_1 + 4a^3D)x^4yz - a^2M_3^2x^4z^2 + (4M_1^3 - 2aeM_1^2)x^3y^3 + (12M_1^2M_3 - 2akM_1^2 - 18aDM_1 + 12a^2eD - 2a^2M_1M_2 - 4aeM_1M_3)x^3y^2z + (12M_1M_3^2 - 4akM_3M_1 - 2a^2M_3M_2 - 18aDM_3 - 2aeM_3^2 + 12a^2kD)x^3yz^2 + (4M_3^3 - 2akM_3^2)x^3z^3 - e^2M_1^2x^2y^4 + (-18eDM_1 - 4aeM_1M_2 - 2e^2M_1M_3 - 2ekM_1^2 + 12M_1^2M_2 + 12ae^2D)x^2y^3z + (24M_1M_2M_3 - 4aeM_3M_2 - 4akM_1M_2 - k^2M_1^2 - 4ekM_1M_3 + 27D^2 - e^2M_3^2 - 18eDM_3 - 18kDM_1 - 18aDM_2 + 24aekD - a^2M_2^2)x^2y^2z^2 + (-18kDM_3 - 2k^2M_3M_1 - 4akM_3M_2 + 12M_2M_3^2 - 2ekM_3^2 + 12ak^2D)x^2yz^3 - k^2M_3^2x^2z^4 + (4e^3D - 2e^2M_1M_2)xy^4z + (12e^2kD - 2e^2M_2M_3 - 18eDM_2 + 12M_1M_2^2 - 2aeM_2^2 - 4eM_1kM_2)xy^3z^2 + (-18kDM_2 + 12k^2eD + 12M_2^2M_3 - 4ekM_3M_2 - 2akM_2^2 - 2k^2M_2M_1)xy^2z^3 + (-2k^2M_3M_2 + 4k^3D)xyz^4 - e^2M_2^2y^4z^2 + (4M_2^3 - 2ekM_2^2)y^3z^3 - k^2M_2^2y^2z^4.$

The sufficiency of the theorem is clear. Indeed, suppose that $\det A = 0$ and every principal minor of A of order 2 is zero. Then $D = M_1 = M_2 = M_3 = 0$. Hence, for every nonsingular diagonal matrix D_3, the characteristic polynomial of D_3A is $p(t) = t^3 - (ax + ey + kz)t^2$, which has 0 as a multiple root. Thus A is not diagonally equivalent to any matrix with 3 distinct eigenvalues.

We now prove the necessity. Assume that the 3×3 matrix A is not diagonally equivalent to any matrix with 3 distinct eigenvalues. Then $D_3A = \text{diag}(x, y, z)A$ has a multiple eigenvalue for all positive integers x, y and z in the set $S = \{1, 2, 3, 4, 5\}$. Then $\text{Res}(p(t), p'(t)) = 0$ for all $(x, y, z) \in S^3$. By Lemma 2.2, all the coefficients of the monomials in x, y and z in $\text{Res}(p(t), p'(t))$ are equal to zero.

Assume that $a \neq 0$. By inspecting the coefficient of x^4y^2 in $\text{Res}(p(t), p'(t))$, we get $-a^2M_1^2 = 0$. Hence, $M_1 = ae - bd = 0$. Similarly, by inspecting the coefficient of x^4z^2, we get $M_3 = 0$. Then by considering the coefficient of x^4yz, we have

$$4a^3D - 2a^2M_1M_3 = 0.$$

It then follows from $a \neq 0$ and $M_1 = 0$ that $D = 0$. Now, all the terms of the coefficient of $x^2y^2z^2$ except possibly $-2a^2M_2^2$ are obviously zero since they involve M_1, M_2 or D as a factor. It follows that $-2a^2M_2^2 = 0$ and hence, $M_2 = 0$. Thus, $D = \det A = 0$ and every principal minor of A of order 2 is zero.

Similarly, if $e \neq 0$, then by inspecting the coefficients of y^4z^2, x^2y^4, xy^4z and $x^2y^2z^2$, we get $M_2 = 0, M_1 = 0, D = 0$ and $M_3 = 0$.

Also, if $k \neq 0$, then by inspecting the coefficients of y^2z^4, x^2z^4, xyz^4 and $x^2y^2z^2$, we get $M_3 = 0, M_2 = 0, D = 0$ and $M_1 = 0$.

We now consider the case of $a = e = k = 0$. By inspecting the coefficient of x^3y^3, we get $4M_1^3 = 2eM_1^2a = 0$ and hence, $M_1 = 0$. Similarly, by inspecting the coefficients of y^3z^3 and x^3z^3, we get $M_2 = 0$ and $M_3 = 0$. It follows that the first term in the coefficient of $x^2y^2z^2$ is 0 since it involves a factor of M_1 while all the other terms except possibly $27D^2$ are 0 since they involve a, e or k as a factor. Thus we have $27D^2 = 0$ and hence, $D = 0$.

Therefore, for every 3×3 matrix A that is not diagonally equivalent to any matrix with three distinct eigenvalues, we have $\det A = 0$ and every principal minor of A of order 2 is 0. □

Observe that as indicated in the Proof of Theorem 2.1, if a 3×3 complex matrix A is diagonally equivalent to a matrix with 3 distinct eigenvalues, then there is diagonal matrix D_3 with each diagonal entry a suitable integer between 1 and 5 such that D_3A has 3 distinct eigenvalues.

More generally, it can be seen that for an $n \times n$ matrix A, the discriminant of the characteristic polynomial of $\text{diag}(x_1, x_2, \ldots, x_n)A$ has degree less than $2n$ in each variable x_i. If an $n \times n$ complex matrix A is diagonally equivalent to a matrix with n distinct eigenvalues, then there is diagonal matrix D_n whose diagonal entries are suitable integers between 1 and $2n$ such that D_nA has n distinct eigenvalues.

Theorem 2.1 may be restated as follows.

Theorem 2.3. *A 3×3 complex matrix A is diagonally equivalent to a matrix with 3 distinct eigenvalues iff* $\det A \neq 0$ *or a principal minor of A of order 2 is nonzero.*

As an immediate consequence of Theorem 2.3, we have the following result.

Corollary 2.4. *Every nonsingular 3×3 complex matrix A is diagonally equivalent to a matrix with 3 distinct eigenvalues.*

Example 2.5. The following 3×3 matrices are not diagonally equivalent to any matrix with 3 distinct eigenvalues:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be easily verified that a 3×3 matrix A with all entries nonzero is not diagonally equivalent to any matrix with 3 distinct eigenvalues iff $\operatorname{rank} A = 1$. In this case, A is diagonally equivalent to a matrix all of whose entries in the first row or column are equal to 1.

3. 4 × 4 matrices and beyond

We conjecture that Theorem 2.1 holds for all orders $n \geq 2$.

Conjecture 3.1. *For all $n \geq 2$, an $n \times n$ complex matrix A is diagonally equivalent to a matrix with n distinct eigenvalues iff* $\det A \neq 0$ *or a principal minor of A of order $n-1$ is nonzero.*

Evidently, if an $n \times n$ matrix A satisfies that every principal minor of A of order $\geq n-1$ is 0, then every matrix diagonally equivalent to A also has this property and hence would have 0 as a multiple eigenvalue.

Conjecture 3.1 claims that if an $n \times n$ complex matrix A satisfies the condition that for every invertible diagonal matrix D_n, D_nA has a multiple eigenvalue, then 0 is a multiple eigenvalue of D_nA for every invertible diagonal matrix D_n.

A weaker version of the above conjecture is the following.

Conjecture 3.2. *For all positive integers n, every nonsingular $n \times n$ complex matrix A is diagonally equivalent to a matrix with n distinct eigenvalues.*

To demonstrate the difficulties that we encounter when we try to resolve the above conjecture for higher orders, let us have a glimpse of what happens when $n = 4$.

Consider a 4×4 matrix

$$B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$$

and a generic invertible diagonal matrix $D_4 = \operatorname{diag}(w, x, y, z)$.

The characteristic polynomial of D_4B is

$$\begin{aligned} p_4(t) = {} & t^4 + (-aw - fx - ky - qz)t^3 + (wxM_1 + wyM_2 + wzM_3 + xyM_4 \\ & + xzM_5 + yzM_6)t^2 - (wxyK_1 + xyzK_2 + yzwK_3 + zwxK_4)t + wxyzD, \end{aligned}$$

where $M_1, M_2, \ldots, M_6$ are the 2×2 principal minors of B, $K_1, \ldots, K_4$ are the 3×3 principal minors of B, and $D = \det B$.

We show a small fraction of the terms in $\operatorname{discr}(p_4(t))$ obtained with the help of Maple, by denoting $p_4(t)$ as poly4 and using the command

```
sort(collect(discrim(poly4, t), [w, x, y, z], distributed), [w, x, y, z]);
```

$\operatorname{discr}(p_4(t)) = a^2M_1^2K_1^2w^6x^4y^2 + (-4a^2DM_1^3 + 2a^2K_4M_1^2K_1)w^6x^4yz + a^2K_4^2M_1^2w^6x^4z^2 + (-4a^3K_1^3 + 2a^2M_1M_2K_1^2)w^6x^3y^3 + (18a^3DM_1K_1 - 12a^3K_4K_1^2 - 12a^2DM_1^2M_2 + 4a^2K_4M_1M_2K_1 + 2a^2M_1^2K_1K_3 + 2a^2M_1M_3K_1^2)w^6x^3y^2z + (18a^3DK_4M_1 - 12a^3K_4^2K_1 - 12a^2DM_1^2M_3 + 2a^2K_4^2M_1M_2 + 2a^2K_4M_1^2K_3 + 4a^2K_4M_1M_3K_1)w^6x^3yz^2 + (-4a^3K_4^3 + 2a^2K_4^2M_1M_3)w^6x^3z^3 + a^2M_2^2K_1^2w^6x^2y^4 + (18a^3DM_2K_1 - 12a^3K_1^2K_3 - 12a^2DM_1M_2^2 + 2a^2K_4M_2^2K_1 + 4a^2M_1M_2K_1K_3 + 2a^2M_2M_3K_1^2)w^6x^2y^3z + (-27a^4D^2 + 18a^3DK_4M_2 + 18a^3DM_1K_3 + 18a^3DM_3K_1 - 24a^3K_4K_1K_3 - 24a^2DM_1M_2M_3 + a^2K_4^2M_2^2 + 4a^2K_4M_1M_2K_3 +$

$4a^2K_4M_2M_3K_1 + a^2M_1^2K_3^2 + 4a^2M_1M_3K_1K_3 + a^2M_3^2K_1^2)w^6x^2y^2z^2 + (18a^3DK_4M_3 - 12a^3K_4^2K_3 - 12a^2DM_1M_3^2 + 2a^2K_4^2M_2M_3 + 4a^2K_4M_1M_3K_3 + 2a^2K_4M_3^2K_1)w^6x^2yz^3 + a^2K_4^2M_3^2w^6x^2z^4 + (-4a^2DM_2^3 + 2a^2M_2^2K_1K_3)w^6xy^4z + (18a^3DM_2K_3 - 12a^3K_1K_3^2 - 12a^2DM_2^2M_3 + 2a^2K_4M_2^2K_3 + 2a^2M_1M_2K_3^2 + 4a^2M_2M_3K_1K_3)w^6xy^3z^2 + (18a^3DM_3K_3 - 12a^3K_4K_3^2 - 12a^2DM_2M_3^2 + 4a^2K_4M_2M_3K_3 + 2a^2M_1M_3K_3^2 + 2a^2M_3^2K_1K_3)w^6xy^2z^3 + (-4a^2DM_3^3 + 2a^2K_4M_3^2K_3)w^6xyz^4 + a^2M_2^2K_3^2w^6y^4z^2 + (-4a^3K_3^3 + 2a^2M_2M_3K_3^2) w^6y^3z^3 + a^2M_3^2K_3^2w^6y^2z^4 + (2afM_1^2K_1^2 - 4M_1^3K_1^2)w^5x^5y^2 + (-8afDM_1^3 + 4afK_4M_1^2K_1 + 16DM_1^4 - 8K_4M_1^3K_1)w^5x^5yz + (2afK_4^2M_1^2 - 4K_4^2M_1^3)w^5x^5z^2 + \cdots + (2fqK_2^2M_5^2 - 4K_2^2M_5^3)x^5y^2z^5 + k^2K_2^2M_4^2x^4y^6z^2 + (-12fk^2K_2^3 + 4fkK_2^2M_4M_6 + 2k^2K_2^2M_4M_5 + 2kqK_2^2M_4^2 + 18kK_2^3M_4 - 12K_2^2M_4^2M_6)x^4y^5z^3 + (f^2K_2^2M_6^2 - 24fkqK_2^3 + 4fkK_2^2M_5M_6 + 4fqK_2^2M_4M_6 + k^2K_2^2M_5^2 + 4kqK_2^2M_4M_5 + q^2K_2^2M_4^2 + 18fK_2^3M_6 + 18kK_2^3M_5 + 18qK_2^3M_4 - 24K_2^2M_4M_5M_6 - 27K_2^4)x^4y^4z^4 + (-12fq^2K_2^3 + 4fqK_2^2M_5M_6 + 2kqK_2^2M_5^2 + 2q^2K_2^2M_4M_5 + 18qK_2^3M_5 - 12K_2^2M_5^2M_6)x^4y^3z^5 + q^2K_2^2M_5^2x^4y^2z^6 + (-4k^3K_2^3 + 2k^2K_2^2M_4M_6)x^3y^6z^3 + (2fkK_2^2M_6^2 - 12k^2qK_2^3 + 2k^2K_2^2M_5M_6 + 4kqK_2^2M_4M_6 + 18kK_2^3M_6 - 12K_2^2M_4M_6^2)x^3y^5z^4 + (2fqK_2^2M_6^2 - 12kq^2K_2^3 + 4kqK_2^2M_5M_6 + 2q^2K_2^2M_4M_6 + 18qK_2^3M_6 - 12K_2^2M_5M_6^2)x^3y^4z^5 + (-4q^3K_2^3 + 2q^2K_2^2M_5M_6)x^3y^3z^6 + k^2K_2^2M_6^2x^2y^6z^4 + (2kqK_2^2M_6^2 - 4K_2^2M_6^3)x^2y^5z^5 + q^2K_2^2M_6^2x^2y^4z^6.$

However, the entire expression for $\mathrm{discr}(p_4(t))$, a polynomial in w, x, y and z of degree at most 6 in each variable, is about 34 pages long in Maple output and is by far more complicated than its counterpart for $n = 3$. The coefficient of $w^3x^3y^3z^3$ alone involves more than 200 terms, one of which is $256D^3$.

Assume that B is not diagonally equivalent to any matrix with 4 distinct eigenvalues. Then all the coefficients (of the monomials in w, x, y and z) in $\mathrm{discr}(p_4(t))$ are equal to 0.

Let us consider the special case that all diagonal entries of B are nonzero. Replacing B by a suitable matrix diagonally equivalent to B if necessary, we may assume that all the diagonal entries of B are equal to 1. Since $a \neq 0$, by inspecting the coefficients of $w^6x^4y^2$ and $w^6x^3y^3$, and considering the two cases of $M_1 \neq 0$ or $M_1 = 0$, we can see that $K_1 = 0$. Similarly, by using $f \neq 0$, $k \neq 0$, $q \neq 0$ and inspecting the coefficients of certain monomials in $\mathrm{discr}(p_4(t))$ whose degree in one variable is 6, it can be easily seen that $K_2 = K_3 = K_4 = 0$. If $M_1 \neq 0$, then an inspection of the coefficient of w^6x^4yz reveals that $D = 0$. If $M_1 = 0$, then note that with the exception of $-27a^4D^2$, every term of the coefficient of $w^6x^2y^2z^2$ involves M_1 or one of K_1, K_2, K_3, and K_4 as a factor and hence, is equal to 0. Thus, $27a^4D^2 = 0$. It follows that $D = 0$.

Showing that $K_1 = K_2 = K_3 = K_4 = 0$ and $D = \det B = 0$ for all 4×4 matrices B that are not diagonally equivalent to any matrix with 4 distinct eigenvalues remains an interesting challenge.

References

[1] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[2] R.A. Horn, A.K. Lopatin, The moment and Gram matrices, distinct eigenvalues and zeroes, and rational criteria for diagonalizability, Linear Algebra Appl. 299 (1999) 153–163.
[3] N. Jacobson, Basic Algebra I, W.H. Freeman, San Francisco, 1974.
[4] M.M. Konstantinov, P.H. Petkov, D.W. Gu, I. Postlethwaite, Nonlocal sensitivity analysis of the eigensystem of a matrix with distinct eigenvalues, Numer. Funct. Anal. Optim. 18 (3–4) (1997) 367–382.
[5] Z. Li, L. Harris, Sign patterns that require all distinct eigenvalues, JP J. Algebra Number Theory Appl. 2 (2) (2002) 161–179.
[6] Y.L. Ling, B.C. Wang, First- and second-order eigensensitivity of matrices with distinct eigenvalues, Int. J. Systems Sci. 19 (7) (1988) 1053–1067.
[7] B. Mishra, Algorithmic Algebra, Springer-Verlag, New York, 1993.
[8] Y.L. Shao, Y.B. Gao, Sign patterns that allow diagonalizability, Linear Algebra Appl. 359 (1–3) (2003) 113–119.
[9] B. Sturmfels, Solving Systems of Polynomial Equations, American Mathematical Society, Providence, 2002.