ELSEVIER

Discrete Mathematics 203 (1999) 271–277

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

Minimal zero-sequences and the strong Davenport constant

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Received 15 September 1997; revised 28 September 1998; accepted 5 October 1998

Abstract

Let G be a finite Abelian group and $\mathscr{U}(G)$ the set of minimal zero-sequences on G. If M_1 and $\mathsf{M}_2 \in \mathscr{U}(G)$, then set $\mathsf{M}_1 \sim \mathsf{M}_2$ if there exists an automorphism φ of G such that $\varphi(\mathsf{M}_1) = \mathsf{M}_2$. Let $\mathscr{O}(\mathsf{M})$ represent the equivalence class of M under $\sim$. In this paper, we consider problems related to the size of an equivalence class of sequences in $\mathscr{U}(G)$ and also examine a stronger form of the Davenport constant of G.

Keywords: Zero-sum problems; Minimal zero-sequence

Let G be a finite Abelian group and $\mathsf{M} = \{g_1, \dots, g_t\}$ a sequence of not necessarily distinct nonzero elements of G. If $\sum_{i=1}^{t} g_i = 0$ then M is called a *zero-sequence* and if no proper subsum of M is 0, then M is called a *minimal zero-sequence*. We denote the length of such a sequence (t in the case of M above) by $\mathscr{L}(\mathsf{M})$ and the number of distinct elements in M by $\mathscr{C}(\mathsf{M})$. The maximum value of $\mathscr{L}(\mathsf{M})$ for G is known as the Davenport constant of G and denoted $\mathsf{D}(G)$. Let $\mathscr{U}(G)$ represent the set of all minimal zero-sequences of G. If $\mathsf{M} \in \mathscr{U}(G)$ and $\varphi \in \mathrm{Aut}(G)$, then notice that $0 = \varphi(0) = \varphi(\sum_{x \in \mathsf{M}} x) = \sum_{x \in \mathsf{M}} \varphi(x)$. Hence, $\mathrm{Aut}(G)$ acts on $\mathscr{U}(G)$ such that $\varphi\mathsf{M} = \{\varphi(x) \mid x \in \mathsf{M}\}$. Set $\mathscr{O}(M) = \{\varphi\mathsf{M} \mid \varphi \in \mathrm{Aut}(G)\}$ to be the orbit of M with respect to G. Thus, if M_1 and M_2 are two minimal zero-sequences of G, then $\mathsf{M}_1 \sim \mathsf{M}_2$ if and only if $\mathsf{M}_2 \in \mathscr{O}(\mathsf{M}_1)$. We use $|\mathscr{O}(\mathsf{M})|$ to denote the size of an equivalence class and set $\mathsf{N}(G)$ equal to the number of distinct equivalences classes under $\sim$.

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0012-365X/99/$ - see front matter
PII: S0012-365X(99)00027-8

The question of considering equivalence classes of minimal zero-sequences arose when the following problem was posed at a recent Algebra Conference in Marseille, France:

> Given a prime integer p and positive integers $x_1,\ldots,x_n$ with $1 \leqslant x_i \leqslant p-1$ for which $\mathsf{S} = \{x_1,\ldots,x_n\} \in \mathscr{U}(\mathbb{Z}_p)$, does there exist an automorphism φ of $\mathbb{Z}_p$ so that $\sum_{i=1}^{n} \varphi(x_i) = p$ when the sum is viewed as an integer?

We show in Theorem 2 that the answer to this question is no. In addition, for a finite Abelian group G of order greater than 2, we show in Theorem 7 that the maximum value of $\mathscr{C}(\mathsf{M})$ is attained by a $\mathsf{M}' \in \mathscr{U}(G)$ with $\mathscr{L}(\mathsf{M}') = \mathscr{C}(\mathsf{M}')$.

Problems involving minimal zero-sequences have been numerous in the literature over the past 30 years. Caro [2] contains a survey of zero-sum problems with an emphasis on applications to graph theory. Gao [6] and Gao and Geroldinger [7] contain some recent results. A survey of known results concerning the Davenport constant can be found in [3]. We also include in the Bibliography a collection of papers which deal with addition of residue classes modulo p or n ([1,5,8,10,14–16]). Refs. [12] and [13] are general references on additive number theory and should be consulted for any undefined notation or terminology.

In the following, $\mathbb{Z}^+$ represents the set of positive integers. Given a finite Abelian group G and a sequence $\{g_1,\ldots,g_t\}$ of not necessarily distinct elements of G, let $\sum\{g_1,\ldots,g_t\}$ denote the set of nonempty sums of elements in $\{g_1,\ldots,g_t\}$.

Definition 1. Let $\mathsf{S} = \{g_1,\ldots,g_t\}$ be a minimal zero-sequence in $\mathbb{Z}_n$. For each $1 \leqslant i \leqslant t$ let g_i' be the unique integer in the equivalence class g_i with $0 \leqslant g_i' \leqslant n-1$. S has type k, denoted type$(\mathsf{S}) = k$, if $\sum_{i=1}^{t} g_i' = kn$ where this sum is considered in $\mathbb{Z}$. Define the index of S by

$$\text{index}(\mathsf{S}) = \min\{\text{type}(\mathsf{S}') \mid \mathsf{S}' \in \mathscr{O}(\mathsf{S})\}.$$

We show that minimal zero-sequences can be constructed with arbitrarily high indices.

Theorem 2. *Let $k \in \mathbb{Z}^+$. There exists $n \in \mathbb{Z}$ and minimal zero-sequence $\mathsf{S} \in \mathscr{U}(\mathbb{Z}_n)$ such that* index$(\mathsf{S}) > k$.

Proof. Pick $a_1 \in \mathbb{Z}^+$. Choose $a_2 \in \mathbb{Z}^+$ such that $a_2 > 2a_1$. Then clearly

$$\sum\{a_1,a_2\} \cap \sum\{2a_1,2a_2\} = \emptyset.$$

Continue this process and create a sequence $\{a_j\}_{j=1}^{\infty}$ by choosing a_i so that $a_i > \sum_{j=1}^{i-1} 2a_j$.

Claim. $\sum\{a_1,\ldots,a_i\} \cap \sum\{2a_1,\ldots,2a_i\} = \emptyset$.

Proof of Claim. We prove the claim by induction. By the previous discussion, the result holds for $i = 1$ and 2. Suppose the result holds for $i = t$. We show it holds for

$i = t+1$. By the induction hypothesis,

$$\sum \bigcup_{j\in T} \{a_j\} \cap \sum \bigcup_{j\in J} \{2a_j\} = \emptyset$$

for any nonempty subsets T and J of $\{1,\ldots,t\}$. So suppose

$$\sum \bigcup_{j\in T} \{a_j\} \cap \sum \bigcup_{j\in J} \{2a_j\} \neq \emptyset$$

for nonempty subsets T and J of $\{1,\ldots,t+1\}$. Setting $T' = T \cap \{1,\ldots,t\}$ and $J' = J \cap \{1,\ldots,t\}$, we must have one of the following:

(1) $\sum_{j\in T'} a_j + a_{t+1} = \sum_{j\in J'} 2a_j$.
(2) $\sum_{j\in T'} a_j = \sum_{j\in J'} 2a_j + 2a_{t+1}$.
(3) $\sum_{j\in T'} a_j + a_{t+1} = \sum_{j\in J'} 2a_j + 2a_{t+1}$.

It is easily seen that none of the equalities given in (1), (2), or (3) can occur by the choice of $a_{t+1} > \sum_{j=1}^{t} 2a_j$, and the proof of the claim is complete.

With the choice of a sequence $\{a_i\}_{i=1}^{k}$ and the choice of $n > \sum_{i=1}^{k} 2a_i$ we get $\mathsf{S} = \{a_1,\ldots,a_k, n-2a_1,\ldots,n-2a_k\}$ to be a zero-free sequence of $\mathbb{Z}_n$ as follows. If T and J are subsets of $\{1,\ldots,k\}$ and $\sum_{j\in T} a_j + \sum_{i\in J}(n-2a_i) \equiv 0 \pmod n$, then summing as ordinary integers we have $\sum_{j\in T} a_j + |J|n - \sum_{i\in J} 2a_i = vn$ for some $v \geqslant 1$. Hence, $|J|n + \sum_{j\in T} a_j = vn + \sum_{i\in J} 2a_i$. If $|J| > v$, then $n \leqslant (|J|-v)n + \sum_{j\in T} a_j = \sum_{i\in J} 2a_i$ which contradicts the selected size of n. Also, $|J| < v$ yields $\sum_{j\in T} a_j \geqslant n$, again a contradiction. Therefore $|J| = v$ and we have $\sum_{j\in T} a_j = \sum_{i\in J} 2a_i$, which contradicts the claim established above.

Let $\mathsf{S}^* = \{a_1,\ldots,a_k, n-2a_1,\ldots,n-2a_k, x\}$ be the completion of S to a minimal zero-sequence of $\mathbb{Z}_n$. We claim that index$(\mathsf{S}^*) > k$. To see this, let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ be any automorphism of $\mathbb{Z}_n$. Then for each integer i we have

$$\varphi(a_i) + \varphi(n-2a_i) \equiv \varphi(a_i) - 2\varphi(a_i) \equiv \varphi(-a_i) \pmod n.$$

We claim the integer $\varphi(a_i) + \varphi(n-2a_i) > n/2$ for each i. If $\varphi(a_i) > n/2$ then this is clear. Otherwise, since $\varphi(a_i) + \varphi(n-2a_i) \equiv \varphi(-a_i) \pmod n$, we must have $\varphi(a_i) + \varphi(n-2a_i) > n/2$. Finally,

$$\sum_{i=1}^{2k} (\varphi(a_i) + \varphi(n-2a_i)) > 2k\frac{n}{2} = kn$$

and thus type$(\varphi(\mathsf{S}^*)) > k$, completing the argument. □

Example 3. Let $m \geqslant 4$ be an integer. An argument similar to that given in Theorem 2 shows that if m is even then the zero-sequence

$$\mathsf{S} = \left\{1, 1, m, \frac{3m}{2} - 1, \frac{3m}{2} - 1\right\}$$

in $\mathbb{Z}_{2m}$ is minimal with index 2 and if m is odd then the same holds for the zero-sequence

$$\mathsf{S} = \left\{1, 1, m, \frac{3m-1}{2}, \frac{3m-3}{2}\right\}.$$

Corollary 4. $\text{Index}(\mathsf{S}) = 1$ *for all* $\mathsf{S} \in \mathscr{U}(\mathbb{Z}_n)$ *if and only if* $n = 2, 3, 4, 5$ *or* 7.

Proof. Theorem 2 provides the argument for $n \geqslant 9$ by setting $a_1 = 1$, $a_2 = 3$ and considering the zero-sequence $\mathsf{S} = \{1, 3, n-2, n-6, 2\}$. Example 3 with $m = 4$ provides the argument for $n = 8$. For $n = 6$ the minimal zero sequence $\{1, 3, 4, 4\}$ has index 2. Computer calculations yield the isomorphism classes for $n = 2$, 3, 4, 5 and 7 as follows:

$\mathbb{Z}_2$:	$\mathcal{O}(\{1,1\})$
$\mathbb{Z}_3$:	$\mathcal{O}(\{1,1,1\})$, $\mathcal{O}(\{1,2\})$
$\mathbb{Z}_4$:	$\mathcal{O}(\{1,1,1,1\})$, $\mathcal{O}(\{1,3\})$, $\mathcal{O}(\{2,2\})$, $\mathcal{O}(\{1,1,2\})$
$\mathbb{Z}_5$:	$\mathcal{O}(\{1,1,1,1,1\})$, $\mathcal{O}(\{1,4\})$, $\mathcal{O}(\{1,1,1,2\})$, $\mathcal{O}(\{1,2,2\})$
$\mathbb{Z}_7$:	$\mathcal{O}(\{1,1,1,1,1,1,1\})$, $\mathcal{O}(\{1,6\})$, $\mathcal{O}(\{1,1,1,2,2\})$, $\mathcal{O}(\{1,2,4\})$, $\mathcal{O}(\{1,1,1,1,1,2\})$, $\mathcal{O}(\{1,1,,2,3\})$, $\mathcal{O}(\{1,1,5\})$, $\mathcal{O}(\{1,1,1,4\})$ $\mathcal{O}(\{1,1,1,1,3\})$.

In regard to the possible sizes of equivalence classes in $\mathscr{U}(G)$, [11, Proposition 5.1, p. 22] implies that $|\mathcal{O}(\mathsf{M})|$ divides $|\text{Aut}(G)|$. Hence, in the special case where $G = \mathbb{Z}_p$ with p prime, if $\mathsf{M} \in \mathscr{U}(\mathbb{Z}_p)$, then $|\mathcal{O}(\mathsf{M})|$ divides $p - 1$.

While it is clear by viewing partitions that the number of minimal zero-sequences modulo a prime p grows rapidly, that the number of isomorphism classes grows rapidly is also true. In fact, one can define specific types of isomorphism classes and argue that their number increases rapidly. This is reflected in the following result.

Theorem 5. *If p is a prime number then*

$$\mathsf{N}(\mathbb{Z}_p) \geqslant \frac{p^2 - 8p + 15}{4}.$$

Proof. Notice that if $\mathsf{M} \cong \mathsf{M}'$, then an element which is repeated n times in M must correspond to some element repeated n times in M'. More specifically, fix a prime p and for each $n \in \mathbb{Z}^+$ with $2 \leqslant n \leqslant p - 4$ consider the minimal zero-sequence

$$\mathsf{M}_n(x, y) = \{\underbrace{1, \ldots, 1}_{n \text{ times}}, x, y\}$$

where $n + x + y = p$, x, $y \in \mathbb{Z}^+$ and $x \leqslant y$. When $n = 2$, there are $(p - 5)/2$ such minimal zero-sequences:

$$\mathsf{M}_2(2, p-4), \mathsf{M}_2(3, p-5), \ldots, \mathsf{M}_2\left(\frac{p-3}{2}, \frac{p-1}{2}\right).$$

Since each $\mathsf{M}_2(x, y)$ contains exactly two copies of 1, no two of these sequences are isomorphic. When $n = 3$ there are still $(p - 5)/2$ such sequences and for $n = 4$ and 5 there are $(p - 7)/2$. By counting the number of 1's which appear in these sequences, it is clear that $\mathsf{M}_n(x,y) \cong \mathsf{M}_m(w,z)$ if and only if $n=m$, $x=w$ and $y=z$. Thus, the total number of such sequences is given by

$$\sum_{i=1}^{\frac{p-5}{2}} (p-(3+2i)) = \frac{p^2-8p+15}{4}$$

and the result follows. □

Consider the following definition.

Definition 6. Let G be a finite Abelian group. Then call

$$\mathsf{SD}(G) = \max\{\mathscr{C}(\mathsf{M}) \mid \mathsf{M} \in \mathscr{U}(G)\},$$

the *strong Davenport constant* of G.

Comments: (1) Clearly, $\mathsf{SD}(G) \leqslant \mathsf{D}(G)$ for every finite Abelian G. For a positive integer $n > 2$, it is easy to see that $\mathsf{SD}(\mathbb{Z}_n) \leqslant (n+1)/2$. Hence, for a cyclic group with $n > 2$, $\mathsf{SD}(\mathbb{Z}_n) < \mathsf{D}(\mathbb{Z}_n)$.

(2) Let $G = \sum_{i=1}^{k} \mathbb{Z}_{n_i}$ where $n_i | n_{i+1}$ for $1 \leqslant i \leqslant k-1$ and set $M(G) = 1 + \sum_{i=1}^{k} (n_i - 1)$. If G is a p-group then $\mathsf{D}(G) = M(G)$ (see [3, Theorem 1.8]). By [7, Theorem 7.3], if p is a prime and G is of the form

$$\mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}} \oplus \cdots \mathbb{Z}_{p^{n_r}} \oplus \sum_{i=1}^{t} \mathbb{Z}_{p^{n_{r+1}}},$$

where $r \geqslant 0$, $t \geqslant 1$, $n_{r+1} > n_r \geqslant \cdots \geqslant n_1$ and $r + (t-1)/2 \geqslant p^{n_{r+1}}$, then there exists $\mathsf{M} \in \mathscr{U}(G)$ with $\mathscr{C}(\mathsf{M}) = M(G)$. Thus, there are finite Abelian groups G with $\mathsf{SD}(G) = \mathsf{D}(G)$.

(3) Using various results from the literature regarding zero-free sequences, we can give some upper bounds on $\mathsf{SD}(\mathbb{Z}_p)$. For example, $2\sqrt{p}+2$ [14], $\sqrt{4p-3}+1$ [14], $\sqrt{4p-7}$ [4], and $\sqrt{2p} + 5\ln(p) + 1$ [9].

(4) There is some further background in the literature related to Definition 6. The papers [14,4] are concerned with finding the critical exponent of a cyclic group $\mathbb{Z}_n$ (i.e., the minimal positive integer m such that if $S \subseteq \mathbb{Z}_n$ is a subset with $|S| \geqslant m$, then every element $g \in \mathbb{Z}_n$ can be written as a sum of elements in S). It is quite natural that this constant is bigger than the strong Davenport constant, since the latter constant merely deals with sums to 0. There is also a beautiful conjecture by Selfridge concerning the exact value of $\mathsf{SD}(\mathbb{Z}_p)$ when p is prime. The precise formulation and further references concerning this conjecture can be found in [9].

We show the existence of a minimal zero-sequence consisting of distinct elements of length $\mathsf{SD}(G)$.

Theorem 7. *Let G be a finite Abelian group with $|G| > 2$. There exists an $\mathsf{M} \in \mathscr{U}(G)$ with $\mathscr{C}(M) = \mathscr{L}(M) = \mathsf{SD}(G)$.*

Proof. Suppose that $|G|>2$. If G is an elementary 2-group, then any minimal zero-sequence consisting of more than two elements must necessarily have $\mathscr{L}(\mathsf{M})=\mathscr{C}(\mathsf{M})$. Hence, suppose G contains an element of order greater than 2 and that there is no minimal zero-sequence $\mathsf{M}\in\mathscr{U}(G)$ with $\mathscr{C}(M)=\mathscr{L}(M)=\mathsf{SD}(G)$. Choose

$$\mathsf{M}=\{g_1^{a_1},\dots,g_k^{a_k}\}\in\mathscr{U}(G),$$

where a_i represents the number of appearances of g_i in M, such that $\mathscr{C}(\mathsf{M})=\mathsf{SD}(G)=k$ and $\mathscr{L}(\mathsf{M})>\mathscr{C}(\mathsf{M})$ where $\mathscr{L}(\mathsf{M})$ is minimal. Assume without loss that $a_1\geqslant 2$. We may also assume without loss that $2g_1\in\mathsf{M}$. To see this, let

$$\mathsf{M}'=\{2g_1,g_1^{a_1-2},g_2^{a_2},\dots,g_k^{a_k}\}\in\mathscr{U}(G)$$

and consider two cases:

Case 1: Suppose $a_1\geqslant 3$. If $2g_1\notin\mathsf{M}$ then $\mathscr{C}(\mathsf{M}')=k+1$, contradicting the maximal cardinality of M.

Case 2: Suppose $a_1=2$. Then $\mathscr{C}(\mathsf{M}')=\mathsf{SD}(G)$ and $\mathscr{L}(\mathsf{M}')=\mathscr{L}(\mathsf{M})-1$, contradicting the minimality of $\mathscr{L}(\mathsf{M})$.

Now, for each $j\geqslant 2$, either $g_1+g_j\in\mathsf{M}$ or $a_j=1$. For suppose there exists a $j\geqslant 2$ with $g_1+g_j\notin\mathsf{M}$ and $a_j\geqslant 2$. Then

$$\mathsf{M}'=\{g_1+g_j,g_1^{a_1-1},g_2^{a_2},\dots,g_j^{a_j-1},\dots,g_k^{a_k}\}$$

is a minimal zero-sequence with $\mathscr{C}(\mathsf{M}')=k+1$, a contradiction. Now the proof involves consideration of two cases:

Case 1: Suppose $a_j=1$ for all $j\geqslant 2$. Without loss we may write $2g_1=g_2$. Then

$$\mathsf{M}=\{g_1^{a_1},2g_1,g_3,\dots,g_k\}$$

with $a_1\geqslant 2$. If $a_1\geqslant 3$, then

$$\mathsf{M}'=\{g_1^{a_1-2},(2g_1)^2,g_3,\dots,g_k\}\in\mathscr{U}(G)$$

with $\mathscr{C}(\mathsf{M}')=k$ and $\mathscr{L}(\mathsf{M}')=\mathscr{L}(\mathsf{M})-1$, another contradiction. On the other hand, if $a_1=2$, then $\mathsf{M}=\{g_1^2,2g_1,g_3,\dots,g_k\}$. If $g_1+g_j\notin\mathsf{M}$ for some $j\geqslant 2$, then

$$\mathsf{M}'=\{g_1+g_j,g_1,2g_1,g_3,\dots,\hat{g}_j,\dots,g_k\}\in\mathscr{U}(G)$$

with $\mathscr{C}(\mathsf{M}')=k=\mathscr{L}(\mathsf{M}')$. Suppose $g_1+g_j\in\mathsf{M}$ for all $j\geqslant 2$. Then, since $2g_1=g_2$, we find that $\mathsf{M}=\{g_1,2g_1,\dots,kg_1\}$. Thus $g_1+kg_1=(k+1)g_1\in\mathsf{M}$ contradicts the minimality of M.

Case 2: Suppose $a_j\neq 1$ for some $j\geqslant 2$. Without loss we may assume $a_2\neq 1$. We know that $g_1+g_2\in\mathsf{M}$. After relabeling if necessary, we may write $g_1+g_2=g_3$. So $\mathsf{M}=\{g_1^{a_1},g_2^{a_2},g_3^{a_3},\dots,g_k^{a_k}\}$ with a_1 and $a_2\geqslant 2$ and $\mathsf{M}'=\{g_1^{a_1-1},g_2^{a_2-1},g_3^{a_3+1},\dots,g_k^{a_k}\}\in\mathscr{U}(G)$ with $\mathscr{C}(\mathsf{M}')=k$ and $\mathscr{L}(\mathsf{M}')=\mathscr{L}(\mathsf{M})-1$.

Hence, in any case, given a minimal zero-sequence M with $\mathscr{C}(\mathsf{M})=\mathsf{SD}(G)$ and $\mathscr{L}(\mathsf{M})>\mathscr{C}(\mathsf{M})$ where $\mathscr{L}(\mathsf{M})$ is minimal, we are able to construct a minimal zero-sequence M' which contradicts the maximality of $\mathscr{C}(\mathsf{M})$ or the minimality of $\mathscr{L}(\mathsf{M})$. □

Acknowledgements

The authors would like to thank the referees for several very helpful comments and suggestions. We would also like to thank Alfred Geroldinger and Weidong Gao for providing us with an example which led to the proof of Theorem 2, and Kala Schrager for her Mathematica routines which were used to verify several of our examples. Scott T. Chapman acknowledges support under a grant from the Trinity University Faculty Development Committee.

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