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Spiral-like functions with respect to a boundary point

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0. Introduction

In the present work we study univalent functions on the unit disk of the complex plane whose image is spiral-shaped with respect to a boundary point.

Although the classes of star-like and spiral-like functions (with respect to an interior point) were studied very extensively, little was known about functions that are holomorphic on the unit disk Δ and star-like with respect to a boundary point [4].

A breakthrough in this matter is due to Robertson [8] who suggested the inequality

$$\operatorname{Re}\left\{2\frac{zh'(z)}{h(z)}+\frac{1+z}{1-z}\right\}>0, \quad z\in\Delta, \tag{0.1}$$

as a characterization of those univalent holomorphic $h:\Delta\mapsto\mathbb{C}$ satisfying $h(0)=1$ such that $h(\Delta)$ is star-like with respect to the boundary point $h(1):=\lim_{r\to 1^-}h(r)=0$ and with image in the right half-plane. This characterization was partially proved by Robertson himself under an additional assumption that h admits holomorphic extension to a neighborhood of the closed unit disk. Furthermore, he proved that this class is closely

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related to the class of close-to-convex functions. In particular, if h satisfying (0.1) is not constant with $h(0) = 1$, then $g(z) = \log h(z)$, $\log h(0) = 0$ is close-to-convex with

$$\operatorname{Re}\left[(1-z)^2 \frac{h'(z)}{h(z)}\right] < 0. \tag{0.2}$$

In 1984 his conjecture was proved by Lyzzaik [6]. Finally, in 1990 Silverman and Silvia [11] gave a full description of the class of univalent functions on Δ, the image of which is star-shaped with respect to a boundary point. Some dynamical characterizations of those functions can be found in [2]. Recently, another representation of star-like functions with respect to a boundary point was obtained by Lecko [5].

Strangely, there seems to be almost no paper on spiral-like functions with respect to a boundary point except for [3] in which it was shown that if h is a univalent spiral-like function with respect to a boundary point which is isogonal at this point then it is, in fact, star-like. However, one can construct a properly spiral-like function by using an appropriate complex power of a star-like function with respect to a boundary point.

We will show, inter alia, that each spiral-like function with respect to a boundary point is a complex power of a star-like function with respect to the same point. Our approach is based on some general conditions similar to (0.1) and (0.2) describing all spiral-like functions and some "angle" characteristics of spiral-shaped domains with respect to a boundary point. Note that these conditions cover the results mentioned above.

1. Spiral-shaped domains with respect to a boundary point

Definition 1.1. A simply connected domain $\Omega \subset \mathbb{C}$, $0 \in \partial\Omega$, is called a spiral-shaped domain with respect to a boundary point if there is a number $\mu \in \mathbb{C}$, $\operatorname{Re}\mu > 0$, such that for any point $w \in \Omega$ the curve $\{e^{-t\mu}w,\ t \geqslant 0\}$ is contained in Ω.

If, in particular, we also have $\mu \in \mathbb{R}$, then Ω is called a star-shaped domain with respect to a boundary point.

Since we intend to study functions which map the unit disk Δ onto spiral-shaped domains, the requirement for Ω to be simply connected is natural in view of the Riemann mapping theorem. For a simply connected domain Ω and $0 \in \partial\Omega$ it is possible to define on Ω a one-valued branch of the function $\arg w$. If, in addition, $1 \in \Omega$ then we can choose this branch in such a way that $\arg 1 = 0$. In this manner, for any number $\lambda \in \mathbb{C}$ the function

$$w^\lambda = \exp\bigl[\lambda\bigl(\ln|w| + i \arg w\bigr)\bigr]$$

is well defined on Ω and attains the value 1 at the point $w = 1$. We will denote the set of all spiral-shaped (respectively, star-shaped) domains with respect to a boundary point which contain the point 1 by $\mathcal{SP}$ (respectively, by $\mathcal{ST}$). It is clear that $\mathcal{ST} \subset \mathcal{SP}$.

To continue our discussion, we find a proper method to measure the "angular size" for spiral-shaped domains. This is down as follows:

Let a domain Ω be spiral-shaped ($\Omega \in \mathcal{SP}$), $w \in \Omega$ and $t \geqslant 0$. Denote the connected component of the set $\{\psi \in \mathbb{R}\colon\ e^{-\mu(t-i\psi)}w \in \Omega\}$ which contains the point $\psi = 0$ by $\Phi_\mu(w,t) = (a_\mu(w,t), b_\mu(w,t))$. In other words,

$$a_\mu(w,t) = \inf\bigl\{\phi \leqslant 0\colon\ e^{-\mu(t-i\psi)}w \in \Omega \text{ for all } \psi \in (\phi,0)\bigr\}, \tag{1.1}$$

$$b_\mu(w,t) = \sup\bigl\{\phi \geqslant 0\colon\ e^{-\mu(t-i\psi)}w \in \Omega \text{ for all } \psi \in (0,\phi)\bigr\}. \tag{1.2}$$

Proposition 1.1. *Let $\Omega \in \mathcal{SP}$ and μ be a complex number with* $\operatorname{Re}\mu > 0$ *such that the curve $\{e^{-t\mu}w,\ t \geqslant 0\} \subset \Omega$ for all $w \in \Omega$. Then the limit*

$$\alpha(w) := \lim_{t\to+\infty}\bigl(b_\mu(w,t) - a_\mu(w,t)\bigr) \tag{1.3}$$

exists (*finitely*)*. Moreover, this limit does not depend on a point $w \in \Omega$, i.e., $\alpha(w) \equiv \alpha =$* constant*. In the particular case when $\Omega \in \mathcal{ST}$ and $\mu \in \mathbb{R}$, this limit is equal to the size θ of the minimal angle in which Ω lies divided by μ, i.e., $\alpha = \theta/\mu$.*

Proof. Definition 1.1 implies that if $e^{-\mu t_0}e^{i\mu\phi}w \in \Omega$ then for all $t \geqslant t_0$ the point $e^{-\mu t}e^{i\mu\phi}w$ is contained in Ω. Consequently, $a_\mu(w,t)$ is decreasing and $b_\mu(w,t)$ is increasing (with respect to t). So, the limit in (1.3) exists. To prove that α is finite, it is enough to show that the functions $a_\mu(w,t)$ and $b_\mu(w,t)$ are bounded. Fix $t \geqslant 0$. We show that

$$b_\mu(w,t) \leqslant \frac{2\pi \operatorname{Re}\mu}{|\mu|^2} \tag{1.4}$$

and

$$a_\mu(w,t) \geqslant -\frac{2\pi \operatorname{Re}\mu}{|\mu|^2} \tag{1.5}$$

in Ω.

This is clear if $\operatorname{Im}\mu = 0$. For if $\phi = -2\pi/\mu \in \Phi_\mu(w,t)$, then Ω contains the circle $\{e^{-\mu(t-i\psi)}w,\ \psi \in [\phi,0]\}$ centered at the origin and Ω is not simply connected. Thus, without loss of generality, assume that $\operatorname{Im}\mu > 0$.

The spiral-shapedness of Ω implies that the curve Γ_1 defined by

$$\Gamma_1(t_1) = e^{-\mu(t+t_1)}w, \quad t_1 \in \left[0, \frac{2\pi \operatorname{Im}\mu}{|\mu|^2}\right],$$

lies in Ω. If inequality (1.4) is not satisfied then $\phi = 2\pi \operatorname{Re}\mu/|\mu|^2 \in \Phi_\mu(w,t)$ and, therefore, the curve Γ_2 defined by

$$\Gamma_2(\psi) = e^{-\mu(t-i\psi)}w, \quad \psi \in [0,\phi],$$

also lies in Ω. Then the curve $\Gamma_2\Gamma_1^{-1}$ lies in Ω winds once about the origin. This contradicts the simply connectedness of Ω, and condition (1.4) is proved.

As the function $a_\mu(w,t)$ is decreasing we can suppose that $t > 2\pi \operatorname{Im}\mu/|\mu|^2$. Once again the spiral-shapedness of Ω implies that the curve Γ_3 defined by

$$\Gamma_3(t_1) = e^{-\mu(t+t_1)}w, \quad t_1 \in \left[-\frac{2\pi \operatorname{Im}\mu}{|\mu|^2}, 0\right],$$

lies in Ω. If inequality (1.5) is not satisfied then $\phi = -2\pi \operatorname{Re}\mu/|\mu|^2 \in \Phi_\mu(w,t)$ and, therefore, the curve Γ_4 defined by

$$\Gamma_4(\psi) = e^{-\mu(t-i\psi)}w, \quad \psi \in [\phi,0],$$

also lies in Ω. Then the curve $\Gamma_4\Gamma_3^{-1}$ lies in Ω winds once about the origin. As above this contradicts the simply connectedness of Ω, and condition (1.5) is also proved.

Now we show that $\alpha(w)$ does not depend on $w \in \Omega$.

Let K be any compact connected subset of Ω. For each point $w_0 \in K$ there exists $\epsilon > 0$ such that the neighborhood

$$U(w_0, \epsilon) = \left\{e^{-\mu(t-i\psi)}w_0,\ -\epsilon < t < \epsilon,\ -\epsilon < \psi < \epsilon\right\}$$

is contained in Ω.

Let $w_1 \in U(w_0, \epsilon)$. Then

$$w_1 = e^{-\mu t'}\hat{w}, \quad \hat{w} = e^{i\mu\psi'}w_0, \quad \text{where } |t'| < \epsilon,\ |\psi'| < \epsilon.$$

By formulae (1.1) and (1.2) we have

$$b_\mu(\hat{w}, t) - a_\mu(\hat{w}, t) = b_\mu(w_0, t) - a_\mu(w_0, t),$$

and thus $\alpha(\hat{w}) = \alpha(w_0)$.

Furthermore, it is clear that

$$b_\mu(w_1, t) - a_\mu(w_1, t) = b_\mu(\hat{w}, t + t') - a_\mu(\hat{w}, t + t').$$

Hence, the limits as $t \to \infty$ in the both sides of the two latter equations coincide, that is, $\alpha(w_1) = \alpha(w_0)$, so it is a constant function on $U(w_0)$. Finding a finite covering system of neighborhoods $U_1, U_2, \ldots, U_n$ of K, we can conclude that $\alpha(w) \equiv$ constant on $U_1 \cup U_2 \cup \cdots \cup U_n \supset K$, so α does not depend on $w \in \Omega$.

In the case when the domain Ω is star-shaped (i.e., $\mu \in \mathbb{R}$), the quantity $b_1(w, t) - a_1(w, t)$ equals exactly the size of the circle arch of the radius $e^{-t\mu}$ (which lies in Ω) divided by μ. The proposition is proved. □

Definition 1.2. Let μ be a complex number with $\operatorname{Re}\mu > 0$. Also let Ω be a simply connected domain such that $0 \in \partial\Omega$. Ω will be called μ-spiral-shaped (with respect to a boundary point) if for any point $w \in \Omega$ the following two conditions hold:

(a) $\{e^{-t\mu}w,\ t \geqslant 0\} \subset \Omega$;
(b) the limit α in (1.3) exists and is equals to 1:

$$\alpha := \lim_{t\to+\infty}\big(b_\mu(w, t) - a_\mu(w, t)\big) = 1.$$

The set of all μ-spiral-shaped domains $\Omega \in \mathcal{SP}$ will be denoted by μ-$\mathcal{SP}$.

It is clear that

$$\mathcal{ST} = \bigcup_{\mu\in\mathbb{R}_+} \mu\text{-}\mathcal{SP}.$$

We investigate some properties of μ-spiral-shaped domains.

Lemma 1.1. (i) *If* $\operatorname{Re}\mu > 0$ *and* Ω *is of the class* $\mathcal{SP}$, *then* $\Omega \in \mu$-$\mathcal{SP}$ *if and only if*

$$\hat{\Omega} = \Omega^{\pi/\mu} := \{z^{\pi/\mu},\ z \in \Omega\} \in \pi\text{-}\mathcal{SP}.$$

Moreover, $\hat{\Omega}$ *is star-shaped.*

(ii) *If exists* $\Omega \in \mu\text{-}\mathcal{SP} \cap \nu\text{-}\mathcal{SP}$, *where* $\mu, \nu \in \mathbb{C}$ *with* $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$, *then* $\mu = \nu$.

Proof. In addition to formulae (1.1) and (1.2) let us denote

$$a_\pi(\hat{w}, t) = \inf\{\phi\colon\ e^{-\pi(t-i\phi)}\hat{w} \in \hat{\Omega}\},$$
$$b_\pi(\hat{w}, t) = \sup\{\phi\colon\ e^{-\pi(t-i\phi)}\hat{w} \in \hat{\Omega}\},$$

where $\hat{w} = w^{\pi/\mu} \in \hat{\Omega}$.

Since the inclusions $e^{-\mu(t-i\phi)}w \in \Omega$ and $e^{-\pi(t-i\phi)}\hat{w} \in \hat{\Omega}$ are one and the same, we have

$$b_\mu(w, t) - a_\mu(w, t) = b_\pi(\hat{w}, t) - a_\pi(\hat{w}, t).$$

Thus the limits of both sides of this equation are either equal to 1 or both differ from 1. Assertion (i) is proved.

In turn, (i) implies that the domains

$$\Omega_1 = \Omega^{\pi/\mu} := \{z^{\pi/\mu},\ z \in \Omega\} \quad \text{and} \quad \Omega_2 = \Omega^{\pi/\nu} := \{z^{\pi/\nu},\ z \in \Omega\}$$

are contained in $\pi\text{-}\mathcal{SP}$. It means that for any point $w \in \Omega$ we have $w_1 = w^{\pi/\mu} \in \Omega_1$ and $w_2 = w^{\pi/\nu} \in \Omega_2$. So, we see: any point $w_2 \in \Omega_2$ if and only if the point $w_1 = w_2^{\nu/\mu}$ lies in Ω_1. In other words, $\Omega_1 = \Omega_2^{\nu/\mu}$.

Suppose now that $\arg\mu \neq \arg\nu$. Lemma 1.2 implies that the domain Ω_1 is $(\mu\pi/\nu)$-spiral-shaped, i.e., by Definition 1.2, it contains the following spiral which goes around the origin:

$$\{e^{-t(\mu\pi/\nu)}w,\ t \geqslant 0\} \subset \Omega_1, \quad \text{when } w \in \Omega_1.$$

This contradicts the inclusion $\Omega_1 \in \pi\text{-}\mathcal{SP}$ (see Proposition 1.1). So $\arg\mu = \arg\nu$.

Suppose now, that $|\mu| \neq |\nu|$, for example, $\mu = R\nu$, $R > 1$. Again we have $\Omega_1 = \Omega_2^{\nu/\mu} = \Omega_2^{1/R}$. Since the domain Ω_2 is contained in some angle which is equal to π, then the domain Ω_1 is contained in the angle which size is of $\pi/R < \pi$ and this contradicts to the inclusion $\Omega_1 \in \pi\text{-}\mathcal{SP}$. Thus we have $\mu = \nu$. □

The proved lemma states that each spiral-shaped domain (with respect to a boundary point) is μ-spiral-shaped with a unique number μ, $\operatorname{Re}\mu > 0$. Now we show that μ can not be arbitrary in the right half-plane.

Proposition 1.2. (i) *If* $\Omega_1 \in \pi\text{-}\mathcal{SP}$ *and* $|\mu/\pi - 1| \leqslant 1$, *then* $\Omega = \Omega_1^{\mu/\pi} \in \mu\text{-}\mathcal{SP}$.

(ii) *In case for some* $\mu \in \mathbb{C}$ *there exists* Ω *that belongs to* $\mu\text{-}\mathcal{SP}$, *then* $|\mu/\pi - 1| \leqslant 1$.

Proof. Without loss of a generality we assume that a domain $\Omega_1 \in \pi\text{-}\mathcal{SP}$ lies in $\Pi_+ := \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\}$. First we will show that for any π-spiral-shaped domain $\Omega_1 \subset \Pi_+$ the domain $\Omega = \Omega_1^\nu$ is simply connected if $|\nu - 1| \leqslant 1$ or $\operatorname{Re}(1/\nu) \geqslant 1/2$.

Since the domain Ω_1 is simply connected and $0 \in \partial\Omega_1$, then Ω_1^ν is simply connected if and only if the mapping $z \mapsto z^\nu$ is one-to-one on Ω_1. It means that for any $w \in \Omega_1$ the following equation:

$$w^\nu = z^\nu \tag{1.6}$$

has no solution $z \in \Omega_1 \backslash w$.

Suppose that $z = re^{i\psi}$, $|\psi| < \pi/2$, is the solution of the latter equation. Substituting $w = \rho e^{i\phi}$, $|\phi| < \pi/2$, we rewrite (1.6) in the following form:

$$\nu(\ln\rho + i\phi) = \nu(\ln r + i\psi) + 2\pi k i, \quad k \in \mathbb{Z}\backslash 0,$$
$$\frac{1}{\nu} = \frac{\ln\rho + i\phi - \ln r - i\psi}{2\pi k i} = \frac{\phi - \psi}{2\pi k} + i\frac{-\ln\rho + \ln r}{2\pi k}.$$

This equality implies that

$$\operatorname{Re}\frac{1}{\nu} < \frac{\pi}{2\pi|k|} = \frac{1}{2|k|} \leqslant \frac{1}{2}.$$

The latter inequality contradicts our supposition that $\operatorname{Re}(1/\nu) \geqslant 1/2$. Thus the domain $\Omega = \Omega_1^\nu$ is simply connected. It is easy to see by Definition 1.1 that $\Omega \in \mathcal{SP}$. By Lemma 1.1, $\Omega \in \mu\text{-}\mathcal{SP}$ with $\mu = \nu\pi$. Assertion (i) is proved.

To prove assertion (ii), we suppose that $\Omega \in \mu\text{-}\mathcal{SP}$, where the number $\nu = \mu/\pi$ satisfies $\operatorname{Re}(1/\nu) < 1/2$, and so $\operatorname{Re}(1/\nu) = (1-\epsilon)/2$ for some $\epsilon \in (0,1)$.

A given point $w \in \Omega$ and t large enough, it follows by Definition 1.2 that:

$$1 - \frac{\epsilon}{2} \leqslant b_{\nu\pi}(w,t) - a_{\nu\pi}(w,t) \leqslant 1.$$

In other words, there exist values $\tilde{\phi}_1$ and $\tilde{\phi}_2$ such that

$$e^{-\nu\pi(t - i\tilde{\phi}_j)}w \in \Omega, \quad j = 1, 2,$$

and

$$1 - \epsilon \leqslant \tilde{\phi}_2 - \tilde{\phi}_1.$$

Thus, for t big enough and for all $\phi \in [\tilde{\phi}_1, \tilde{\phi}_2]$

$$e^{-\nu\pi(t - i\phi)}w \in \Omega. \tag{1.7}$$

In particular, the points $e^{-\nu\pi(t-i\phi_1)}w$ and $e^{-\nu\pi(t-i\phi_2)}w$, where $\phi_1 = \tilde{\phi}_1$ and $\phi_2 = \tilde{\phi}_1 + 1 - \epsilon$, belong to Ω.

It follows by Definition 1.2 that

$$e^{-\nu\pi(t_j - i\phi_j)}w \in \Omega, \quad j = 1, 2,$$

for all $t_1, t_2 \geqslant t$. Hence we can choose those numbers t_1 and t_2 such that

$$\frac{1}{\nu} = \frac{1-\epsilon}{2} + i\frac{t_2 - t_1}{2}.$$

This implies that

$$e^{-\nu\pi(t_1 - i\phi_1)}w = e^{-\nu\pi(t_2 - i\phi_2)}w. \tag{1.8}$$

It follows by Lemma 1.1 that the domain $\Omega_1 = \Omega^{1/\nu} \in \pi\text{-}\mathcal{SP}$. Thus the domain $\Omega_2 = \Omega_1^2 = \Omega^{2/\nu} \in 2\pi\text{-}\mathcal{SP}$, therefore a one-valued branch of the function $\arg w$ is correctly defined on the domain Ω_2. Further, Eq. (1.8) implies that

$$e^{-2\pi(t_1 - i\phi_1)}w_2 = e^{-2\pi(t_2 - i\phi_1)}e^{2\pi i(1-\epsilon)}w_2 \in \Omega_2, \tag{1.9}$$

where $w_2 = w^{2/\nu} \in \Omega_2$. Equality (1.9) means that for the same point of the simply connected domain Ω_1 there are two values of its argument. That is a contradiction which proves assertion (ii). □

2. The class Snail(Δ)

Definition 2.1. A univalent function $f:\Delta \mapsto \mathbb{C}$ on the unit disk Δ is said to be of class Snail(Δ) (respectively, μ-Snail(Δ)) if

(a) $f(0)=1$ and $\lim_{r\to 1^-} f(r)=0$;
(b) $f(\Delta)\in \mathcal{SP}$ (respectively, $f(\Delta)\in \mu$-$\mathcal{SP}$).

In the particular case where μ is positive (that is, $f(\Delta)$ is star-shaped) we write $f\in$ μ-Fan(Δ).

Observe that $f\in\mu$-Fan(Δ) if and only if f is univalent, its image $f(\Delta)$ is star-shaped with respect to the origin and the smallest wedge containing $f(\Delta)$ is of angle μ.

Now we formulate our main result.

Theorem 2.1. *Let $f:\Delta\mapsto\mathbb{C}$ be a holomorphic function and $f(0)=1$. Let $\mu\in\mathbb{C}$, $|\mu/\pi-1|\leqslant 1$. The following assertions are equivalent.*

(I) $f\in\mu$-Snail(Δ).
(II) $f_1(z)=f(z)^{\pi/\mu}\in\pi$-Fan($\Delta$), *i.e., f_1 is star-like with respect to the boundary point $z=1$ function and the smallest wedge which contains its image is exactly of angle π.*
(III) *The function f satisfies the following condition*:

$$\operatorname{Re}\left(\frac{2\pi}{\mu}\cdot\frac{zf'(z)}{f(z)}+\frac{1+z}{1-z}\right)>0,\quad z\in\Delta, \tag{2.1}$$

and it is possible to replace the number μ in this inequality with a number ν only if $\nu=R\mu$, $R>1$.
(IV) *The function $s(z):=zf(z)/(1-z)^{\mu/\pi}$ is ϕ-spiral-like of order $\cos\phi-r/(2\pi)$, where $\mu=re^{i\phi}$, i.e., s is a univalent function satisfying the condition*

$$\operatorname{Re}\left(e^{-i\phi}\frac{zs'(z)}{s(z)}\right)>\cos\phi-\frac{r}{2\pi},\quad z\in\Delta, \tag{2.2}$$

and it is possible to replace the number μ in this inequality with a number ν only if $\nu=R\mu$, $R>1$.
(V) *The function f satisfies three following conditions*:
 (a) *f is univalent in Δ*;
 (b) $\operatorname{Re}(\mu(f(z)/f'(z))\bar z)\geqslant \operatorname{Re}(\mu(f(0)/f'(0))\bar z)(1-|z|^2)$;
 (c) $\angle\lim_{z\to 1}(f(z)/f'(z)(z-1))=\pi/\mu$, *where $\angle$ means that the limit considered is the angular limit.*

Moreover, if f is a univalent function on Δ which satisfies one of the conditions (II)–(V) *with a some complex number μ, $\operatorname{Re}\mu>0$, then μ lies in the disk $|\mu/\pi-1|\leqslant 1$ and $f\in\mu$-Snail(Δ).*

Remark. Note that if $f(1):=\angle\lim_{z\to 1} f(z)$ exists then one can define $Q(z):=f'(z)\times (z-1)/(f(z)-f(1))$ which is called the Wisser–Ostrowski quotient (see, for example, [7]). Thus, it follows by the above assertion (Vc) that $f \in \text{Snail}(\Delta)$ is star-like whenever $\angle\lim_{z\to 1} Q(z)$ is a real number (cf., [3,10]).

The proof of the theorem is done in several steps.

Step 1. (I)⇔(II) By Lemma 1.1, it is immediate that if $f \in \mu\text{-Snail}(\Delta)$ then $f_1(z) = f(z)^{\pi/\mu} \in \pi\text{-Fan}(\Delta)$, and if $f_1(z) \in \pi\text{-Fan}(\Delta)$ then $f(z) = f_1(z)^{\mu/\pi} \in \mu\text{-Snail}(\Delta)$.

Step 2. (III)⇔(IV) This equivalence is verified by the substituting $s(z) = zf(z)/(1-z)^{\mu/\pi}$ in (2.2) and $f(z) = (1-z)^{\mu/\pi} s(z)/z$ in (2.1). Indeed, it is easy to see that

$$\left(\frac{2\pi}{\mu}\cdot\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) = \frac{2\pi}{|\mu|}\left(e^{-i\phi}\cdot\frac{zs'(z)}{s(z)} + \frac{|\mu|}{2\pi} - e^{-i\phi}\right),$$

and this equality proves our assertion.

Step 3. (II)⇔(III) To prove this equivalence we need some lemmata. The first lemma is a reformulation of a result of Silverman and Silvia [11, Theorem 9] (see also [6,8]) in terms of classes $\mu\text{-Fan}(\Delta)$:

Lemma 2.1. *Let μ be a positive number, $\mu \leqslant 2\pi$. A function $f : \Delta \mapsto \mathbb{C}$, $f(0) = 1$, belongs to $\bigcup_{l\leqslant\mu} l\text{-Fan}(\Delta)$ if and only if*

$$\operatorname{Re}\left(\frac{2\pi}{\mu}\cdot\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) > 0, \quad z \in \Delta, \tag{2.3}$$

and $f \neq 1$ identically.

Let us assume now that (II) holds. Then by Lemma 2.1

$$\operatorname{Re}\left(2\cdot\frac{zf_1'(z)}{f_1(z)} + \frac{1+z}{1-z}\right) > 0, \quad z \in \Delta. \tag{2.4}$$

If inequality (2.1) holds for some $\nu \in \mathbb{C}$, then the function f_1 satisfies the inequality

$$\operatorname{Re}\left[\frac{2\mu}{\nu}\cdot\frac{zf_1'(z)}{f_1(z)} + \frac{1+z}{1-z}\right] > 0.$$

Therefore, the function $f_2(z) = f_1(z)^{\mu/\nu}$ satisfies inequality (2.3). Thus, Lemma 2.1 implies that $f_2 \in l\text{-Fan}(\Delta)$ for some positive number $l \leqslant \pi$. Hereby, $f_1 = f_2(z)^{\nu/\mu} \in (l\nu/\mu)\text{-Fan}(\Delta)$. Hence, by Lemma 1.1 $l\nu/\mu = \pi$ or $\nu = \pi/l\cdot\mu$. As $l \leqslant \pi$ assertion (III) holds.

Assume now that condition (III) holds. By substitution $f(z) = f_1(z)^{\mu/\pi}$ we get inequality (2.4). Using Lemma 2.1 we obtain $f_1 \in \bigcup_{l\leqslant\pi} l\text{-Fan}(\Delta)$. Suppose that $f_1 \in l\text{-Fan}(\Delta)$ with $l < \pi$. Again by Lemma 2.1 f_1 satisfies inequality (2.3) with μ replaced by l. Returning to the function $f(z) = f_1(z)^{\mu/\pi}$ we have

$$\operatorname{Re}\left(\frac{2\pi}{\mu}\cdot\frac{\pi}{l}\cdot\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) > 0,$$

which contradicts our assumption. Thus, $l = \pi$, i.e., $f_1 \in \pi\text{-Fan}(\Delta)$, and we are done.

To proceed, we note that the inclusion $f \in \mu\text{-Snail}(\Delta)$ implies that for any $z \in \Delta$ and $t \geqslant 0$

$$e^{-t\mu} f(z) \in f(\Delta).$$

This means that for each $t \geqslant 0$ the function $u(t,\cdot)$ defined by

$$u(t,z) := f^{-1}\bigl(e^{-t\mu} f(z)\bigr)$$

is well-defined self-mapping of Δ. Differentiating $u(t,z)$ with respect to t, one can see that this is a solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + \mu \dfrac{f(u(t,z))}{f'(u(t,z))} = 0, \\ u(0,z) = z, \quad z \in \Delta. \end{cases} \tag{2.5}$$

Lemma 2.2 (see [1]). *Let $g \in \mathrm{Hol}(\Delta)$. The for each $z \in \Delta$, the Cauchy problem*

$$\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + g\bigl(u(t,z)\bigr) = 0, \\ u(0,z) = z, \end{cases}$$

has a unique solution $\{u(t,z), t \geqslant 0\} \subset \Delta$ if and only if the function g satisfies the following inequality:

$$\mathrm{Re}\bigl(g(z)\bar{z}\bigr) \geqslant \mathrm{Re}\bigl(f(0)\bar{z}\bigr)\bigl(1-|z|^2\bigr),$$

for all $z \in \Delta$.

Step 4. (I)⇒(V) Let f be a μ-spiral-like function. Then condition (Va) follows at once. Hence, as mentioned above, the Cauchy problem (2.5) can be solved for all $t \geqslant 0$ and $z \in \Delta$. Applying Lemma 2.2 for the function

$$g(z) = \mu \frac{f(z)}{f'(z)},$$

we get inequality (Vb).

Therefore, it remains to check condition (Vc).

As shown above, (I) is equivalent to (III) (Steps 1 and 3). Then for any ν of the form $\nu = R\mu$, $R > 1$, the following inequality holds:

$$\mathrm{Re}\left(\frac{2\pi z\mu}{\nu g(z)} + \frac{1+z}{1-z}\right) > 0, \quad z \in \Delta.$$

Note also that this inequality no longer holds for other values of ν. By the Riesz–Herglotz formula there exists a probability measure $d\sigma$ on the unit circle such that

$$\frac{2\pi z\mu}{\nu g(z)} + \frac{1+z}{1-z} = \int\limits_{|\zeta|=1} \frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\, d\sigma(\zeta), \quad z \in \Delta,$$

or, equivalently,

$$\frac{\pi\mu(z-1)}{\nu g(z)} = \int\limits_{|\zeta|=1} \frac{1-\bar{\zeta}}{1-z\bar{\zeta}}\, d\sigma(\zeta), \quad z \in \Delta. \tag{2.6}$$

Note that the integral representation (2.6) is not valid in case ν is different from $R\mu$, $R > 1$. Decomposing σ with respect to the Dirac measure δ at the point $\zeta = 1 \in \partial\Delta$, one can write $\sigma = (1-a)\sigma_1 + a\delta$, where $0 \leqslant a \leqslant 1$, and σ_1 and δ are mutually singular probability measures. Also, Eq. (2.6) with $\nu = \mu$ implies that

$$\frac{\pi\mu(z-1)}{((a-1)\mu)g(z)} = \int_{|\zeta|=1} \frac{1-\bar{\zeta}}{1-z\bar{\zeta}}\, d\sigma_1(\zeta), \quad z \in \Delta,$$

which is valid only if $1-a \geqslant 1$. Hence, $a = 0$ and $\sigma = \sigma_1$ is singular with respect to δ.

Let $\{z_n\}$ be any sequence in Δ nontangentially convergent to 1. This means that there is a positive number K such that for all $n = 1, 2, \ldots,$

$$\frac{|1-z_n|}{1-\operatorname{Re} z_n} < K.$$

We now consider the functions $f_n : \partial\Delta \mapsto \mathbb{C}$ defined by

$$f_n(\zeta) := \frac{1-\bar{\zeta}}{1-z\bar{\zeta}}, \quad \zeta \in \partial\Delta.$$

It is easy to see that each function f_n maps the unit circle $\partial\Delta$ onto the circle $|w - c_n| = c_n$, where

$$c_n(\zeta) = \frac{1-\bar{z}_n}{1-|z|^2}, \quad n = 1, 2, \ldots.$$

Hence,

$$\left|f_n(\zeta)\right| \leqslant 2|c_n| = \frac{2|1-z_n|}{1-\operatorname{Re} z_n} < 2K$$

for all $n = 1, 2, \ldots$. Setting $\nu = \mu$ in (2.6) and applying Lebesgue's bounded convergence theorem we obtain

$$\lim_{n\to\infty} \frac{\pi(z_n-1)}{g(z_n)} = \int_{|\zeta|=1} \lim_{n\to\infty} \frac{1-\bar{\zeta}}{1-z_n\bar{\zeta}}\, d\sigma(\zeta) = 1.$$

Therefore,

$$\angle \lim_{z\to 1} \frac{f(z)}{f'(z)(z-1)} = \angle \lim_{z\to 1} \frac{g(z)}{\mu(z-1)} = \frac{\pi}{\mu},$$

and condition (Vc) follows.

Step 5. (V)⇒(I) Note that by condition (Va) the image $f(\Delta)$ is a simply connected domain. By Lemma 2.2, condition (Vb) implies that the Cauchy problem (2.5) is solved, and its solution is a self-mapping of the unit disk Δ for each $t \geqslant 0$. Solving directly the Cauchy problem (2.5) and using the univalence of f we get

$$u(t,z) := f^{-1}\big(e^{-t\mu} f(z)\big) \in \Delta \quad \text{for each } t \geqslant 0.$$

Thus for all $z \in \Delta$ the curve $\{e^{-t\mu} f(z),\, t \geqslant 0\}$ is contained in $f(\Delta)$, i.e., $f \in \operatorname{Snail}(\Delta)$.

Assume that for some $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu > 0$ the function f belongs to ν-$\operatorname{Snail}(\Delta)$. We have seen already in Step 4 that in this case

$$\angle \lim_{z\to 1} \frac{f(z)}{f'(z)(z-1)} = \frac{\pi}{\nu}.$$

Comparing this equality with (Vc), we get $\nu = \mu$. This completes the proof of the theorem. □

3. An application

In this section we use the well-known notion of subordinated functions.

Definition 3.1. A function $s_1 \in \mathrm{Hol}(\Delta, \mathbb{C})$ is said to be subordinated to $s_2 \in \mathrm{Hol}(\Delta, \mathbb{C})$ ($s_1 \prec s_2$) if there exists a holomorphic function ω with $|\omega(z)| \leqslant |z|$, $z \in \Delta$, such that $s_1 = s_2 \circ \omega$.

The following description of spiral-like functions (with respect to the origin) is due to Ruscheweyh (see [9, Corollaries 1 and 2]).

Lemma 3.1. *Let $s \in \mathrm{Hol}(\Delta, \mathbb{C})$ with $s(0) = s'(0) - 1 = 0$. Let $\alpha \in (-\pi/2, \pi/2)$ and $0 \leqslant \beta < \cos\alpha$. Then*

$$\mathrm{Re}\left(\exp(i\alpha)\frac{z s'(z)}{s(z)}\right) > \beta, \quad z \in \Delta, \tag{3.1}$$

if and only if one of the following two conditions holds:

(a) *for all $u, v \in \bar{\Delta}$ we have*

$$\frac{us(vz)}{vs(uz)} \prec \left(\frac{1-uz}{1-vz}\right)^{2(\cos\alpha-\beta)\exp(-i\alpha)}; \tag{3.2}$$

(b) *for all $t \in (0, 2\cos\alpha)$ the function s satisfies the inequality*

$$\left|s\big(z\big(1-\exp(i\alpha)t\big)\big)\right| \leqslant F(t,\alpha,\beta)\left|s(z)\right| \quad \textit{for all } z \in \Delta, \tag{3.3}$$

where

$$F(t,\alpha,\beta) = \left|1-\exp(i\alpha)t\right|\left(1-\frac{t}{2\cos\alpha}\right)^{2\cos\alpha(\beta-\cos\alpha)}. \tag{3.4}$$

Moreover, this bound is sharp.

By using this result and Theorem 2.1 one can characterize the class $\mathrm{Snail}(\Delta)$ in terms of subordinated functions. Indeed, to do this we just have to substitute $s(z) = zf(z)/(1-z)^{\mu/\pi}$ in (3.2) and (3.3), where $f \in \mu\text{-}\mathrm{Snail}(\Delta)$. Now by Theorem 2.1, we already know that s satisfies the inequality

$$\mathrm{Re}\left(e^{-i\phi}\frac{zs'(z)}{s(z)}\right) > \cos\phi - \frac{|\mu|}{2\pi}, \quad z \in \Delta,$$

where $\phi = \arg\mu$ if and only if $f \in \mu\text{-}\mathrm{Snail}(\Delta)$. Therefore, setting

$$\alpha := -\phi = -\arg\mu \quad \text{and} \quad \beta := \cos\phi - \frac{|\mu|}{2\pi}$$

in (3.2)–(3.4), we get $2(\cos\alpha - \beta) = |\mu|/\pi$. Thus one can rewrite conditions (a) and (b) of Lemma 3.1 in the form

$$\frac{u\left(\frac{vzf(vz)}{(1-vz)^{\mu/\pi}}\right)}{v\left(\frac{uzf(vz)}{(1-uz)^{\mu/\pi}}\right)} = \frac{f(vz)(1-uz)^{\mu/\pi}}{f(uz)(1-vz)^{\mu/\pi}} \prec \left(\frac{1-uz}{1-vz}\right)^{(|\mu|/\pi)\exp(i\arg\mu)} \tag{3.2'}$$

and

$$\left|\frac{z(1-\exp(-i\phi)t)f(z(1-\exp(-i\phi)t))}{(1-z(1-\exp(-i\phi)t))^{\mu/\pi}}\right| \leqslant F(t,-\phi,\beta)\left|\frac{zf(z)}{(1-z)^{\mu/\pi}}\right|$$
$$= \left|1-\exp(-i\phi)t\right|\left(1-\frac{t}{2\cos\phi}\right)^{-\cos\phi|\mu|/\pi}\left|\frac{zf(z)}{(1-z)^{\mu/\pi}}\right|. \tag{3.3′}$$

So, we have proved the following characterization of the class μ-Snail(Δ).

Corollary 3.1. *Let $f:\Delta\mapsto\mathbb{C}$ be a holomorphic function and $f(0)=1$. Let $\mu\in\mathbb{C}$, $|\mu/\pi-1|\leqslant 1$ and $\phi=\arg\mu\in(-\pi/2,\pi/2)$. Then $f\in\mu$-Snail(Δ) if and only if one of the following conditions holds*:

(a) *for all $u,v\in\bar{\Delta}$*

$$\left(\frac{1-uz}{1-vz}\right)^{\mu/\pi}\frac{f(vz)}{f(uz)} \prec \left(\frac{1-uz}{1-vz}\right)^{\mu/\pi};$$

(b) *for all $t\in(0,2\cos\phi)$*

$$\left|\frac{f(z(1-e^{-i\phi}t))}{f(z)}\right| \leqslant \left|\left(\frac{1-z(1-e^{-i\phi}t)}{1-z}\right)^{\mu/\pi}\right|\left(1-\frac{t}{2\cos\phi}\right)^{-\operatorname{Re}\mu/\pi}.$$

Furthermore, setting in Corollary 3.1 $u=0$, $v=1$, we obtain

Corollary 3.2. *If $f\in\mu$-Snail(Δ), then*

$$\left(\frac{1}{1-z}\right)^{\mu/\pi}f(z) \prec \left(\frac{1}{1-z}\right)^{\mu/\pi}.$$

In particular, if $f\in\mu$-Fan(Δ) with $\mu\leqslant\pi$, then $\operatorname{Re}(f(z)/(1-z)^{\mu/\pi})>1/2$.

The case of star-like functions (i.e., $\mu\in\mathbb{R}$) is of a special interest (see, for example, [3]). In this situation one can formulate the following consequence of Corollary 3.1.

Corollary 3.3. *Let $f:\Delta\mapsto\mathbb{C}$ be a holomorphic function and $f(0)=1$. Let $l\in(0,2)$. Then $f\in(l\pi)$-Fan(Δ) if and only if for all $t\in(-1,1)$*

$$\left|\frac{f(zt)}{f(z)}\right| \leqslant \left(\left|\frac{1-zt}{1-z}\right|\frac{2}{1+t}\right)^{l}.$$

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