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Some applications of the Hermite matrix polynomials series expansions[1]

Emilio Defez*, Lucas Jódar
Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, Spain

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Abstract

This paper deals with Hermite matrix polynomials expansions of some relevant matrix functions appearing in the solution of differential systems. Properties of Hermite matrix polynomials such as the three terms recurrence formula permit an efficient computation of matrix functions avoiding important computational drawbacks of other well-known methods. Results are applied to compute accurate approximations of certain differential systems in terms of Hermite matrix polynomials. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction and preliminaries

The evaluation of matrix functions is frequent in the solution of differential systems. So, the system

$$Y' = AY, \qquad Y(0) = y_0, \tag{1}$$

where A is matrix and y_0 is a vector, arises in the semidiscretization of the heat equation [17]. The matrix differential problem

$$Y'' + A^2 Y = 0, \qquad Y(0) = P, \qquad Y'(0) = Q, \tag{2}$$

* Corresponding author. E-mail: edefez@mat.upv.es.

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where A is a matrix and P and Q are vectors, arises from semidiscretization of the wave equation [15]. The Sylvester matrix differential equation

$$X' = AX + XB, \qquad X(0) = C, \tag{3}$$

where A, B and C are matrices, appears in systems stability and control [4, p. 226]. The solutions of problems (1)–(3) can be expressed in terms of $\exp(At)$, $\cos(At)$, $\sin(At)$ and $\exp(Bt)$ and the computation of these matrix functions has motivated many and varied approaches. An excelent survey about the matrix exponential is [11] and the study of $\cos(At)$ is treated in [1, 14, 15]. Some of the drawbacks of the existing methods are

(i) The computation of eigenvalues or eigenvectors [11, 16];
(ii) The computation of the inverses of matrices (see Padé methods, [5, p. 557]) [11, 14];
(iii) Storage problems and expensive computational time [5, Chap. 1];
(iv) Round-off accumulation errors [5, p. 551];
(v) Difficulties for computing approximations of matrix functions with a prefixed accuracy [15, 16].

In this paper we propose a new method for computing the above matrix functions using Hermite matrix polynomials which avoids the quoted computational difficulties. Results are applied to construct approximations of problems (1)–(3) with a prefixed accuracy in a bounded domain.

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z, [13, p. 72], then $z^{1/2}$ represents $\exp(\frac{1}{2}\log(z))$. If A is a matrix in $\mathscr{C}^{r\times r}$, its 2-norm denoted $\|A\|_2$ is defined by $\|A\|_2 = \|Ax\|_2/\|x\|_2$, where for a vector y in $\mathscr{C}^r$, $\|y\|_2$ denotes the usual euclidean norm of y, $\|y\|_2 = (y^T y)^{1/2}$. The set of all the eigenvalues of A is denoted by $\sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathscr{C}^{r\times r}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus, [3, p. 558], it follows that $f(A)g(A) = g(A)f(A)$. If A is a matrix with $\sigma(A) \subset D_0$, then $A^{1/2} = \sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A. We say that A is a positive stable matrix if $\mathrm{Re}(z) > 0$ for all $z \in \sigma(A)$.

For the sake of clarity in the presentation we recall some properties of the Hermite matrix polynomials which will be used below and that have been established in [7], see also [8]. If A is a positive stable matrix in $\mathscr{C}^{r\times r}$, the nth Hermite matrix polynomial is defined by

$$H_n(x,A) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (\sqrt{2A})^{n-2k}}{k!(n-2k)!} x^{n-2k}, \tag{4}$$

and satisfy the three terms recurrence relationship

$$\begin{aligned} &H_n(x,A) = Ix\sqrt{(2A)}H_{n-1}(x,A) - 2(n-1)H_{n-2}(x,A), \quad n \geqslant 1 \\ &H_{-1}(x,A) = 0, \qquad H_0(x,A) = I, \end{aligned} \tag{5}$$

where I is the identity matrix in $\mathscr{C}^{r\times r}$. By [7] we also have

$$\mathrm{e}^{xt\sqrt{2A} - t^2 I} = \sum_{n\geqslant 0} \frac{1}{n!} H_n(x,A) t^n, \quad |t| < +\infty, \tag{6}$$

and if $H_n(x)$ denotes the classical nth scalar Hermite polynomial, then one gets

$$\|H_n(x,A)\|_2 \leqslant \frac{H_n(\frac{\mathrm{i}a\|\sqrt{2A}\|_2}{2})}{i^n}, \quad |x|<a, \quad n\in N, \tag{7}$$

$$\sum_{n\geqslant 0} \frac{H_n(\frac{\|\sqrt{2A}\|_2 \mathrm{i}a}{2})}{n!\mathrm{i}^n}|t|^n = \exp(|t|a\|\sqrt{2A}\|_2 + |t|^2), \quad |t|<\infty. \tag{8}$$

If $A(k,n)$ and $B(k,n)$ are matrices on $\mathscr{C}^{r\times r}$ for $n\geqslant 0$, $k\geqslant 0$, in an analogous way to the proof of Lemma 11 of [12, p. 57] it follows that

$$\sum_{n\geqslant 0}\sum_{k\geqslant 0} A(k,n) = \sum_{n\geqslant 0}\sum_{k=0}^{[n/2]} A(k,n-2k), \qquad \sum_{n\geqslant 0}\sum_{k\geqslant 0} B(k,n) = \sum_{n\geqslant 0}\sum_{k=0}^{n} B(k,n-k). \tag{9}$$

If B is a matrix in $\mathscr{C}^{r\times r}$ and n_0 is a positive integer we denote by $\mathscr{A}(B,n_0)$, $\mathscr{D}(B,n_0)$ and $\mathscr{E}(B,n_0)$ the real numbers:

$$\mathscr{A}(B,n_0) = \sum_{n=0}^{n_0}\sum_{k=0}^{[n/2]} \frac{(\|B\|)^{n-2k}}{k!(n-2k)!}, \qquad \mathscr{D}(B,n_0) = \sum_{n=0}^{n_0}\sum_{k=0}^{n} \frac{(\|B\|)^{2(n-k)}}{k!(2(n-k))!}, \tag{10}$$

$$\mathscr{E}(B,n_0) = \sum_{n=0}^{n_0}\sum_{k=0}^{n} \frac{(\|B\|)^{2(n-k)+1}}{k!(2(n-k)+1)!}. \tag{11}$$

This paper is organized as follows. In Section 2 some new properties of Hermite matrix polynomials are established. Section 3 deals with the Hermite matrix polynomial series expansions of e^{At}, $\sin(At)$ and $\cos(At)$ of an arbitrary matrix as well as with theirs finite series truncation with a prefixed accuracy in a bounded domain. Finally, in Section 4 analytic-numerical approximations of problems (1)–(3) are contructed in terms of Hermite matrix polynomial series expansions. Given an admissible error $\varepsilon>0$ and a bounded domain D, an approximation in terms of Hermite matrix polynomials is contructed so that the error with respect to the exact solution is uniformly upper bounded by ε in D.

2. On Hermite matrix polynomials

Let A be a positive stable matrix. By (6) it follows that

$$\mathrm{e}^{xt\sqrt{2A}+t^2 I} = \sum_{n\geqslant 0} \frac{H_n(x,A)}{n!}t^n.$$

Hence

$$\sum_{n\geqslant 0} \frac{(\sqrt{2A}x)^n}{n!}t^n = \left(\sum_{n\geqslant 0}\frac{t^{2n}}{n!}\right)\left(\sum_{n\geqslant 0}\frac{H_n(x,A)}{n!}t^n\right) \tag{12}$$

and by (9) one gets

$$\left(\sum_{n\geqslant 0}\frac{t^{2n}}{n!}\right)\left(\sum_{n\geqslant 0}\frac{H_n(x,A)}{n!}t^n\right)=\sum_{n\geqslant 0}\sum_{k=0}^{[n/2]}\frac{H_{n-2k}(x,A)}{k!(n-2k)!}t^n. \tag{13}$$

By identification of the coefficient of t^n in (12) and (13) one gets

$$x^nI=(\sqrt{2A})^{-n}\sum_{k=0}^{[n/2]}\frac{n!}{k!(n-2k)!}H_{n-2k}(x,A),\quad -\infty<x<+\infty. \tag{14}$$

Lemma 2.1. *Let be A a positive stable matrix, $K>2$ and $n\geqslant 0$ integer. Then*

$$\|H_n(x,A)\|_2\leqslant\sqrt{n!}\sqrt{2^n}K^n\mathrm{e}^{x^2},\quad |x|<\frac{K}{\|\sqrt{2A}\|_2}. \tag{15}$$

Proof. It is clear that for $n=0$ the inequality (15) holds true. Suppose that (15) is true for $k=0,1,\ldots,n$. Taking norms in the three terms formula (5) and using the induction hypothesis one gets

$$\begin{aligned}\|H_{n+1}(x,A)\|_2 &\leqslant |x|\ \|\sqrt{(2A)}\|_2\|H_n(x,A)\|_2+2n\|H_{n-1}(x,A)\|_2\\ &\leqslant K\sqrt{n!}\sqrt{2^n}K^n\mathrm{e}^{x^2}+2n\sqrt{(n-1)!}\sqrt{2^{n-1}}K^{n-1}\mathrm{e}^{x^2}\\ &=\sqrt{n!}\sqrt{2^n}K^{n-1}\mathrm{e}^{x^2}(K^2+\sqrt{2n}\,)\\ &=\sqrt{(n+1)!}\sqrt{2^n}K^{n-1}\mathrm{e}^{x^2}\left(\frac{K^2}{\sqrt{n+1}}+\sqrt{\frac{2n}{n+1}}\right).\end{aligned} \tag{16}$$

Since

$$\left(\frac{K^2}{\sqrt{n+1}}+\sqrt{\frac{2n}{n+1}}\right)\leqslant\left(\frac{K^2}{\sqrt{2}}+\sqrt{2}\right)=\frac{K^2+2}{\sqrt{2}}\leqslant\sqrt{2}K^2,$$

by (16) one gets (15) for $n+1$. □

The following result that may be regarded as a matrix version of the algorithm given in [10], provides an efficient procedure for computing finite matrix polynomial series expansions in terms of matrix polynomials satisfying a three terms formula. In particular, it is applicable to the Hermite matrix polynomials. Note the remarkable fact from a computational point of view that the evaluation of a matrix polynomial at $\bar{x}$ is only expressed in terms of $P_0(\bar{x})$.

Theorem 2.1. *Let $\{P_n(x)\}_{n\geqslant 0}$ be a sequence of matrix polynomials such that*

$$A_nP_n(x)=(xI-B_n)P_{n-1}(x)-C_nP_{n-2}(x),\quad n\geqslant 1,$$

where A_n is an invertible matrix in $\mathscr{C}^{r\times r}$ and the degree of $P_n(x)$ is n. Let $Q(x)$ be a matrix polynomial defined by

$$Q(x)=\sum_{j=0}^{n} E_jP_j(x),$$

where E_j is a matrix in $\mathscr{C}^{r\times r}$. Let $\bar{x}$ be any real number and consider the sequence of matrices defined by

$$D_n=E_n,$$

$$D_{n-1}=E_{n-1}+D_nA_n^{-1}(\bar{x}I-B_n),$$

and for $j=n-2,\ldots,0$,

$$D_j=E_j+D_{j+1}A_{j+1}^{-1}(\bar{x}I-B_{j+1})-D_{j+2}A_{j+2}^{-1}C_{j+2}.$$

Then $Q(\bar{x})=D_0P_0(\bar{x})$.

Proof. See [10].

3. Hermite matrix polynomials series expansions

We begin this section with Hermite matrix polynomial series expansion of $\exp(Bt)$, $\sin(Bt)$ and $\cos(Bt)$ for matrices satisfying the spectral property

$$|\mathrm{Re}(z)|>|\mathrm{Im}(z)| \quad \text{for all } z\in\sigma(B). \tag{17}$$

Theorem 3.1. *Let B be a matrix in $\mathscr{C}^{r\times r}$ satisfying* (17). *Then*

$$\mathrm{e}^{Bx}=e\sum_{n\geqslant 0}\frac{H_n(x,\frac{1}{2}B^2)}{n!}, \quad -\infty<x<+\infty, \tag{18}$$

$$\cos(Bx)=\frac{1}{e}\sum_{n\geqslant 0}\frac{(-1)^n}{(2n)!}H_{2n}\left(x,\frac{1}{2}B^2\right), \quad -\infty<x<+\infty, \tag{19}$$

$$\sin(Bx)=\frac{1}{e}\sum_{n\geqslant 0}\frac{(-1)^n}{(2n+1)!}H_{2n+1}\left(x,\frac{1}{2}B^2\right), \quad -\infty<x<+\infty. \tag{20}$$

Let $E(B,x,n)$, $C(B,x,n)$ and $S(B,x,n)$ be respectively the nth partial sum of series (18)–(20) *respectively. Given $\varepsilon>0$ and $c>0$, let $a=c+\varepsilon$ and let n_0, n_1 and n_2 be the first positive integers satisfying*

$$\mathscr{A}(Ba,n_0)\geqslant \exp(a\|B\|_2+1)-\frac{\varepsilon}{e}, \tag{21}$$

$$\mathscr{D}(Ba,n_1)\geqslant e(\cosh(a\|B\|_2)-\varepsilon), \tag{22}$$

$$\mathscr{E}(Ba,n_2)\geqslant e(\sinh(a\|B\|_2)-\varepsilon), \tag{23}$$

where $\mathscr{A}$, $\mathscr{D}$ *and* $\mathscr{E}$ *are defined by* (10) *and* (11). *Then for* $|x|<c$ *and* $n \geqslant \max\{n_0,n_1,n_2\}$ *one gets*

$$\|e^{Bx}-E(B,x,n)\|_2<\varepsilon \tag{24}$$

$$\|\cos(Bx)-C(B,x,n)\|_2<\varepsilon, \tag{25}$$

$$\|\sin(Bx)-S(B,x,n)\|_2<\varepsilon. \tag{26}$$

Proof. Let $A=\frac{1}{2}B^2$. By the spectral mapping theorem [3, p. 569] and (17) it follows that

$$\sigma(A)=\left\{\frac{1}{2}b^2; b\in\sigma(B)\right\},\quad \operatorname{Re}\left(\frac{1}{2}b^2\right)=\frac{1}{2}\{(\operatorname{Re}(b))^2-(\operatorname{Im}(b))^2\}>0,\quad b\in\sigma(B).$$

Thus A is a positive stable matrix and taking $t=1$ in (6), $B=\sqrt{2A}$, one gets

$$e^{Bx}=\sum_{n\geqslant 0}\frac{e}{n!}H_n\left(x,\frac{1}{2}B^2\right),\quad -\infty<x<+\infty.$$

If $n_0\geqslant 1$ one gets

$$\left\|e^{Bx}-\sum_{k=0}^{n_0}\frac{e}{k}H_{k!}\left(x,\frac{1}{2}B^2\right)\right\|_2\leqslant\sum_{k>n_0}\frac{e}{k!}\left\|H_k\left(x,\frac{1}{2}B^2\right)\right\|_2=e\sum_{k>n_0}\frac{\|H_k(x,\frac{1}{2}B^2)\|_2}{k!}. \tag{27}$$

By (7) for $\varepsilon>0$, $c>0$, $a=c+\varepsilon$ and $|x|<a$, it follows that

$$e\sum_{k>n_0}\frac{\|H_k(x,\frac{1}{2}B^2)\|_2}{k!}\leqslant e\sum_{k\geqslant 0}\frac{H_k(\frac{\|B\|_2ai}{2})}{k!i^k}-e\sum_{k=0}^{n_0}\frac{H_k(\frac{\|B\|_2ai}{2})}{k!i^k}. \tag{28}$$

By (8), (10), (27) and (28) one gets

$$\begin{aligned}\left\|e^{Bx}-\sum_{k=0}^{n_0}\frac{e}{k!}H_k\left(x,\frac{1}{2}B^2\right)\right\|_2&\leqslant\exp(a\|B\|_2+1)-e\sum_{k=0}^{n_0}\frac{H_k(\frac{\|B\|_2ai}{2})}{k!i^k}\\&=\exp(a\|B\|_2+1)-\mathscr{A}(Ba,n_0).\end{aligned} \tag{29}$$

By [12, p. 57] one gets

$$\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{l=0}^{[k/2]}\frac{(\|B\|_2a)^{k-2l}}{l!(k-2l)!}=\exp(a\|B\|_2+1).$$

Hence, taking the first positive integer n_0 satisfying (21), then by (29) one gets

$$\left\|e^{Bx}-\sum_{k=0}^{n_0}\frac{e}{k!}H_k\left(x,\frac{1}{2}B^2\right)\right\|_2\leqslant\varepsilon,\quad |x|<c,\ n\geqslant n_0.$$

Considering (14) for the positive stable matrix $A=\frac{1}{2}B^2$, it follows that

$$x^{2n}I=B^{-2n}\sum_{k=0}^{n}\frac{(2n)!}{k!(2(n-k))!}H_{2(n-k)}\left(x,\frac{1}{2}B^2\right).$$

Taking into account the series expansion of $\cos(Bx)$ and (9) we can write

$$\cos(Bx) = \sum_{n\geqslant 0} \frac{(-1)^n B^{2n} x^{2n}}{(2n)!} = \sum_{n\geqslant 0}\sum_{k=0}^{n} \frac{(-1)^n}{k!\,(2(n-k))!} H_{2(n-k)}\left(x, \frac{1}{2}B^2\right)$$

$$= \sum_{n\geqslant 0}\sum_{k\geqslant 0} \frac{(-1)^{n+k}}{k!\,(2n)!} H_{2n}\left(x, \frac{1}{2}B^2\right) = \sum_{n\geqslant 0}\left(\sum_{k\geqslant 0}\frac{(-1)^k}{k!}\right)\frac{(-1)^n}{(2n)!} H_{2n}\left(x, \frac{1}{2}B^2\right).$$

Hence

$$\cos(Bx) = \frac{1}{e}\sum_{n\geqslant 0}\frac{(-1)^n}{(2n)!} H_{2n}\left(x, \frac{1}{2}B^2\right), \quad -\infty < x < +\infty. \tag{30}$$

By [12, p. 57] one gets

$$\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(\|B\|_2 a)^{2(k-l)}}{l!(2(k-l))!} = e\ \cosh(a\|B\|_2).$$

Let us consider the n_1th partial sum of the series (30). Then in an analogous way to the previous computations, for $|x| < c$ one gets

$$\|\cos(Bx) - C(B,x,n_1)\|_2 \leqslant \cosh(a\|B\|_2) - \frac{1}{e}\mathscr{D}(aB,n_1), \quad a = c + \varepsilon,$$

where $\mathscr{D}(aB,n_1)$ is given by (10). Taking the first integer $n_1 \geqslant 1$ satisfying (22) one gets (25). By (14) we also have

$$x^{2n+1}I = B^{-(2n+1)}\sum_{k=0}^{n}\frac{(2n+1)!}{k!(2(n-k)+1)!} H_{2(n-k)+1}\left(x, \frac{1}{2}B^2\right).$$

and in an analogous way to the series expansion of $\cos(Bx)$, one gets (20). By [12, p. 57] one gets

$$\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(\|B\|_2 a)^{2(k-l)+1}}{l!(2(k-l)+1)!} = e\ \sinh(a\|B\|_2).$$

Then in an analogous way to the previous computations, one gets (26). Hence the result is established. □

Now we show that condition (17) can be removed in the construction of accurate Hermite matrix polynomial expansions of $\exp(At)$, $\cos(At)$ and $\sin(At)$ of an arbitrary matrix A in $\mathscr{C}^{r\times r}$.

Lemma 3.1. *Let A be a matrix in $\mathscr{C}^{r\times r}$ and let γ any positive number such that*

$$\gamma > \max\{|\mathrm{Re}(z)| + |\mathrm{Im}(z)|;\ z \in \sigma(A)\}.$$

Then the matrix $B = A + \gamma I$ satisfies (17).

Proof. By the spectral mapping theorem $\sigma(A+\gamma I) = \{z+\gamma;\ z\in\sigma(A)\}$ and for $z \in \sigma(A)$ one gets $|\mathrm{Re}(z+\gamma)| = |\mathrm{Re}(z)+\gamma| \geqslant \gamma - |\mathrm{Re}(z)| > |\mathrm{Im}(z)| = |\mathrm{Im}(z+\gamma)|$. Hence the result is established. □

Corollary 3.1. *Let A be a matrix in $\mathscr{C}^{r\times r}$ and let γ be a number satisfying the condition of Lemma* 3.1. *Let* $\varepsilon>0$, $c>0$, $a=c+\varepsilon$. *With the notacion of Theorem* 3.1 *it follows that*
(i) *If n_0 satisfies* (21) *for $B=A+\gamma I$, then*

$$\|e^{Ax}-e^{-\gamma x}E(A+\gamma I,x,n)\|_2<\varepsilon, \quad n\geqslant n_0, \quad |x|<c. \tag{31}$$

(ii) *Let n_1 and n_2 be positive integers satisfying* (22) *and* (23) *respectively for $\varepsilon/2=\varepsilon'$, and $B=A+\gamma I$, then for $|x|<c$ and $n\geqslant \max\{n_1,n_2\}$,*

$$\|\sin(Ax)-\{\cos(\gamma x)S(A+\gamma I,x,n)-\sin(\gamma x)C(A+\gamma I,x,n)\}\|_2<\varepsilon, \tag{32}$$

$$\|\cos(Ax)-\{\cos(\gamma x)C(A+\gamma I,x,n)+\sin(\gamma x)S(A+\gamma I,x,n)\}\|_2<\varepsilon. \tag{33}$$

Proof. (i) The result is a consequence of Theorem 3.1 and the formula $e^{Ax}=e^{-\gamma x}e^{(A+\gamma I)x}$. The proof of part (ii) follows from the expressions

$$\sin(Ax)=\sin[(A+\gamma I)x]\cos(\gamma x)-\cos[(A+\gamma I)x]\sin(\gamma x),$$

$$\cos(Ax)=\cos[(A+\gamma I)x]\cos(\gamma x)+\sin[(A+\gamma I)x]\sin(\gamma x),$$

and Theorem 3.1. □

In the following example we illustrate the use of Hermite matrix polynomials for computing e^A, $\sin A$ and $\cos A$ for a matrix A satisfying (17). As it has been proved in Corollary 3.1, condition (17) is not necessary taking an appropiate value of γ in accordance with Lemma 3.1. It is interesting to point out that algorithm of Theorem 2.1 has been adapted for the Hermite matrix polynomials using (5). Computations have been perfomed using Mathematica version 2.2.1.

Example 3.1. Consider the matrix

$$A=\begin{pmatrix} 3 & -1 & 1\\ 2 & 0 & 1\\ 1 & -1 & 2\end{pmatrix},$$

where $\sigma(A)=\{1,2,2\}$. Using the minimal theorem [3, p. 571] one gets the exact value of e^A:

$$e^A=\begin{pmatrix} 2e^2 & -e^2 & e^2\\ -e+2e^2 & e-e^2 & e^2\\ -e+e^2 & e-e^2 & e^2\end{pmatrix}$$

$$=\begin{pmatrix} 14.7781121978613 & -7.38905609893065 & 7.38905609893065\\ 12.05983036940225 & -4.670774270471604 & 7.38905609893065\\ 4.670774270471604 & -4.670774270471604 & 7.38905609893065\end{pmatrix}.$$

Given an admissible error $\varepsilon = 10^{-5}$, from (21), taking $n_0 = 30$ the approximation $E(A,1,30)$ provides the required accuracy

$$E(A,1,30) = e\sum_{n=0}^{30} \frac{1}{n!} H_n\left(1, \frac{1}{2}A^2\right)$$

$$= \begin{pmatrix} 14.77811219786323 & -7.389056098931726 & 7.389056098931725 \\ 12.0598303694045 & -4.670774270472999 & 7.389056098931723 \\ 4.670774270472778 & -4.670774270472778 & 7.389056098931503 \end{pmatrix},$$

$$\|e^A - E(A,1,30)\|_2 = 3.181762178061584 \times 10^{-12}.$$

Of course, in practice the number of terms required to obtain a prefixed accuracy uses to be smaller than the one provided by (21), because the error bounds given by Theorem 3.1 is valid for any matrix satisfying (17). So for instance taking $n_0 = 19$ one gets

$$E(A,1,19) = \begin{pmatrix} 14.77810950722812 & -7.389054626492605 & 7.389054626492605 \\ 12.05982687088079 & -4.670771990145276 & 7.389054626492603 \\ 4.670772244388193 & -4.670772244388193 & 7.389054880735518 \end{pmatrix},$$

$$\|e^A - E(A,1,19)\|_2 = 6.356409123149743 \times 10^{-6}.$$

In an analogous way we have

$$\sin A = \begin{pmatrix} \sin(2)+\cos(2) & -\cos(2) & \cos(2) \\ -\sin(1)+\sin(2)+\cos(2) & \sin(1)-\cos(2) & \cos(2) \\ -\sin(1)+\sin(2) & \sin(1)-\sin(2) & \sin(2) \end{pmatrix}$$

$$= \begin{pmatrix} 0.4931505902785393 & 0.4161468365471424 & -0.4161468365471424 \\ -0.3483203945293571 & 1.257617821355039 & -0.4161468365471424 \\ 0.06782644201778521 & -0.06782644201778521 & 0.909297426825682 \end{pmatrix}.$$

Taking $n_0 = 8$ one gets the Hermite matrix polynomial approximation

$$S(A,1,8) = \frac{1}{e}\sum_{n=0}^{8} \frac{(-1)^n}{(2n+1)!} H_{2n+1}\left(1, \frac{1}{2}A^2\right)$$

$$= \begin{pmatrix} 0.4931487870648995 & 0.4161486588860599 & -0.4161486588860598 \\ -0.3483221953348546 & 1.257619641285814 & -0.4161486588860596 \\ 0.0678264635512047 & -0.06782646355120475 & 0.90929744595096 \end{pmatrix},$$

with

$$\|\sin A - S(A,1,8)\|_2 = 4.446422404096298 \times 10^{-6}.$$

The number of terms provided by (23) for the accuracy $\varepsilon = 10^{-5}$ is $n_0 = 19$. Finally approximating $\cos A$ by Hermite matrix polynomial series we have

$$\cos A = \begin{pmatrix} \cos(2) - \sin(2) & \sin(2) & -\sin(2) \\ -\cos(1) + \cos(2) - \sin(2) & \cos(1) + \sin(2) & -\sin(2) \\ -\cos(1) + \cos(2) & \cos(1) - \cos(2) & \cos(2) \end{pmatrix}$$
$$= \begin{pmatrix} -1.325444263372824 & 0.909297426825682 & -0.909297426825682 \\ -1.865746569240964 & 1.449599732693821 & -0.909297426825682 \\ -0.956449142415282 & 0.956449142415282 & -0.4161468365471424 \end{pmatrix}.$$

Taking $n_0 = 8$ one gets the approximation

$$C(A,1,8) = \frac{1}{e}\sum_{n=0}^{8} \frac{(-1)^n}{(2n)!} H_{2n}\left(1, \frac{1}{2}A^2\right)$$
$$= \begin{pmatrix} -1.325448071525842 & 0.909299412639782 & -0.909299412639782 \\ -1.865751647963166 & 1.449602989077106 & -0.909299412639782 \\ -0.956452235323384 & 0.956452235323384 & -0.4161486588860598 \end{pmatrix},$$

with error

$$\|\cos A - C(A,1,8)\|_2 = 9.1659488356359 \times 10^{-6}.$$

For a prefixed accuracy $\varepsilon = 10^{-5}$, the expression (22) gives $n_0 = 14$.

4. Applications

In this section we construct matrix polynomial approximations of problems (1)–(3) expressed in terms of Hermite matrix polynomials. It is well known that the solution of problem (1) is

$$Y(x) = \mathrm{e}^{Ax} y_0.$$

Let γ be as in Lemma 3.1, $\varepsilon > 0$, $c > 0$, $a = c + \varepsilon$. Let n_0 be the first positive integer such that

$$\mathscr{A}((A + \gamma I)a, n_0) \geqslant \exp(a\|A + \gamma I\|_2 + 1) - \frac{\varepsilon}{e(1 + \|y_0\|_2)}.$$

Then by Corollary 3.1 it follows that

$$\|\mathrm{e}^{Ax} y_0 - \mathrm{e}^{-\gamma x} E(A + \gamma I, x, n) y_0\|_2 < \varepsilon, \quad |x| < c, \quad n \geqslant n_0.$$

Thus

$$\widetilde{Y}_n(x) = \mathrm{e}^{(1-\gamma x)} \left[\sum_{k=0}^{n} \frac{H_k(x, \frac{1}{2}(A + \gamma I)^2)}{k!}\right] y_0, \tag{34}$$

is an approximate solution of problem (1) such that if $Y(x)$ is the exact solution one gets

$$\|Y(x)-\widetilde{Y}_n(x)\|_2<\varepsilon, \quad |x|<c, \quad n\geqslant n_0. \tag{35}$$

Problem (2) can be solved considering the extended system

$$Z=\begin{bmatrix} Y \\ Y' \end{bmatrix}, \qquad Z'=\begin{bmatrix} 0 & I \\ -A^2 & 0 \end{bmatrix} Z; \qquad Z(0)=\begin{bmatrix} P \\ Q \end{bmatrix},$$

but such an approach increases the computational cost [1], and involves a lack of explicitness in terms of two vector parameters that is very interesting to study boundary value problems associated to (2) using the shooting method [9].

Consider problem (2) where A is an invertible matrix in $\mathscr{C}^{r\times r}$. By [6] the pair $\{\cos(Ax),\sin(Ax)\}$ is a fundamental set of solutions of the equation

$$Y''+A^2Y=0, \quad -\infty<x<+\infty,$$

because $Y_1(x)=\cos(Ax)$, $Y_2(x)=\sin(Ax)$ satisfy

$$W(0)=\begin{bmatrix} Y_1(0) & Y_2(0) \\ Y_1'(0) & Y_2'(0) \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \quad \text{is invertible in } \mathscr{C}^{2r\times 2r}.$$

Hence, the unique solution of problem (2) is given by

$$Y(x)=\cos(Ax)P+\sin(Ax)A^{-1}Q, \quad -\infty<x<+\infty. \tag{36}$$

If γ is given by Lemma 3.1, the expresion (36) and Corollary 3.1 suggest the approximation

$$\begin{aligned}\widetilde{Y}_n(x)&=\{\cos(\gamma x)C(A+\gamma I,x,n)+\sin(\gamma x)(A+\gamma I,x,n)\}P\\ &\quad+\{\cos(\gamma x)S(A+\gamma I,x,n)-\sin(\gamma x)C(A+\gamma I,x,n)\}A^{-1}Q\\ &=C(A+\gamma I,x,n)(\cos(\gamma x)P-\sin(\gamma x)A^{-1}Q)\\ &\quad+S(A+\gamma I,x,n)(\sin(\gamma x)P+\cos(\gamma x)A^{-1}Q),\end{aligned}$$

$$\begin{aligned}\widetilde{Y}_n(x)=&\left[\sum_{k=0}^{n}\frac{(-1)^kH_{2k}(x,\frac{1}{2}(A+\gamma I)^2)}{(2k)!}\right]\left(\cos(\gamma x)\frac{P}{e}-\sin(\gamma x)\frac{A^{-1}Q}{e}\right)\\ &+\left[\sum_{k=0}^{n}\frac{(-1)^kH_{2k+1}(x,\frac{1}{2}(A+\gamma I)^2)}{(2k+1)!}\right]\left(\sin(\gamma x)\frac{P}{e}+\cos(\gamma x)\frac{A^{-1}Q}{e}\right).\end{aligned} \tag{37}$$

By Corollary 3.1, for $n\geqslant\max\{n_1,n_2\}$, where n_1 and n_2 are given by the Corollary 3.1, one gets

$$\|Y(x)-\widetilde{Y}_n(x)\|_2\leqslant\varepsilon(\|P\|_2+\|A^{-1}\|_2\|Q\|_2), \quad |x|<c. \tag{38}$$

We conclude this section with the construction of Hermite matrix polynomials approximations of the solution of problem (3). By [2, p. 195] the solution of (3) is given by

$$X(t)=e^{A^{\prime}t}Ce^{Bt}. \tag{39}$$

Let $\gamma \geqslant \max\{\gamma_A, \gamma_B\}$ where

$$\gamma_A > \max\{|\mathrm{Re}(z) + |\mathrm{Im}(z)|;\ z \in \sigma(A)\},$$

and

$$\gamma_B > \max\{|\mathrm{Re}(z) + |\mathrm{Im}(z)|;\ z \in \sigma(B)\}.$$

Corollary 3.1 suggests the approximation

$$\widetilde{X}_n(t) = \mathrm{e}^{-\gamma t}E(A + \gamma I, t, n)C\mathrm{e}^{-\gamma t}E(B + \gamma I, t, n), \tag{40}$$

and note that

$$X(t) - \widetilde{X}_n(t) = (\mathrm{e}^{At} - \mathrm{e}^{-\gamma t}E(A + \gamma I, t, n))C\mathrm{e}^{Bt} + \mathrm{e}^{-\gamma t}E(B + \gamma I, t, n)C(\mathrm{e}^{Bt} - \mathrm{e}^{-\gamma t}E(B + \gamma I, t, n)). \tag{41}$$

Consider the domain $|t| < c$ and take

$$K = \max\{c(\|B\|_2 + \gamma), 2\} + 1. \tag{42}$$

By (15) and (18) it follows that $|t| < K/(\|B\|_2 + \gamma)$ and

$$\|E(B + \gamma I, t, n)\|_2 \leqslant \mathrm{e}^{c^2+1} \sum_{j \geqslant 0} \sqrt{\frac{(2K)^j}{j!}} = L. \tag{43}$$

Taking norms in (41) and using (43) one gets

$$\|X(t) - \widetilde{X}_n(t)\|_2 \leqslant \|\mathrm{e}^{At} - \mathrm{e}^{-\gamma t}E(A + \gamma I, t, n)\|_2 \|C\|_2 \mathrm{e}^{c\|B\|_2} + L\|C\|_2 \|\mathrm{e}^{Bt} - \mathrm{e}^{-\gamma t}E(B + \gamma I, t, n)\|_2. \tag{44}$$

Taking the first positive integer n_0' such that

$$\mathscr{A}((A + \gamma I)a, n_0') \geqslant \exp(w\|A + \gamma I\|_2 a + 1) - \frac{\varepsilon \mathrm{e}^{-c\|B\|_2}}{2\mathrm{e}(\|C\|_2 + 1)},$$

and the first positive integer n_1' such that

$$\mathscr{A}((B + \gamma I)a, n_1') \geqslant \exp(\|B + \gamma I\|_2 a + 1) - \frac{\varepsilon}{2L(\|C\|_2 + 1)e},$$

then for $n \geqslant \max\{n_0', n_1'\}$ and $|t| < c$, by Corollary 3.1 and (39), (40), (43) it follows that

$$\|X(t) - \widetilde{X}_n(t)\|_2 < \varepsilon, \quad |t| < c.$$

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