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ACYCLIC LOCALIZATIONS

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0. Introduction and statement of results

This paper studies the following problem: Given a projective chain complex C_* over a ring Λ, and a surjective homomorphism of rings $F:\Lambda\to\Lambda'$, what conditions must be satisfied by the homology of C_* so that $C_*\otimes_\Lambda\Lambda'$ will be *acyclic*? This problem arises naturally in connection with many topological questions. For example, in high-dimensional knot theory the integral homology of the complement vanishes above the first dimension while the homology with respect to suitable local coefficients may be very complex.

The most direct method of attacking this problem is to equate the E_∞-term of the universal coefficient spectral sequence to zero and consider what this implies about the E_2-term. This procedure has several drawbacks, not the least of which is that Λ' may be of *infinite* homological dimension over Λ so that the spectral sequence will have an infinite number of terms. Even when this spectral sequence collapses, one doesn't usually get an *explicit* (and easily verifiable) condition on the homology of C_*.

The present paper will show that, in many cases, the homology of such *relatively acyclic* complexes can be characterized in *torsion-theoretic* terms — i.e., a module will occur as a homology module of some relatively acyclic complex if and only if it is a *torsion module* in a suitable sense.

This characterization is applied in Section 3 to prove that in many cases *homology* surgery obstruction groups, defined in [3], are *canonically isomorphic* to suitable *ordinary* surgery obstruction groups, defined in [13].

Before we state our main result, note that all chain complexes will be assumed to be *finitely generated*, *projective*, and *bounded from below*, and the ring will be assumed to act on the right in this paper.

Theorem 1. *Let $f: G \to H$ be a surjective homomorphism of groups with H a finite extension of a polycyclic group and with kernel a finitely generated nilpotent group K. In addition, suppose the action of H on K defined by conjugation factors through a finite group.*[1] *Let I be the kernel of the homomorphism of group-rings $F: \mathbf{Z}G \to \mathbf{Z}H$ induced by f, and let C_* be a $\mathbf{Z}G$-chain complex. Then $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ is acyclic if and only if $C_* \otimes_{\mathbf{Z}G} \Lambda$ is acyclic, where $\Lambda = \mathbf{Z}G[S^{-1}]$ and S is the multiplicatively closed set of elements of $\mathbf{Z}G$ of the form $1+i$, $i \in I$. In particular, $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ is acyclic if and only if $H_i(C_*) \otimes_{\mathbf{Z}G} \Lambda = 0$ for all i and a finitely generated right $\mathbf{Z}G$-module, M, is a homology module of a chain complex that is acyclic over $\mathbf{Z}H$ if and only if for every $x \in M$ there exists $i \in I$ such that $x(1+i) = 0$.*

Remarks. This theorem will be proved in Section 2. We will also show that, under the hypotheses of this theorem, if S if the multiplicatively closed set of elements of $\mathbf{Z}G$ of the form $(1+i)$, $i \in I$ the classical ring of fractions, $\Lambda = \mathbf{Z}G[S^{-1}]$, is well defined. The proof of Theorem 1, in Section 2 of this paper, actually implies the corresponding statement for *left* complexes as well.

Corollary 2. *Let $f: G \to H$ be a surjective homomorphism of groups with H a finite group and with kernel K a finitely generated abelian group.*

A finitely generated right $\mathbf{Z}G$-module M is a homology module of a finitely generated projective complex that is acyclic over $\mathbf{Z}H$ if and only if there exists an element i of the augmentation ideal of K such that $M \cdot (1+i) = 0$.

Proof. First note that M is a finitely generated module over $\mathbf{Z}K$. If we regard $\mathbf{Z}G$ as a finitely generated free module over $\mathbf{Z}K$, we get $\mathbf{Z}H = \mathbf{Z}G \otimes_{\mathbf{Z}K} \mathbf{Z}$ as a left $\mathbf{Z}G$-module where $\mathbf{Z}K$ acts on $\mathbf{Z}$ via multiplication by the image under the · augmentation. This implies that if C_* is a finitely generated right projective complex over $\mathbf{Z}G$, $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ will be acyclic if and only if $C_* \otimes_{\mathbf{Z}K} \mathbf{Z}$ is acyclic, after regarding C_* as a complex over $\mathbf{Z}K$. It follows that M will satisfy the conditions of Theorem 1 with $G = K$ and H the trivial group. Consequently, M can be a homology module of a finitely generated projective complex over $\mathbf{Z}G$ that is acyclic over $\mathbf{Z}H$ if and only if every element of M is annihilated by an element of $\mathbf{Z}K$ of the form $(1+i)$, where i is contained in the augmentation ideal of $\mathbf{Z}K$. If $(1+i_j)$ are annihilators of a finite generating set for M over $\mathbf{Z}K$, the fact that K is *abelian* implies that the *product* of the $(1+i_j)$ will annihilate *all* of M.

The following corollary shows that, in special cases, the statement of Theorem 1 can be strengthened considerably.

Corollary 3. *Let $f: G \to H$ be a surjective homomorphism of groups with kernel isomorphic to $\mathbf{Z}$. If we regard $\mathbf{Z}[\mathbf{Z}]$ as the ring of Laurent polynomials in a variable t then a finitely generated right $\mathbf{Z}G$-module M can be a homology module of a*

[1] A recent result of J. Roseblade (Theorem A in "Applications of the Artin–Rees Lemma to Group Rings", Symposia Mathematica, XVII, 471–478) makes it possible to eliminate this condition entirely. I am grateful to Professor Patrick Smith for pointing this out to me.

projective chain complex C_* over $\mathbf{Z}G$ such that $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ is acyclic if and only if there exists a Laurent polynomial $p(t)$ such that $p(1) = \pm 1$ and $M \cdot p(t) = 0$.

Proof. The universal coefficient spectral sequence (see [4], p. 356) shows that, in this case, M can be a homology module of a relatively acyclic complex if and only if $M \otimes_{\mathbf{Z}G} \mathbf{Z}H = 0$ and the set of elements of M fixed by the action of t is zero. These conditions are equivalent to the condition that $(t-1)$ induce an automorphism of M. If $M \cdot p(t) = 0$ as in the conclusion of the theorem, $(t-1)$ will induce an automorphism of M since $(t-1) \mid p(t) - 1$ and $p(t) - 1$ must induce an automorphism of M. Thus the conditions on M are *sufficient.*

Now suppose that M is a finitely generated $\mathbf{Z}G$-module such that $(t-1)$ induces an automorphism of M. Let M_0 be the $\mathbf{Z}[\mathbf{Z}]$-submodule of M spanned by a generating set for M over $\mathbf{Z}G$ under the action of $\mathbf{Z}[\mathbf{Z}]$. Clearly $(t-1)$ will induce an automorphism of M_0 and this implies, by the universal coefficient spectral sequence (see [4], p. 356), that M_0 is a homology module of a finitely generated projective chain complex over $\mathbf{Z}[\mathbf{Z}]$ such that $C'_* \otimes_{\mathbf{Z}[\mathbf{Z}]} \mathbf{Z}$ is acyclic. Corollary 2 now implies that there exists an element $i_0 \in I_0$, where I_0 is the augmentation ideal of $\mathbf{Z}[\mathbf{Z}]$ such that $M_0 \cdot (1 + i_0) = 0$. If we set $p(t)$ to $(1 + i_0)$ the existence of a surjection of $\mathbf{Z}[\mathbf{Z}]$-modules $M_0 \otimes_{\mathbf{Z}[\mathbf{Z}]} \mathbf{Z}G \to M$ ($m_0 \otimes k$ maps to $m_0 \cdot k$, where $k \in \mathbf{Z}G$) implies that $M \cdot p(t) = 0$ and the conclusion follows.

Theorem 1 and its corollaries also have the following interesting consequence:

Corollary 4. *Let G and H be as in Theorem 1 or Corollaries 2 or 3. If M is a finitely generated right $\mathbf{Z}G$-module such that $\mathrm{Tor}_i^{\mathbf{Z}G}(M, \mathbf{Z}H) = 0$ for all i, and M' is any finitely generated submodule or quotient, then $\mathrm{Tor}_i^{\mathbf{Z}G}(M', \mathbf{Z}H) = 0$ for all i.*

The results of this paper, and particularly of Theorem 1 and its corollaries, will be applied to geometric problems in [8] and homotopy-theoretic problems in [9] — both of these applications arise in connection with the codimension-two placement problem for compact manifolds.

Another interesting consequence of Theorem 1 and its corollaries is that, in many cases, *homology* surgery obstruction groups are canonically isomorphic to suitable *ordinary* surgery obstruction groups. To make this statement precise we make the following definition:

Definition 5. The symbol S will denote one of the following multiplicatively closed sets of elements of $\mathbf{Z}G$:

(1) If G, K and H satisfy the conditions of Theorem 1, but not those of the corollaries, S consists of all elements of $\mathbf{Z}G$ of the form $1 + i$ where $i \in I$.

(2) If G, H and K satisfy the conditions of Corollary 2, but not Corollary 3, S consists of all elements of $\mathbf{Z}G$ of the form $1 + i'$ where $i' \in I'$, the augmentation ideal of $\mathbf{Z}k$.

(3) Otherwise, S consists of all Laurent polynomials (in the notation of Corollary 3) $p(t)$ such that $p(1) = \pm 1$.

The symbol Λ will denote $\mathbf{Z}G[S^{-1}]$ (whose existence is implied by Theorem 2.1) and $g:\Lambda\to\mathbf{Z}H$ will denote the canonical extension of F defined by the universal property of classical rings of quotients (see [12], p. 50).

Remark. In cases (2) and (3) one must couple Theorem 2.1 with an argument like that used in proposition 6.2 of [12] in order to prove the existence of Λ.

Let $L_i^B(\Lambda)$, B a subgroup of $K_1(\Lambda)$, be the group defined algebraically in [14], p. 286, and let $\Gamma_i^B(F)$ and $\Gamma_i^B(\mathbf{Z}G\to\Lambda)$, where A is a subgroup of $K_1(\mathbf{Z}H)$, be generalizations of the Γ-groups defined algebraically in Chapter I of [3] with Wh(F) and Wh($\mathbf{Z}G\to\Lambda$) replaced by $K_1(\mathbf{Z}H)/A$ and $K_1(\Lambda)/B$, respectively, in the definition. Recall that F is the homomorphism of integral group-rings induced by f.

Theorem 6. *Let G, H, S, F, Λ and g be as in Definition* 5 *and Theorem* 1 *and its corollaries. Let A be a subgroup of $K_1(\mathbf{Z}H)$ and let $B=\bar{g}^{-1}(A)$, where $\bar{g}:K_1(\Lambda)\to K_1(\mathbf{Z}H)$ is the map induced by g. Then the canonical homomorphisms $j_i:\Gamma_i^B(\mathbf{Z}G\to\Lambda)\to L_i^B(\Lambda)$, defined in Chapter* I *of* [3], *and the homomorphism $p_i:\Gamma_i^B(\mathbf{Z}G\to\Lambda)\cong L_i^B(\Lambda)\to\Gamma_i^A(F)$, induced by the identity map of $\mathbf{Z}G$ and $g:\Lambda\to\mathbf{Z}H$ are isomorphisms for all i.*

This will be proven in Section 3.

Remarks. (1) This result was known to Cappell and Shaneson in the case where all groups are *abelian* (though a proof has never appeared).

(2) Theorem 6 will be used to obtain a torsion-theoretic formulation for relative homology surgery theory in [10].

Some of the results in this paper are an extension of part of my doctoral dissertation and I would like to take this opportunity to thank my advisor, Professor Sylvain Cappell, for his encouragement and guidance in that work.

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1. The Artin–Rees condition

In this section we will verify that the ring $\mathbf{Z}G$ has a certain algebraic property that will be used to prove Theorem 1.

Let Λ be a ring and let $\mathfrak{a}$ be a two-sided ideal of Λ. Then $\mathfrak{a}$ is said to satisfy the right *Artin–Rees* condition if, for every right ideal $\mathfrak{b}$ of Λ, there exists an integer n such that $\mathfrak{b}\cap\mathfrak{a}^n\subset\mathfrak{b}\mathfrak{a}$. The ideal $\mathfrak{a}$ will be said to satisfy the *symmetric* Artin-Rees condition if it also satisfies the condition above with $\mathfrak{b}$ a *left* ideal and $\mathfrak{b}\mathfrak{a}$ replaced by $\mathfrak{a}\mathfrak{b}$.

Suppose $\mathfrak{a}$ is a finitely generated ideal of the ring Λ. Then it will be said to have a *centralizing set of generators* in Λ if there exists a set of elements $\{x_i\}$ of Λ such that $\mathfrak{a}=(x_1,\dots,x_n)$ and

(a) x_1 is contained in the center of Λ;

(b) $x_i + (x_1, \ldots, x_{i-1})$ is contained in the center of $\Lambda/(x_1, \ldots, x_{i-1})$.

In [5, 2.7], Nouazé and Gabriel have shown that an ideal in a noetherian ring that has a centralizing set of elements will satisfy the symmetric Artin–Rees condition. Throughout the remainder of this section and of Section 2, G, H, K, F and S will be as in part 1 of Definition 5:

(1) $f: G \to H$ is a surjective homomorphism of groups with kernel a finitely generated nilpotent group K such that the action of H on K induced by conjugation by elements of G factors through a finite group.

(2) $F: \mathbf{Z}G \to \mathbf{Z}H$ is the homomorphism induced by f with kernel I.

(3) S is the multiplicatively closed set of all elements of $\mathbf{Z}G$ of the form $1+i$, $i \in I$.

This section will show that the ideal I of $\mathbf{Z}G$ satisfies the symmetric Artin–Rees condition. This will be applied in Section 2 to prove Theorem 1. Since the action of H on K factors through a finite group, it follows that there exists a subgroup H' of H of finite index such that H' acts trivially on K. If $G' = f^{-1}(H')$, the definition of the action of H' on K implies that conjugation by elements of G' induces *inner automorphisms* of K.

Proposition 1.1. *Let G' be as defined above and let I_0 be the augmentation ideal of $\mathbf{Z}K$. Then the ideal $I' = I_0 \cdot \mathbf{Z}G' = \mathbf{Z}G' \cdot I_0$ has a centralizing set of generators in $\mathbf{Z}G'$.*

Proof. This is an immediate consequence of Lemma 2.3 of [11] which states that the ideal I_0 has a centralizing set of generators $\{x_i\}$ in $\mathbf{Z}K$. Since conjugation by elements of G' induces *inner automorphisms* of K, the $\{x_i\}$ will clearly be a centralizing set of generators for I' in $\mathbf{Z}G'$.

Proposition 1.2. *The ideal I satisfies the symmetric Artin–Rees condition in $\mathbf{Z}G$.*

Proof. We first remark that I' satisfies the symmetric Artin–Rees condition in $\mathbf{Z}G'$. To see this, note that G' is a finite extension of a polycyclic group since it is a subgroup of G, which is an extension of a finitely generated nilpotent group by a finite extension of a polycyclic group. The lemma on p. 136 of [6] implies that $\mathbf{Z}G'$ is a noetherian ring so that the property of I' follows from 2.7 of [5].

Since G' is of finite index in G, $\mathbf{Z}G$ can be regarded as a (right or left) free module of finite rank over $\mathbf{Z}G'$. Theorem 4.4 in Chapter VII of [12] implies that if M is any (right or left) $\mathbf{Z}G'$ — submodule of $\mathbf{Z}G$ (hence, in particular, any *ideal* of $\mathbf{Z}G$), there exists an integer n such that

(a) $((I')^n \cdot \mathbf{Z}G) \cap M \subset M \cdot I'$ if M is a *right* module, or

(b) $(\mathbf{Z}G \cdot (I')^n) \cap M \subset I' \cdot M$ if M is a *left* module.

The result follows from the fact that $I^k = \mathbf{Z}G \cdot (I')^k = (I')^k \cdot \mathbf{Z}G$ for all k.

2. The acyclic localization

In this section, we define the acyclic localization of the ring $\mathbf{Z}G$ and prove Theorem 1. A basic result of this section is:

Theorem 2.1. *Let S be the multiplicatively closed set of all elements of $\mathbf{Z}G$ of the form $1+i$ where $i \in I$. Then, under the hypotheses of Theorem 1 the classical rings of quotients $\mathbf{Z}G[S^{-1}]$ and $[S^{-1}]\mathbf{Z}G$, defined in Chapter II of [12], exist and are well defined. Furthermore:*

(1) *$\mathbf{Z}G[S^{-1}] \simeq [S^{-1}]\mathbf{Z}G$ as rings — henceforth, this ring will be denoted by Λ.*

(2) *The ring Λ is a flat (right and left) $\mathbf{Z}G$-module, where the $\mathbf{Z}G$-action is defined by multiplication by the image under the canonical homomorphism $j:\mathbf{Z}G \to \Lambda$.*

Proof. In order that the classical rings of quotients, $\mathbf{Z}G[S^{-1}]$ and $[S^{-1}]\mathbf{Z}G$, be defined, it is necessary and sufficient that S satisfy the following conditions (see [12], Chapter II, section1):

(1) S is multiplicatively closed, i.e, $x, y \in S$ implies that $xy \in S$.

(2) $0 \notin S$.

(3) S is a *right* divisor set, i.e., if $a \in \mathbf{Z}G$, $t \in S$, there exist $b \in \mathbf{Z}G$, $s \in S$ such that $sa = bt$.

(4) S is a *left* divisor set — this condition is the same as the preceding except that we require that $as = tb$.

(5) $sa = 0$ for $a \in \mathbf{Z}G$, $s \in S$ if and only if there exists $t \in S$ such that $at = 0$.

Statement (1) follows from the fact that $(1+i)(1+i') = (1+i+i'+ii')$ and if $i, i' \in I$, then $i+i'+ii'$ will be contained in I. Statement (2) follows from the fact that I is not the whole ring $\mathbf{Z}G$. Statements (3) and (4) follow from Theorem 1.1 of [11] and the fact that I satisfies the *symmetric* Artin–Rees condition in $\mathbf{Z}G$ — see Proposition 1.2. Statement (5) follows from the fact that $\mathbf{Z}G$ is a *noetherian* ring (by p. 136 of [6]), statements (3) and (4), and from Proposition 1.5 in Chapter II of [12]. The statement that $\mathbf{Z}G[S^{-1}]$ is isomorphic to $[S^{-1}]\mathbf{Z}G$ is just Corollary 1.3 in Chapter II of [12]. The statement that Λ is a *flat* module is Proposition 3.5 in Chapter II of [12].

Since the image of the set S under $F:\mathbf{Z}G \to \mathbf{Z}H$ consists of invertible elements of $\mathbf{Z}H$ (in fact, it consists of the *identity* element), it follows from the universal property of classical rings of quotients that there exists a *unique* homomorphism (see [12], Chapter II) $\hat{F}:\Lambda \to \mathbf{Z}H$ such that the diagram

$$\begin{array}{ccc} & \Lambda & \\ {}^{j}\nearrow & & \searrow^{\hat{F}} \\ \mathbf{Z}G & \xrightarrow[F]{} & \mathbf{Z}H \end{array}$$

commutes. Here j is the canonical homomorphism from $\mathbf{Z}G$ to $\mathbf{Z}G[S^{-1}]$. Since F is surjective, it follows that $\hat{F}$ is surjective and, in fact,

Proposition 2.2. *If Λ, $\hat{F}$ and j are as defined above:*

(1) $\ker \hat{F} = \hat{I} = \Lambda \cdot j(I) = j(I) \cdot \Lambda$;

(2) *the ideal $\hat{I}$ is contained in the Jacobson radical of Λ.*

Remark. The second statement describes the crucial property of the localization Λ.

Proof. By abuse of notation, we will use the symbols for elements of $\mathbf{Z}G$ to denote their images under j — thus elements of Λ are of the form as^{-1} where $a \in \mathbf{Z}G$ and $s \in S$, and the homomorphism F is defined by $\hat{F}(as^{-1}) = F(a)$. Statement 1 then follows from the fact that I is two-sided and the fact that the isomorphism between $\mathbf{Z}G[S^{-1}]$ and $[S^{-1}]\mathbf{Z}G$ commutes with j and $\hat{F}$ (by the universal property of classical rings of quotients). In order to prove the second statement, we must show that, if $\hat{i} \in \hat{I}$, then $1 + \hat{i}$ is invertible in Λ. Since statement 1 implies that $\hat{i}$ is of the form $i_1(1 + i_2)^{-1}$ with $i_1, i_2 \in I$ is follows that $(1 + i_2)(1 + i_1 + i_2)^{-1}$ is an inverse for $1 + \hat{i}$ and the conclusion follows.

The following proposition *proves* Theorem 1.

Proposition 2.3. *Let C_* be a finitely generated right projective chain complex over $\mathbf{Z}G$. Then $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ is acyclic if and only if $C_* \otimes_{\mathbf{Z}G} \Lambda$ is acyclic — this will happen if and only if $H_i(C_*) \otimes_{\mathbf{Z}G} \Lambda = 0$ for all i, i.e., the homology modules of C_* are Λ-torsion modules.*

Proof. We begin by proving the second statement. Since Λ is *flat* over $\mathbf{Z}G$, $H_i(C_* \otimes_{\mathbf{Z}G} \Lambda) = H_i(C_*) \otimes_{\mathbf{Z}G} \Lambda$ and this implies the second statement.

If $C_* \otimes_{\mathbf{Z}G} \Lambda$ is acyclic, the fact that $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H = (C_* \otimes_{\mathbf{Z}G} \Lambda) \otimes_\Lambda \mathbf{Z}H$ shows that $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ will be acyclic. Now suppose $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ is acyclic. Then all of its boundary maps will be split surjections and, if C_0 is the lowest dimensional nonvanishing chain module, we get the following commutative diagram:

$$\begin{array}{ccc} C_1 \bigotimes_{\mathbf{Z}G} \Lambda & \xrightarrow{1 \otimes \hat{F}} & C_1 \bigotimes_{\mathbf{Z}G} \mathbf{Z}H \\ {\scriptstyle \hat{\partial}} \big\downarrow & & {\scriptstyle \bar{\partial}} \big\downarrow \\ C_0 \bigotimes_{\mathbf{Z}G} \Lambda & \xrightarrow{1 \otimes \hat{F}} & C_0 \bigotimes_{\mathbf{Z}G} \mathbf{Z}H. \end{array}$$

Since $\bar{\partial}$ is surjective, Nakayama's Lemma and the fact that $\ker \hat{F}$ lies in the Jacobson radical of Λ (by Proposition 2.2) imply that $\hat{\partial}$ is also surjective, hence *split*. The conclusion follows by induction — i.e., the preceding argument shows that $C_* \otimes_{\mathbf{Z}G} \Lambda$ is chain-homotopy equivalent to a Λ-chain complex whose lowest dimensional nonvanishing chain module lies in dimension 1.

3. Applications to homology surgery theory

In this section we will prove Theorem 6. We will use the notation of Definition 5.

Proposition 3.1. *Let $\{s_i\}$ be a finite set of elements of S. Then there exist $\bar{s} \in S$ and $a_i \in \mathbf{Z}G$ such that $s_i a_i = \bar{s}$ for all i.*

Proof. Since S is a right divisor set for every $s \in S$ and $a \in \mathbf{Z}G$, there exist $b \in \mathbf{Z}G$ and $t \in S$ such that $sb = at$. If we set $s = s_1$ and $a = s_2$ we get a common multiple of s_1 and s_2 contained in S and the result follows by induction.

It is not hard to see that this implies that the canonical map $\mathbf{Z}G \to \Lambda$ is *locally epic* in the sense of [3], p. 288.

Proposition 3.2. *Let $\bar{g}: K_1(\Lambda) \to K_1(\mathbf{Z}H)$ be the map induced by $g: \Lambda \to \mathbf{Z}H$, defined in Definition 5. Then $\bar{g}$ is surjective and its kernel contains the classes of $K_1(\Lambda)$ represented by multiplication by elements of S. In particular, $S \subset B$ (see Theorem 6 for the definition of B).*

Proof. This is a direct consequence of the definition of the homomorphism g and Proposition 1.3 in Chapter IX of [1].

The following proposition proves the first part of Theorem 6:

Proposition 3.3. *The canonical homomorphisms $j_i: \Gamma_i^B(\mathbf{Z}G \to \Lambda) \to L_i^B(\Lambda)$, defined in Chapter I of [3], are isomorphisms.*

Proof. Suppose i is even. Lemmas 1.2 and 1.3 of Chapter I of [3] imply that it will suffice to prove that every special η-form (in the terminology of [3]) with *free* underlying module that maps to a kernel under j_i is *strongly equivalent to zero.* Let $x = (H, \varphi, \mu)$ be such a special η-form and let $\{y_i\}$ be a set of elements of H that map to a preferred basis of a subkernel of $x \otimes_{\mathbf{Z}G} \Lambda$. It follows, by the properties of a subkernel, that $\varphi(y_i, y_j) = b_{ij}$, and $\mu(y_i) = c_i$ where the $b_{ij}, c_i \in \ker \mathbf{Z}G \to \Lambda$.

Proposition 1.4 on p. 51 of [12] implies that there exist s_i, s_{ij} such that $c_i s_i = b_{ij} s_{ij} = 0$ and Proposition 3.1 implies that there exists a common multiple $\bar{s}$ of the s_i, s_{ij}, and clearly $c_i \bar{s} = b_{ij} \bar{s} = 0$. If we define $y'_i = y_i \bar{s}$ it follows, by Proposition 3.2, that the y'_i also map to a preferred basis of a subkernel of $x \otimes_{\mathbf{Z}G} \Lambda$. Since, by the identities Q1–Q6 on p. 286 of [3], $\mu(y'_i) = \varphi(y'_i, y'_j) = 0$, it follows that the span of y'_i is a pre-subkernel of x.

Now suppose i is odd. In this case j_i is known to be *injective* — see Chapter I, section 2 of [3]. The formation-theoretic interpretation of Wall groups in [7] and section I.2 of [3] imply that to prove the result, it will be sufficient to prove that every triple $(H; K_1, K_2)$, where H is a kernel over Λ, K_1 is a standard subkernel and K_2 is any other subkernel, is *of the form* $(H', K'_1, K'_2) \otimes_{\mathbf{Z}G} \Lambda$ where H' is a kernel over $\mathbf{Z}G$, K_1' is a standard subkernel, and K_2' is a *pre-subkernel*, modulo B-simple basis changes. Clearly $(H; K_1) = (H'; K_1') \otimes_{\mathbf{Z}G} \Lambda$ so that we need only construct K_2'. Let $\{x_j\}$ be a preferred basis of H and let $y_i = \sum_j x_j a_{ij} s_{ij}^{-1}$ be a basis of K_2. If $\bar{s}$ is a common multiple of the s_{ij} (which exists, by Proposition 3.1) it is clear that $K_2 \cdot \bar{s}$ is the image of a submodule K_2' of H'. Since, by Proposition 3.2, multiplication by $\bar{s}$ constitutes a simple basis change, the proposition follows by an argument similar to the one used in the even-dimensional case.

Now we will prove the remaining statements of Theorem 6 in a series of propositions and corollaries. Recall that $p_i : \Gamma_i^B(\mathbf{Z}G \to \Lambda) \to \Gamma_i^A(F)$ are the maps induced by the identity map of $\mathbf{Z}G$ and $g : \Lambda \to \mathbf{Z}H$ and that our objective is to prove that the p_i are isomorphisms for all i.

Proposition 3.4. *Let $x = (F, \varphi, \mu)$ satisfy conditions Q1–Q5 on p. 286 of* [3] *over $\mathbf{Z}G$ with F a free module and let C_* denote the following chain complex*:

$0 \to F \xrightarrow{\mathrm{ad}\,\varphi} \mathrm{Hom}_{\mathbf{Z}G}(F, \mathbf{Z}G) \to 0$, i.e., $C_0 = 0$, $C_1 = F$,

$C_2 = \mathrm{Hom}_{\mathbf{Z}G}(F, \mathbf{Z}G)$, $C_3 = 0$.

Then x defines an element of $\Gamma_{2k}^B(\mathbf{Z}G \to \Lambda)$ if and only if the complex $C_* \otimes_{\mathbf{Z}G} \Lambda$ is acyclic with Whitehead torsion in B with respect to preferred bases.

Note that the kernel of g is contained in the Jacobson radical of Λ because of the property of Λ proved in Theorem 1 or the appropriate corollary. It is necessary to point this out because, although this property of the kernel of g was proved to hold in Section 2 (and, in fact, is the basis of Theorem 1), the multiplicatively closed set S that we are using in Section 3 corresponds to one of the three cases of Definition 5 and may (in cases (2) and (3)) be *strictly smaller* than the set S used in Sections 1 and 2.

To prove the proposition it will clearly suffice to prove that the corresponding map of Wall groups $L_{2k-1}^B(\Lambda) \to L_{2k-1}^A(\mathbf{Z}H)$ is injective since the Γ-groups are subgroups of the Wall groups — see section 1.2 of [3]. Recall that if R is an arbitrary ring and $W \subset K_1(R)$ is a suitable subgroup, the Wall group $L_{2k-1}^W(R)$ is defined as $d^{-1}(W)/h(W)$ (see [14], p. 286), where $d : KU_1(R) \to K_1(R)$ maps an element to the automorphism of the underlying module and $h : K_1(R) \to KU(R)$ is the hyperbolic homorphism defined on p. 267 of [14]. Consider the commutative exact diagram:

$$\begin{array}{ccccc} K_1(g) & \longrightarrow & K_1(\Lambda) & \xrightarrow{g} & K_1(\mathbf{Z}H) \\ \downarrow r & & \downarrow h & & \downarrow h \\ KU_1(g) & \longrightarrow & KU_1(\Lambda) & \xrightarrow{g'} & KU_1(\mathbf{Z}H) \end{array}$$

where *all vertical maps* are *hyperbolic homomorphisms* and the rows are portions of the exact sequences induced in K-theory and KU-theory by g — see [2], Chapter

III, section 1. Theorem 7.9, p. 178 of [2] implies that, since $\ker g$ is contained in the Jacobson radical of Λ, the map r *is surjective*. A diagram chase using the fact that $\bar{g}$ is surjective (see Proposition 3.2) shows that $\bar{g}'^{-1}(h(A)) \subset h(B)$ and this completes the proof.

Proposition 3.7. *Let $x = (H, \varphi, \mu)$ represent an element of $\Gamma_{2k}(\mathbf{Z}G \xrightarrow{r} \theta)$, where θ is some ring and r is a local epimorphism, H is a free module, and suppose $\{y_i\}$ are a finite set of elements of H such that $\varphi(y_i, y_j) = \mu(y_i) = 0$ for all i and j. Let C_* denote the following chain complex: $0 \to F \xrightarrow{i} H \xrightarrow{a} \mathrm{Hom}_{\mathbf{Z}G}(F, \mathbf{Z}G) \to 0$, where F is the free module on the set $\{y_i\}$, i is the canonical map to the span of the $\{y_i\}$ in H, and a is defined by $a(h) = \sum \varphi(h, i(y_n))y_n^*$ for $h \in H$, where $\{y_i^*\}$ is a dual basis of* $\mathrm{Hom}_{\mathbf{Z}G}(F, \mathbf{Z}G)$ *corresponding to the $\{y_i\}$.*

Then the span of the $\{y_i\}$ is a pre-subkernel of x, with the $\{y_i\}$ mapping to a preferred basis of the corresponding subkernel of $x \otimes_{\mathbf{Z}G} \theta$, if and only if $C_* \otimes_{\mathbf{Z}G} \theta$ is simply acyclic.

Proof. This is a direct consequence of the definition of a pre-subkernel given on p. 287 of [3] and Lemma 5.3 on p. 47 of [13].

We are now in a position to complete the proof of Theorem 6.

Corollary 3.8. *The homomorphisms $p_{2k} : \Gamma^B_{2k}(\mathbf{Z}G \to \Lambda) \to \Gamma^A_{2k}(F)$ are injective.*

Recall that they have already been shown to be surjective in Corollary 3.5.

Proof. In view of Lemmas 1.2 and 1.3 in Chapter I of [3] it will suffice to prove that if $x = (F, \varphi, \mu)$ represents an element of $\Gamma^A_{2k}(F)$ that is strongly equivalent to zero, then x, *regarded as an element of $\Gamma^B_{2k}(\mathbf{Z}G \to \Lambda)$* (that x could be so regarded was proved in Corollary 3.5) is *still* strongly equivalent to zero. Let $\{y_i\}$ be elements of a pre-subkernel of x that map to a preferred basis of $x \otimes_{\mathbf{Z}G} \mathbf{Z}H$ — here we are regarding x as an element of $\Gamma^A_{2k}(F)$ and *assuming* that it is strongly equivalent to zero. If we form the complex C_* from x and the $\{y_i\}$, as in Proposition 3.7, it follows that $C_* \otimes_{\mathbf{Z}G} \mathbf{Z}H$ is *acyclic*. Theorem 1 or the appropriate corollary (see Definition 5) now implies that $C_* \otimes_{\mathbf{Z}G} \Lambda$ is *also acyclic* which, with Proposition 3.7, completes the proof of this proposition.

Corollary 3.9. *The homomorphisms*

$$p_{2k-1} : \Gamma^B_{2k-1}(\mathbf{Z}G \to \Lambda) \to \Gamma^A_{2k-1}(F)$$

are surjective.

These homomorphisms have been proved to be injective in Proposition 3.6.

Proof. Here it will be convenient to use the *formation-theoretic* interpretation of odd-dimensional Γ-groups — see [7] and Chapter I, section 2 of [3]. An element of

$\Gamma^A_{2k-1}(F)$ is represented by a triple $(H; K_1, K_2)$ where H is a kernel over $\mathbf{Z}H$, K_1 is a standard subkernel, and K_2 is some other subkernel and, by the definition of $\Gamma^A_{2k-1}(F)$, $(H; K_1, K_2) = (H'; K'_1, K'_2) \otimes_{\mathbf{Z}G} \mathbf{Z}H$ where H' is a kernel over $\mathbf{Z}G$ with a standard subkernel K'_1 and K'_2 is a *pre-subkernel* with respect to the map F. It will clearly suffice to show that $(H'; K'_1, K'_2)$ also represents an element of $\Gamma^B_{2k-1}(\mathbf{Z}G \to \Lambda)$. The only way it could fail to represent an element of this Γ-group is that K'_2 could fail to be a pre-subkernel with respect to the map $\mathbf{Z}G \to \Lambda$, i.e., $K'_2 \otimes_{\mathbf{Z}G} \Lambda$ might not be a *direct summand* of $H' \otimes_{\mathbf{Z}G} \Lambda$. That this *does not* happen follows from Proposition 3.7 by an argument *exactly like* that used in the proof of Corollary 3.8.

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