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On Free Conformal and Vertex Algebras

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Vertex algebras and conformal algebras have recently attracted a lot of attention due to their connections with physics and Moonshine representations of the Monster. See, for example, [6, 10, 17, 15, 19].

In this paper we describe bases of free conformal and free vertex algebras (as introduced in [6]; see also [20]).

All linear spaces are over a field $\Bbbk$ of characteristic 0. Throughout this paper $\mathbb{Z}_+$ will stand for the set of non-negative integers.

In Sections 1 and 2 we give a review of conformal and vertex algebra theory. All statements in these sections are either in [9, 17, 16, 15, 18, 20] or easily follow from results therein. In Section 3 we investigate free conformal and vertex algebras.

1. CONFORMAL ALGEBRAS

1.1. *Definition of Conformal Algebras*

We first recall some basic definitions and constructions; see [16, 15, 18, 20]. The main object of investigation is defined as follows:

DEFINITION 1.1. A *conformal algebra* is a linear space C endowed with a linear operator D: $C \to C$ and a sequence of bilinear products $\textcircled{n}$: $C \otimes C \to C$, $n \in \mathbb{Z}_+$, such that for any $a, b \in C$ one has

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(i) (Locality) There is a non-negative integer $N = N(a,b)$ such that $a \circled{n} b = 0$ for any $n \geq N$;

(ii) $D(a \circled{n} b) = (Da) \circled{n} b + a \circled{n} (Db)$;

(iii) $(Da) \circled{n} = -na \circled{n-1} b$.

1.2. *Spaces of Power Series*

Now let us discuss the main motivation for Definition 1.1. We closely follow [14, 18].

1.2.1. *Circle Products*

Let A be an algebra. Consider the space of power series $A[[z, z^{-1}]]$. We will write series $a \in A[[z, z^{-1}]]$ in the form

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \qquad a(n) \in A.$$

On $A[[z, z^{-1}]]$ there is an infinite sequence of bilinear products $\circled{n}$, $n \in \mathbb{Z}_+$, given by

$$\big(a \circled{n} b\big)(z) = \operatorname{Res}_w\big(a(w)b(z)(z-w)^n\big). \tag{1.1}$$

Explicitly, for a pair of series $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$ and $b(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n-1}$ we have

$$\big(a \circled{n} b\big)(z) = \sum_m \big(a \circled{n} b\big)(m) z^{-m-1},$$

where

$$\big(a \circled{n} b\big)(m) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} a(n-s) b(m+s). \tag{1.2}$$

There is also the linear derivation $D = d/dz$: $A[[z, z^{-1}]] \to A[[z, z^{-1}]]$. It is easy to see that D and $\circled{n}$ satisfy conditions (ii) and (iii) of Definition 1.1.

We can consider formula (1.2) as a system of linear equations with unknowns $a(k)b(l)$, $k \in \mathbb{Z}_+$, $l \in \mathbb{Z}$. This system is triangular, and its unique solution is given by

$$a(k)b(l) = \sum_{s=0}^{k} \binom{k}{s} \big(a \circled{s} b\big)(k+l-s). \tag{1.3}$$

Remark. The term "circle products" appears in [18], where the product "$\textcircled{n}$" is denoted by "$\circ_n$." In [15] this product is denoted by "$_{(n)}$."

1.2.2. *Locality*

Next we define a very important property of power series, which makes them form a conformal algebra. Let again A be an algebra.

DEFINITION 1.2 (See [1, 17, 15, 18, 20].) A series $a \in A[[z, z^{-1}]]$ is called *local of order N* to $b \in A[[z, z^{-1}]]$ for some $N \in \mathbb{Z}_+$ if

$$a(w)b(z)(z-w)^N = 0. \tag{1.4}$$

If a is local to b and b is local to a then we say that a and b are *mutually local*.

Remark. In [18, 20] the property (1.4) is called *quantum commutativity*.

Note that (1.4) implies that for every $n \geq N$ one has $a \textcircled{n} b = 0$. We will denote the order of locality by $N(a, b)$, i.e.,

$$N(a,b) = \min\left\{n \in Z_+ \mid \forall k \geq n, a \textcircled{k} b = 0\right\}.$$

Note also that if A is a commutative or skew-commutative or skew-commutative algebra, e.g., a Lie algebra, then locality is a symmetric relation. In this case we say "a and b are local" instead of "mutually local."

Let $a(z) = \sum_{m \in \mathbb{Z}} a(m) z^{-m-1}$ and $b(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n-1}$ be some series. Then the locality condition (1.4) reads

$$\sum_s (-1)^s \binom{N}{s} a(m-s)b(n+s) = 0 \qquad \text{for any } n, m \in \mathbb{Z}. \tag{1.5}$$

The locality condition (1.4) is known to be equivalent to the formula

$$a(m)b(n) = \sum_{s=0}^{N(a,b)-1} \binom{m}{s} (a \textcircled{s} b)(m+n-s). \tag{1.6}$$

The following statement is a trivial consequence of the definitions.

PROPOSITION 1.1. *Let A be an algebra and let $S \subset A[[z, z^{-1}]]$ be a space of pairwise mutually local power series, which is closed under all the circle products and ∂. Then S is a conformal algebra.*

One can prove (see, for example, [15]) that such families exhaust all conformal algebras.

Finally, we state here a trivial property of local series:

LEMMA 1.1. *Let* $a, b \in A[[z, z^{-1}]]$ *be a pair of formal power series and assume* a *is local to* b. *Then each of the series* a, Da, za *is local to each of* b, Db, zb.

1.3. *Construction of the Coefficient Algebra of a Conformal Algebra*

Given a conformal algebra C, we can build its *coefficient algebra* Coeff C in the following way. For each integer n take a linear space $\hat{A}(n)$ isomorphic to C. Let $\hat{A} = \oplus_{n \in \mathbb{Z}} \hat{A}(n)$. For an element $a \in C$ we will denote the corresponding element in $\hat{A}(n)$ by $a(n)$. Let $E \subset \hat{A}$ be the subspace spanned by all elements of the form

$$(Da)(n) + na(n-1) \qquad \text{for any } a \in C, n \in \mathbb{Z}. \tag{1.7}$$

The underlying linear space of Coeff C is $\hat{A}/E$. By an abuse of notation we will denote the image of $a(n) \in \hat{A}$ in Coeff C again by $a(n)$. The following proposition defines the product on Coeff C.

PROPOSITION 1.2. *Formula* (1.6) *unambiguously defines a bilinear product on* Coeff C.

Clearly (1.6) defines a product on $\hat{A}$. To show that the product is well defined on Coeff C, it is enough to check only that

$$(Da)(m)b(n) = -ma(m-1)b(n)$$

and

$$a(m)(Db)(n) = -na(m)b(n-1),$$

which is a straightforward calculation.

1.4. *Examples of Conformal Algebras*

1.4.1. *Differential Algebras*

Take a pair (A, δ), where A is an associative algebra and $\delta: A \to A$ is a locally nilpotent derivation,

$$\delta(ab) = \delta(a)b + a\delta(b), \qquad \delta^n(a) = 0 \qquad \text{for } n \gg 0.$$

Consider the ring $A[\delta, \delta^{-1}]$. Its elements are polynomials of the form $\sum_{i \in \mathbb{Z}} a_i \delta^i$, where only a finite number of $a_i \in A$ are nonzero. Here we put $a\delta^{-n} = a(\delta^{-1})^n$ and $a\delta^0 = a$. The multiplication is defined by the formula

$$a\delta^k \cdot b\delta^l = \sum_{i \geq 0} \binom{k}{i} a\delta^i(b)\delta^{k+l-i}.$$

It is easy to check that $A[\delta, \delta^{-1}]$ is a well-defined associative algebra. In fact, $A[\delta, \delta^{-1}]$ is the Ore localization of the ring of differential operators $A[\delta]$. If in addition A has an identity element 1, then $\delta(1) = 0$ and $\delta\delta^{-1} = \delta^{-1}\delta = 1$.

For $a \in A$ denote $\tilde{a} = \sum_{n\in\mathbb{Z}} a\delta^n z^{-n-1} \in A[\delta, \delta^{-1}][[z, z^{-1}]]$.

One easily checks that for any $a, b \in A$, $\tilde{a}$ and $\tilde{b}$ are local and

$$\tilde{a}\,\textcircled{n}\,\tilde{b} = \widetilde{a\delta^n(b)}. \tag{1.8}$$

So by Lemma 1.1 and Proposition 1.1 the series $\{\tilde{a} \mid a \in A\} \subset A[\delta, \delta^{-1}][[z, z^{-1}]]$ generate an (associative) conformal algebra; see Section 1.6.

One can instead consider $A[\delta, \delta^{-1}]$ to be a Lie algebra, with respect to the commutator $[p, q] = pq - qp$. If two series $\tilde{a}$ and $\tilde{b}$ are local in the associative sense they are local in the Lie sense, too. One computes also

$$\tilde{a}\,\textcircled{n}\,\tilde{b} = \widetilde{a\delta^n(b)} - \sum_{s\geq 0} (-1)^{n+s} \frac{1}{s!} \partial^s\big(b\delta^{n+s}(a)\big)^{\sim}, \tag{1.9}$$

where $\partial = d/dz$. Note that in (1.9) the circle products are defined by

$$\big(\tilde{a}\,\textcircled{n}\,\tilde{b}\big)(m) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} [a\delta^{n-s}, b\delta^{m+s}]. \tag{1.10}$$

Again, it follows that $\{\tilde{a} \mid a \in A\} \subset A[\delta, \delta^{-1}][[z, z^{-1}]]$ generate a (Lie) conformal algebra; see Section 1.6.

An important special case is when there is an element $v \in A$ such that $\delta(v) = 1$. Then $\tilde{v} = \sum_n v\delta^n z^{-n-1} \in A[\delta, \delta^{-1}][[z, z^{-1}]]$ generates with respect to the product (1.10) a (centerless) Virasoro conformal algebra. It satisfies the relations

$$\tilde{v}\,\textcircled{0}\,\tilde{v} = \partial\tilde{v}, \qquad \tilde{v}\,\textcircled{1}\,\tilde{v} = 2\tilde{v},$$

and the rest of the products are 0.

1.4.2. *Loop Algebras*

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\Bbbk$ and let σ: $\mathfrak{g} \to \mathfrak{g}$ be an automorphism of finite order, $\sigma^p = \mathrm{id}$. Then $\mathfrak{g}$ is decomposed into a direct sum of eigenspaces of σ:

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}/p\mathbb{Z}} \mathfrak{g}_k, \qquad \sigma|_{\mathfrak{g}_k} = e^{2\pi ik/p}.$$

Define a *twisted loop algebra* $\tilde{\mathfrak{g}} \subset \mathfrak{g} \otimes \Bbbk[t, t^{-1}]$ by

$$\tilde{\mathfrak{g}} = \left\{ \sum_j a_j t^j \,\middle|\, a_j \in \mathfrak{g}_{j \bmod p} \right\}.$$

The Lie product in $\tilde{\mathfrak{g}}$ is given by $[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n}$. If $p = 1$, then $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \Bbbk[t, t^{-1}]$, of course.

Now for any $a \in \mathfrak{g}_k$, $0 \le k < p$, define

$$\tilde{a} = \sum_{j \in \mathbb{Z}} a t^{pj+k} z^{-j-1} \in \tilde{\mathfrak{g}}[[z, z^{-1}]].$$

It is easy to see that any two $\tilde{a}, \tilde{b}$ are local with $N(\tilde{a}, \tilde{b}) = 1$ and if $a \in \mathfrak{g}_k$ and $b \in \mathfrak{g}_l$ we have

$$\tilde{a} \textcircled{0} \tilde{b} = \begin{cases} \widetilde{[a, b]} & \text{if } k + l < p, \\ z\widetilde{[a, b]} & \text{if } k + l \ge p. \end{cases}$$

As in Section 1.4.1, we conclude that $\{\tilde{a} \mid a \in \mathfrak{g}\} \subset \tilde{\mathfrak{g}}[[z, z^{-1}]]$ generate a (Lie) conformal algebra. Again, see Section 1.6 for the definition of varieties of conformal algebras.

1.5. *More on Coefficient Algebras*

Let C be a conformal algebra and let $A = \operatorname{Coeff} C$. Define

$$A_+ = \operatorname{Span}\{a(n) \mid a \in C, n \ge 0\},$$

$$A_- = \operatorname{Span}\{a(n) \mid a \in C, n < 0\},$$

$$A(n) = \operatorname{Span}\{a(n) \mid a \in C\}.$$

Define also for each $n \in \mathbb{Z}$ linear maps $\phi(n)$: $C \to A(n)$ by $a \mapsto a(n)$, and let $\phi = \sum_{n \in \mathbb{Z}} \phi(n) z^{-n-1}$: $C \to A[[z, z^{-1}]]$ so that $\phi a = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$.

Here we summarize some general properties of conformal algebras and their coefficient algebras.

PROPOSITION 1.3. (a) $A = A_- \oplus A_+$ *is a direct sum of subalgebras.*

(b) A_+ *and* A_- *are filtered algebras with filtrations given by*

$$A(0) \subseteq A(1) \subseteq \cdots \subseteq A_+, \qquad A_- = A(-1) \supseteq A(-2) \supseteq \cdots,$$

$$\bigcup_{n \ge 0} A(n) = A_+, \qquad \bigcap_{n < 0} A(n) = 0.$$

(c) $$\operatorname{Ker} \phi(n) = \begin{cases} D^{n+1} C + \bigcup_{k \ge 1} \operatorname{Ker} D^k & \textit{if } n \ge 0, \\ \operatorname{Ker} D^{-n-1} & \textit{if } n < 0. \end{cases}$$

In particular, $\phi(-1)$ *is injective.*

(d) *The map* $\phi\colon C \to A[[z, z^{-1}]]$, *given by* $a \mapsto \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$, *is an injective homomorphism of conformal algebras*; *i.e., it preserves the circle products and agrees with the derivation*

$$\phi\big(a \textcircled{n} b\big) = \phi(a) \textcircled{n} \phi(b), \qquad \phi(Da) = D\phi(a). \tag{1.11}$$

(e) *The map* $\phi\colon C \to A[[z, z^{-1}]]$ *has the following universal property*: *For any homomorphism* $\psi\colon C \to B[[z, z^{-1}]]$ *of* C *to an algebra of formal power series, there is the unique algebra homomorphism* $\rho\colon A \to B$ *such that the corresponding diagram commutes*

$$\begin{array}{ccc} A\big[[z, z^{-1}]\big] & \xrightarrow{\ \rho\ } & B\big[[z, z^{-1}]\big] \\ {\scriptstyle\phi}\nwarrow & & \nearrow{\scriptstyle\psi} \\ & C & \end{array}$$

(f) *The formula* $D(a(n)) = -na(n-1)$ *defines a derivation* $D\colon A \to A$ *such that* $DA_- \subset A_-$, $DA_+ \subset A_+$.

Proof. From formula (1.6) for the product in A it easily follows that A_+ and A_- are indeed subalgebras. Also none of the linear identities (1.7) contains both generators with negative and non-negative index. This proves (a). Similar arguments establish also (b).

Now we prove that $\operatorname{Ker} \phi(n)$ is included in the right-hand side of (c). Take some $a \in C$, $a \neq 0$, and assume that $a(n) = 0$. Then $a(n) \in \hat{A}$ is a linear combination of identities (1.7) (see Section 1.3) so we must have in $\hat{A}$

$$a(n) = \sum_{k=k_{\min}}^{k_{\max}} \lambda_k\big((Da_k)(k) + ka_k(k-1)\big).$$

We can assume that $\lambda_k \neq 0$ for all $k_{\min} \leq k \leq k_{\max}$ and that $a_k \neq 0$ for $k = k_{\min}$ and for $k = k_{\max}$. Assume also that $\lambda_k = 0$ if $k > k_{\max}$ or $k < k_{\min}$.

Comparing terms with index k for $k_{\min} \leq k \leq k_{\max}$, we get

$$\delta_{kn} a = \lambda_k Da_k + \lambda_{k+1}(k+1) a_{k+1}. \tag{1.12}$$

Taking in (1.12) $k = k_{\min} - 1$, we see that there are two cases: Either (1) $k_{\min} = 0$ and $n \geq 0$ or (2) $n + 1 = k_{\min} \neq 0$.

Case 1. Taking in (1.12) $k = 0, \dots, n-1$, we get that $a_k \in D^k C$ for $0 \leq k \leq n$. Now we have two subcases: $k_{\max} > n$ and $k_{\max} \leq n$.

If $k_{\max} > n$ we substitute in (1.12) $k = k_{\max}, k_{\max} - 1, \dots, n + 1$ and get that $D^{k_{\max}-k+1}a_k = 0$. Now take $k = n$ in (1.12) and get that $a \in D^{n+1}C + \operatorname{Ker} D^{k_{\max}-n}$.

If $k_{\max} \leq n$ we have $\lambda_{n+1} = 0$, and hence substitution $k = n$ in (1.12) gives $a \in D^{n+1}C$.

Case 2. Here we again have two subcases: $n \geq 0$ or $n < 0$.

If $n \geq 0$ then, as in the previous case, we get $D^{k_{\max}-k+1}a_k = 0$ for $n + 1 \leq k \leq k_{\max}$. Now taking $k = n$ in (1.12), we get $a \in \operatorname{Ker} D^{k_{\max}-n}$.

Finally, if $n < 0$ then, since $\lambda_n = 0$, we have $a = \lambda_{n+1}(n+1)a_{n+1}$. Then we substitute $k = n + 1, n + 2, \dots$ into (1.12) until for some $k_0 \leq -1$ we get $\lambda_{k_0+1}(k_0 + 1)a_{k_0+1} = 0$. It follows that $D^{k_0-k+1}a_k = 0$ for $n + 1 \leq k \leq k_0$; therefore $a \in \operatorname{Ker} D^{k_0-n} \subset \operatorname{Ker} D^{-n-1}$. This proves one inclusion in (c). It also follows that $\operatorname{Ker} \phi(-1) = 0$.

Next we show that ϕ is a homomorphism of conformal algebras, that is, formulas (1.11) hold. For the first identity we have

$$\begin{aligned}\phi(Da) &= \sum_{n\in\mathbb{Z}} (Da)(n) z^{-n-1} = \sum_{n\in\mathbb{Z}} (-n) a(n-1) z^{-n-1}\\ &= \frac{d}{dz}\sum_n a(n) z^{-n-1}.\end{aligned}$$

The second identity reads

$$\big(a\,\textcircled{n}\,b\big)(m) = \sum_s (-1)^s \binom{n}{s} a(n-s) b(m+s),$$

which is precisely the formula (1.2).

Now (d) is done after we notice that ϕ is injective, since $\phi(-1)$ is injective.

Now we can prove the other inclusion in (c). If $a \in \operatorname{Ker} D^k C$, then ϕa is a solution of the differential equation $\partial_z^k \phi a(z) = 0$. Hence ϕa is a polynomial of degree at most $k - 1$, and therefore $\phi(n)a = 0$ for $n \geq 0$ and for $n < -k$. If $a \in D^k C$, then $\phi(n)a = 0$ for $0 \leq n \leq k - 1$, by induction and (1.7).

Statement (e) is clear, since identities (1.7) hold for any homomorphism $\psi\colon C \to B[[z, z^{-1}]]$.

Finally, the formula $D(a(n)) = -na(n-1)$ defines a derivative of $\hat{A}$. So in order to prove (f) we have to show that D agrees with the identities (1.7). This is indeed the case:

$$D((Da)(n) + na(n-1)) = -n((Da)(n-1) + (n-1)a(n-2)).$$

∎

1.6. *Varieties of Conformal Algebras*

Consider now a variety of algebras $\mathfrak{A}$ (see [8, 13]).

DEFINITION 1.3. A conformal algebra C is a $\mathfrak{A}$-*conformal algebra* if $\operatorname{Coeff} C$ lies in the variety $\mathfrak{A}$.

The identities in $\mathfrak{A}$-conformal algebras are all the circle product identities R such that for any integer m, $R(m)$ becomes an $\mathfrak{A}$-algebra identity after substitution of (1.2) for every circle product in R. Conversely, given a classical algebra identity r, we can substitute (1.6) for all products in r and get an identity of $\mathfrak{A}$-conformal algebras. This way we get a correspondence between classical and conformal identities. See the next section for examples.

Combining Propositions 1.1 and 1.3(d), we get the following well-known fact:

PROPOSITION 1.4. *$\mathfrak{A}$-conformal algebras are exhausted (up to isomorphism) by conformal algebras of formal power series $S \subset A[[z, z^{-1}]]$ for $\mathfrak{A}$-algebras A.*

1.7. *Associative and Lie Conformal Algebras*

The following theorem gives the explicit correspondence between conformal and classical algebras in some important cases.

THEOREM 1.1 [16]. *Let C be a conformal algebra and let $A = \operatorname{Coeff} C$ be its coefficient algebra.*

(a) *A is associative if and only if the following identity holds in C:*

$$\left(a \textcircled{n} b\right) \textcircled{m} c = \sum_{s=0}^{n} (-1)^s \binom{n}{s} a \textcircled{$n-s$} \left(b \textcircled{$m+s$} c\right). \tag{1.13}$$

(b) *The Jacoby identity $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ in A is equivalent to the following conformal Jacoby identity in C:*

$$\left(a \textcircled{n} b\right) \textcircled{m} c = \sum_{s=0}^{n} (-1)^s \binom{n}{s} \times \left(a \textcircled{$n-s$} \left(b \textcircled{$m+s$} c\right) - b \textcircled{$m+s$} \left(a \textcircled{$n-s$} c\right)\right). \tag{1.14}$$

(c) *The skew-commutativity identity* $[a, b] = -[b, a]$ *in* A *corresponds to the quasi-symmetry identity*

$$a \circled{n} b = \sum_{s \geq 0} (-1)^{n+s+1} \frac{1}{s!} D^s (b \circled{n+s} a). \tag{1.15}$$

(d) *The commutativity of* A *is equivalent to*

$$a \circled{n} b = \sum_{s \geq 0} (-1)^{n+s} \frac{1}{s!} D^s (b \circled{n+s} a). \tag{1.16}$$

The identities (1.13), (1.14), and (1.15) immediately imply the following:

COROLLARY 1.1. *Let* C *be a Lie conformal or an associative conformal algebra and let* $A = \operatorname{Coeff} C$ *be its coefficient algebra. Then* C *is an* A_+*-module with the action given by* $a(n)c = a \circled{n} c$ *for* $a, c \in C$, $n \in \mathbb{Z}_+$. *Moreover, this action agrees with the derivations on* A_+ *and* C: $(Da(n))c = [D, a(n)]c$.

From now on we will deal only with associative or Lie conformal algebras.

1.8. *Dong's Lemma*

We end this section by stating a very important property of formal power series over associative or Lie algebras. This property allows us to construct conformal algebras by taking a collection of generating series.

LEMMA 1.2. *Let* A *be an associative or a Lie algebra and let* $a, b, c \in A[[z, z^{-1}]]$ *be three formal power series. Assume that they are pairwise mutually local. Then for all* $n \in \mathbb{Z}_+$, $a \circled{n} b$ *and* c *are mutually local. Moreover, in the Lie algebra case,*

$$N(a \circled{n} b, c) = N(c, a \circled{n} b) \leq N(a, b) + N(b, c) + N(c, a) - n - 1, \tag{1.17}$$

and, in the associative case,

$$N(a \circled{n} b, c) \leq N(b, c), \qquad N(c, a \circled{n} b) \leq N(c, a) + N(a, b) - n - 1.$$

2. VERTEX ALGEBRAS

2.1. *Fields*

Let now V be a vector space over $\Bbbk$. Denote by $gl(V)$ the Lie algebra of all $\Bbbk$-linear operators on V. Consider the space $\mathrm{F}(V) \subset gl(V)[[z, z^{-1}]]$ of *fields* on V, given by

$$\mathrm{F}(V) = \left\{ \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \middle| \forall v \in V, a(n)v = 0 \text{ for } n \gg 0 \right\}.$$

For $a(z) \in \mathrm{F}(V)$ denote

$$a_-(z) = \sum_{n<0} a(n) z^{-n-1}, \qquad a_+(z) = \sum_{n \geq 0} a(n) z^{-n-1}.$$

Denote also by $\mathbb{1} = \mathbb{1}_{\mathrm{F}(V)} \in \mathrm{F}(V)$ the identity operator, such that $\mathbb{1}(-1) = \mathrm{Id}_V$; all other coefficients are 0.

Remark. In [18, 20] the elements of $\mathrm{F}(V)$ are called *quantum operators* on V.

We view $gl(V)$ as a Lie algebra, and Section 1.2.1 gives a collection of products $\textcircled{n}$, $n \in \mathbb{Z}_+$, on $\mathrm{F}(V)$. Now in addition to these products we introduce products $\textcircled{n}$ for $n < 0$. Define first $\textcircled{-1}$ by

$$a(z) \textcircled{-1} b(z) = a_-(z) b(z) + b(z) a_+(z). \tag{2.1}$$

Note that the products in (2.1) make sense, since for any $v \in V$ we have $a(n)v = b(n)v = 0$ for $n \gg 0$. The -1st product is also known as the *normally ordered product* (or *Wick product*) and is usually denoted by $:a(z)b(z):$.

Next, for any $n < 0$ set

$$a(z) \textcircled{n} b(z) = \frac{1}{(-n-1)!} :\big(D^{-n-1} a(z)\big) b(z):, \tag{2.2}$$

where $D = \frac{d}{dz}$. Taking $b = \mathbb{1}$, we get

$$a \textcircled{-1} \mathbb{1} = a, \qquad a \textcircled{-2} \mathbb{1} = Da. \tag{2.3}$$

It is easy to see that

$$\mathbb{1} \textcircled{n} a = \delta_{-1,n} a.$$

We have the following explicit formula for the circle products: If $(a \textcircled{n} b)(z) = \sum_m (a \textcircled{n} b)(m) z^{-m-1}$, then

$$\left(a \textcircled{n} b\right)(m) = \sum_{s \le n} (-1)^{s+n} \binom{n}{n-s} a(s) b(m+n-s) - \sum_{s \ge 0} (-1)^{s+n} \binom{n}{s} b(m+n-s) a(s). \tag{2.4}$$

Note that if $n > 0$ then (2.4) becomes

$$\left(a \textcircled{n} b\right)(m) = \sum_{s \ge 0} (-1)^{n+s} \binom{n}{s} \left[a(s), b(m+n-s)\right],$$

which is precisely formula (1.2) for Lie algebras.

It is easy to see that D is a derivation of all the circle products:

$$D\left(a \textcircled{n} b\right) = Da \textcircled{n} b + a \textcircled{n} Db. \tag{2.5}$$

Note also that the Dong's Lemma 1.2 remains valid for negative n and the estimate (1.17) still holds.

2.2. *Definition of Vertex Algebras*

Instead of giving a formal definition of a vertex algebra in the spirit of Definition 1.1, we present a description of these algebras similar to Proposition 1.4. For a more abstract approach see, e.g., [9, 15, 18, 20].

DEFINITION 2.1. A *vertex algebra* is a subspace $S \subset \mathrm{F}(V)$ of fields over a vector space V such that

(i) Any two fields $a, b \in S$ are local (in the Lie sense).

(ii) S is closed under all the circle products $\textcircled{n}$, $n \in \mathbb{Z}$, given by (2.4).

(iii) $\mathbb{1} \in S$.

Note that from (2.3) it follows that a vertex algebra is closed under the derivation $D = d/dz$.

Note also that a vertex algebra is a Lie conformal algebra.

Let $S \subset \mathrm{F}(V)$ be a vertex algebra. We introduce the left action map Y: $S \to \mathrm{F}(S)$ defined by

$$Y(a) = \sum_{n \in \mathbb{Z}} \left(a \textcircled{n} \cdot\right) \zeta^{-n-1}. \tag{2.6}$$

Clearly, $Y(\mathbb{1}_S) = \mathbb{1}_{\mathrm{F}(S)}$.

We state here the following characterizing property of Y (see [15, 20]):

PROPOSITION 2.1. *The left action map* $Y\colon S \to \mathrm{F}(S)$ *is an isomorphism of vertex algebras, i.e.,* $Y(S) \subset \mathrm{F}(S)$ *is a vertex algebra and*

$$Y\big(a \,\textcircled{n}\, b\big) = Y(a) \,\textcircled{n}\, Y(b), \qquad Y(\mathbb{1}_S) = \mathbb{1}_{\mathrm{F}(S)}. \tag{2.7}$$

From (2.3) and (2.5) it follows that Y also agrees with D:

$$Y(Da) = \partial_\zeta Y(a) = \left[D, Y(a)\right].$$

2.3. *Enveloping Vertex Algebras of a Lie Conformal Algebra*

Let C be a Lie conformal algebra and let $L = \operatorname{Coeff} C$ be its coefficient Lie algebra.

DEFINITION 2.2 [14, 15]. (a) An L-module M is called *restricted* if for any $a \in C$ and $v \in M$ there is some integer N such that for any $n \geq N$ one has $a(n)v = 0$

(b) An L-module M is called a *highest weight* module if it is generated over L by a single element $m \in M$ such that $L_+ m = 0$. In this case m is called the *highest weight vector*.

Clearly any submodule and any factor module of a restricted module are restricted.

Let M be a restricted L-module. Then the representation $\rho\colon L \to gl(M)$ could be extended to the map $\rho\colon L[[z, z^{-1}]] \to \mathrm{F}(M)$ which combined with the canonical embedding $\phi\colon C \to L[[z, z^{-1}]]$ (see Proposition 1.3)(d) gives a conformal algebra homomorphism $\psi\colon C \to \mathrm{F}(M)$. Then $\psi(C) \subset \mathrm{F}(M)$ consists of pairwise local fields, and by Dong's Lemma 1.2, $\psi(C)$ together with $\mathbb{1} \in \mathrm{F}(M)$ generates a vertex algebra $S_M \subset F(M)$.

The following proposition is well known; see, e.g., [11].

PROPOSITION 2.2. (a) *The vertex algebra* $S = S_M$ *has the structure of a highest weight module over* L *with the highest weight vector* $\mathbb{1}$. *The action is given by*

$$a(n)\,\beta = \psi(a) \,\textcircled{n}\, \beta, \qquad a \in C, n \in \mathbb{Z}, \beta \in S_M.$$

Moreover this action agrees with the derivations:

$$\big(Da(n)\big)\,\beta = \left[D, a(n)\right]\beta.$$

(b) *Any* L*-submodule of* S *is a vertex algebra ideal. If* M_1 *and* M_2 *are two restricted* L*-modules,* $S_1 = S_{M_1}$, $S_2 = S_{M_2}$, *and* $\mu\colon S_1 \to S_2$ *is an* L*-mod-*

ule homomorphism such that $\mu(\mathbb{1}) = \mathbb{1}$, *then* μ *is a vertex algebra homomorphism.*

2.4. *Universal Enveloping Vertex Algebras*

Now we build a universal highest weight module V over L, which is often referred to as a *Verma module*. Take the one-dimensional trivial L_+-module $\Bbbk\, \mathbb{1}_V$, generated by an element $\mathbb{1}_V$. Then let

$$Y = \operatorname{Ind}_{L_+}^{L} \Bbbk\, \mathbb{1}_V = U(L) \otimes_{U(L_+)} \Bbbk\, \mathbb{1}_V \cong U(L)/U(L)L_+ .$$

It is easy to see that V is a restricted module and hence we get an enveloping vertex algebra $S = S_V \subset \mathrm{F}(V)$ and a homomorphism $\psi: C \to S$. Clearly, ψ is injective, since $\rho: L \to gl(V)$ is injective.

THEOREM 2.1. (a) *The map* $\chi: S \to V$ *given by* $\alpha \mapsto \alpha(-1)\mathbb{1}_V$ *is an* L*-module isomorphism, and* $\chi(\mathbb{1}_S) = \mathbb{1}_V$.

(b) S *is the universal enveloping vertex algebra of* C *in the following sense: If* $\mu: C \to U$ *is another homomorphism of* C *to a vertex algebra* U, *then there is the unique map* $\hat{\mu}: S \to U$ *which makes up the following commutative triangle*:

$$\begin{array}{ccc} S & \xrightarrow{\hat{\mu}} & U \\ \nwarrow_{\psi} & & \nearrow_{\mu} \\ & C & \end{array}$$

From now on we identify V and $S = S_V$ via χ and write $V = V(C)$ for the universal enveloping vertex algebra of a Lie conformal algebra C and $\mathbb{1}_S = \mathbb{1}_V = \mathbb{1}$. The embedding $\psi: C \to V = U(L)/U(L)L_+$ is then given by $a \mapsto a(-1)\mathbb{1}$. By Proposition 1.3(c), the map $\phi(-1): C \to L_-$, defined by $a \mapsto a(-1)$, is an isomorphism of linear spaces. Therefore, the image of C in V is equal to $\psi(C) = L_- \mathbb{1} = L\mathbb{1} \subset V$.

3. FREE CONFORMAL ALGEBRAS

3.1. *Definition of Free Conformal and Free Vertex Algebras*

Let $\mathscr{B}$ be a set of symbols. Consider a function $N: \mathscr{B} \times \mathscr{B} \to \mathbb{Z}_+$, which will be called a *locality function*.

Let $\mathfrak{A}$ be a variety of algebras. In all the applications $\mathfrak{A}$ will be either Lie or associative algebras. Consider the category $\mathfrak{Conf}(N)$ of $\mathfrak{A}$-conformal algebras (see Section 1.6) generated by the set $\mathscr{B}$ such that in any conformal algebra $C \in \mathfrak{C}onf(N)$ one has

$$a \textcircled{n} b = 0 \qquad \forall a, b \in \mathcal{B}\ \forall n \geq N(a, b).$$

By an abuse of notation we will not make a distinction between $\mathcal{B}$ and its image in a conformal algebra $C \in \mathfrak{C}onf(N)$.

The morphisms of $\mathfrak{Conf}(N)$ are, naturally, conformal algebra homomorphisms f: $C \to C'$ such that $f(a) = a$ for any $a \in \mathcal{B}$.

We claim that $\mathfrak{Conf}(N)$ has the universal object, a conformal algebra $C = C(N)$, such that for any other $C' \in \mathfrak{Conf}(N)$ there is the unique morphism f: $C \to C'$. We call $C(N)$ a *free conformal algebra*, corresponding to the locality function N.

In order to build $C(N)$, we first build the corresponding coefficient algebra $A = \operatorname{Coeff} C$ (see Section 1.3).

Let $A \in \mathfrak{A}$ be the algebra presented by the set of generators

$$X = \{b(n) \mid b \in \mathcal{B}, n \in \mathbb{Z}\} \tag{3.1}$$

with relations

$$\left\{ \sum_s (-1)^s \binom{N(b,a)}{s} b(n-s)a(m+s) = 0 \,\middle|\, a, b \in \mathcal{B}, m, n \in \mathbb{Z} \right\}. \tag{3.2}$$

For any $b \in \mathcal{B}$ let $\tilde{b} = \sum_n b(n) z^{-n-1} \in A[[z, z^{-1}]]$. From (3.2) it follows that any two $\tilde{a}$ and $\tilde{b}$ are mutually local; therefore by Dong's Lemma 1.2 they generate a conformal algebra $C \subset A[[z, z^{-1}]]$.

PROPOSITION 3.1. (a) $A = \operatorname{Coeff} C$.

(b) *The conformal algebra C is the free conformal algebra corresponding to the locality function N.*

Proof. (a) Clearly, there is a surjective homomorphism $A \to \operatorname{Coeff} C$, since relations (3.2) must hold in $\operatorname{Coeff} C$. Now the claim follows from the universal property of $\operatorname{Coeff} C$ (see Proposition 1.3(e)).

(b) Take another algebra $C' \in \mathfrak{Conf}(N)$ and let $A' = \operatorname{Coeff} C'$. Obviously, there is an algebra homomorphism f: $A \to A'$ such that $f(b(n)) = b(n)$ for any $b \in \mathcal{B}$ and $n \in \mathbb{Z}$. It could be extended to a map f: $A[[z, z^{-1}]] \to A'[[z, z^{-1}]]$. Now it is easy to see that the restriction $f|_C$

gives the desired conformal algebra homomorphism $C \to C'$:

$$\begin{array}{ccc} A[[z, z^{-1}]] & \xrightarrow{f} & A'[[z, z^{-1}]] \\ \uparrow & & \uparrow \\ C & \xrightarrow{f} & C' \end{array}$$

Indeed, due to formula (1.2), f preserves the circle products, and, since ∂ is a derivation of the products, and $f(\partial\tilde{a}) = \partial f(\tilde{a})$, for $a \in C$ one has $f(\partial\phi) = \partial f(\phi)$ for any $\phi \in C$. ∎

In the case when $\mathfrak{A}$ is the variety of Lie algebras, we may consider the universal vertex enveloping algebra $V(C)$ of a free Lie conformal algebra $C = C(N)$. In accordance with Theorem 2.1, we call $V(C)$ a *free vertex algebra*.

Though the construction of free conformal and vertex algebras makes sense for an arbitrary locality function $N: \mathscr{B} \times \mathscr{B} \to \mathbb{Z}_+$, the results of Sections 3.4–3.7 are valid only for the case when N is constant.

3.2. *The Positive Subalgebra of* Coeff $C(N)$

Let again $C = C(N)$ be a free conformal algebra corresponding to a locality function $N: \mathscr{B} \times \mathscr{B} \to \mathbb{Z}_+$, $\mathscr{B}$ being an alphabet, and let $A =$ Coeff C. Recall that by Proposition 1.3(a) we have the decomposition $A = A_- \oplus A_+$ of the coefficient algebra into the direct sum of two subalgebras. Denote $X_i = \{b(n) \mid b \in \mathscr{B}, n \geq i\} \subset X$.

LEMMA 3.1. *The subalgebra $A_+ \subset A$ is isomorphic to the algebra $\hat{A}_+$ presented by the set of generators X_0 and those of relations* (3.2) *which contain only elements of X_0*:

$$\left\{ \sum_s (-1)^s \binom{N(b,a)}{s} b(n-s)a(m+s) = 0 \,\middle|\, a, b \in \mathscr{B}, m \geq 0, \right.$$
$$\left. n \geq N(b,a) \right\}. \qquad (3.3)$$

Proof. Clearly, there is a surjective homomorphism $\varphi: \hat{A}_+ \to A_+$ which maps X_0 to itself. We prove that φ is in fact an isomorphism. We proceed in four steps.

Step 1. First we prove that A_+ is generated by X_0 in A. Indeed, we have $X_0 \subset A_+$. On the other hand, A_+ is spanned by elements of the form

$a(m)$, where $m \geq 0$ and $a \in C$ is a circle product monomial in $\mathscr{B}$. By induction on the length of a it is enough to check that if $a = a_1 \textcircled{k} a_2$, then $a(m)$ is in the subalgebra, generated by X_0, which follows from (1.2).

Step 2. Let $\hat{\tau}$: $\hat{A}_+ \to \hat{A}_+$ be the homomorphism, which acts on the generators X_0 by $a(n) \mapsto a(n+1)$, so that $\hat{\tau}(\hat{A}_+)$ is the subalgebra of $\hat{A}_+$ generated by X_1. We claim that $\hat{\tau}$ is injective, and therefore $\hat{\tau}(\hat{A}_+) \cong \hat{A}_+$. Indeed, $\hat{\tau}$ acts on the free associative algebra $\Bbbk\langle X_0 \rangle$. Assume that for some $p \in \hat{A}_+$ we have $\hat{\tau}(p) = 0$. Take any preimage $P \in \Bbbk\langle X_0 \rangle$ of p. Then we have $\hat{\tau}(P) = \sum_i \xi_i R_i$, where $\xi_i \in \Bbbk\langle X_0 \rangle$ and R_i are relations (3.3), such that in all ξ_i and R_i there appear only indexes greater than or equal to 1. But then P itself must be of the form $\sum_i \xi_i' R_i'$, where "$'$" stands for decreasing all indexes by 1; hence $p = 0$.

Step 3. Next we claim that there is an automorphism τ of the algebra A which acts on the generators X by the shift $a(n) \mapsto a(n+1)$. Indeed, relations (3.2) are invariant under the shift, and clearly, τ is invertible. For any integer n denote $A_n = \tau^n A_+$. We have $A_n \cong A_+ = A_0$ for every n.

Step 4. Now for each integer n take a copy $\hat{A}_n$ of $\hat{A}_+$. Let $\hat{\tau}_n$: $\hat{A}_n \to \hat{A}_{n-1}$ be the isomorphism of $\hat{A}_+$ onto $\hat{\tau}(\hat{A}_+)$, built in Step 1. Let $\hat{A}$ be the limit of all these $\hat{A}_n$ with respect to the maps $\hat{\tau}_n$. We identify generators of $\hat{A}_n$ with the set X_n. It is easy to see that φ: $\hat{A}_0 \to A_0$ extends to the homomorphism φ: $\hat{A} \to A$, such that $\varphi(\hat{A}_n) = A_n$ and $\varphi|_X = \mathrm{id}$. Now we observe that all the defining relations (3.2) of A hold in $\hat{A}$; hence there is an inverse map φ^{-1}: $A \to \hat{A}$, and therefore φ is an isomorphism. ▮

3.3. *The Diamond Lemma*

For future purposes we need a digression on the diamond lemma for associative algebras. We closely follow [2], but use more modern terminology.

Let X be some alphabet and let K be some commutative ring. Consider the free associative algebra $K\langle X \rangle$ of non-commutative polynomials with coefficients in K. Denote by X^* the set of words in X, i.e., the free semigroup with 1 generated by X.

A *rule* on $K\langle X \rangle$ is a pair $\rho = (w, f)$, consisting of a word $w \in X^*$ and a polynomial $f \in K\langle X \rangle$. The left-hand side w is called *the principal part* of rule ρ. We will denote $w = \bar{\rho}$.

Let $\mathfrak{R}$ be a collection of rules on $K\langle X \rangle$. For a rule $\rho = (w, f) \in \mathscr{R}$ and a pair of words $u, v \in X^*$ consider the K-linear endomorphism $r_{u\rho v}$: $K\langle X \rangle \to K\langle X \rangle$, which fixes all words in X^* except for uwv, and sends the latter to ufv.

A rule $\rho = (w, f)$ is said to be *applicable* to a word $v \in X^*$ if w is a subword of v, i.e., $v = v'wv''$. The result of application of ρ to v is, naturally, $r_{v'\rho v''}(v) = v'fv''$. If $p \in K\langle X \rangle$ is a polynomial which involves a word v, such that a rule ρ is applicable to v, then we say that ρ is applicable to p.

A polynomial $p \in K\langle X \rangle$ is called *terminal* if no rule from $\mathscr{R}$ is applicable to v; that is, no term of p is of the form $u\bar{\rho}v$ for $\rho \in \mathscr{R}$.

Define a binary relation "$\to$" on $K\langle X \rangle$ in the following way: Set $p \to q$ if and only if there is a finite sequence of rules $\rho_1, \dots, \rho_n \in \mathscr{R}$, and a pair of sequences of words $u_i, v_i \in X^*$ such that $q = r_{u_n\rho_n v_n} \cdots r_{u_1\rho_1 v_1}(p)$.

DEFINITION 3.1. (a) A set of rules $\mathscr{R}$ is a *rewriting system* on $K\langle X \rangle$ if there are no infinite sequences of the form

$$p_1 \to p_2 \to \cdots;$$

i.e., any polynomial $p \in K\langle X \rangle$ can be modified only finitely many times by rules from $\mathscr{R}$.

(b) A rewriting system is *confluent* if for any polynomial $p \in K\langle X \rangle$ there is the unique terminal polynomial t such that $p \to t$.

Any rule $\rho = (w, f) \in \mathscr{R}$ gives rise to an identity $w - f \in K\langle X \rangle$. Let $I(\mathscr{R}) \subset K\langle X \rangle$ be the two-sided ideal generated by all such identities.

Let $v_1, v_2 \in X^*$ be a pair of words. A word $w \in X^*$ is called a *composition* of v_1 and v_2 if $w = w'uw''$, $v_1 = w'u$, $v_2 = uw''$, and $u \neq 0$.

Finally, take a word $v \in X^*$. Let us call it an ambiguity if there are two rules $\rho, \sigma \in \mathscr{R}$ such that either v is a composition of $\bar{\rho}$ and $\bar{\sigma}$ or if $v = \bar{\rho}$ and $\bar{\sigma}$ is a subword of ρ.

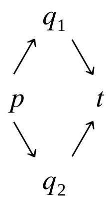
Now we can state the lemma.

LEMMA 3.2 (Diamond Lemma). (a) *A rewriting system $\mathscr{R}$ is confluent if and only if all terminal monomials form a basis of $K\langle X \rangle / I(\mathscr{R})$.*

(b) *A rewriting system is confluent if and only if it is confluent on all the ambiguities; that is, for any ambiguity $v \in X^*$ there is the unique terminal $t \in K\langle X \rangle$ such that $v \to t$.*

Remark. Statement (a) appears in [21]. A variant of Lemma 3.2 appears in [3, 4]. It was also known to Shirshov (see [25]). The name "diamond" is due to the following graphical description of the confluency property; see [21]. Let $\mathscr{R}$ be a rewriting system in the sense of Definition 3.1(a), and let "$\to$" be defined as above. Assume $p, q_1, q_2 \in K\langle X \rangle$ are such that $p \to q_1$ and $p \to q_2$. Then there is some $t \in K\langle X \rangle$ such that $q_1 \to t$ and

$q_2 \to t$:



Bergman in [2] uses the existence of a semigroup order with descending chain condition on the set of words X^*. Though in our case there is an order on the set (3.1), this order does not satisfy the descending chain condition, so we slightly modify the argument in [2].

Proof of Lemma 3.2. (a) Assume that the rewriting system $\mathscr{R}$ is confluent. Define a map r: $K\langle X\rangle \to K\langle X\rangle$ by taking $r(p)$ to be the unique terminal monomial such that $p \to r(p)$. The crucial observation is that r is a K-linear endomorphism of $K\langle X\rangle$. So if $p = \sum_i \xi_i u_i(w_i - f_i)v_i \in I(\mathscr{R})$, $\xi_i \in K$, $u_i, v_i \in X^*$, $(w_i, f_i) \in \mathscr{R}$, then $r(p) = \sum_i \xi_i r(u_i(w_i - f_i)v_i) = 0$; therefore the terminal monomials are linearly independent modulo $I(\mathscr{R})$.

Form the other side, if $\mathscr{R}$ is not confluent, then there are a polynomial $p \in K\langle X\rangle$ and terminals $q_1, q_2 \in K\langle X\rangle$ such that $p \to q_1$, $p \to q_2$, and $q_1 \neq q_2$, and then $q_1 - q_2 \in I(\mathscr{R})$.

(b) Take a polynomial $p \in K\langle X\rangle$. We prove that there is the unique terminal t such that $p \to t$ by induction on the number $n(p) = \#\{q \mid p \to q\}$. Condition (a) of Definition 3.1 assures that $n(p)$ is always finite.

If $n(p) = 0$ then p is a terminal itself and there is nothing to prove. By induction, without loss of generality we can assume that there are at least two different rules $\rho, \sigma \in \mathscr{R}$ which are applicable to p. This means that there are some words $u, v, x, y \in X^*$ such that $r_{u\rho v}(p) \neq p$, $r_{x\sigma y}(p) \neq p$, and $r_{u\rho v}(p) \neq r_{x\sigma y}(p)$. By induction, both $r_{u\rho v}(p)$ and $r_{x\sigma y}(p)$ are uniquely reduced to terminals, say, $r_{u\rho v}(p) \to t_1$ and $r_{x\sigma y}(p) \to t_2$. We need to show that $t_1 = t_2$.

Consider two cases: when $\bar{\rho}$ and $\bar{\sigma}$ have common symbols in p, and thus $u\bar{\rho}v = x\bar{\sigma}y$ is a word in p; and when $\bar{\rho}$ and $\bar{\sigma}$ are disjoint.

In the first case, let $w \in X^*$ be the union of $\bar{\rho}$ and $\bar{\sigma}$ in p. Then w is an ambiguity. By assumption, there is the unique terminal $s \in K\langle X\rangle$ such that $w \to s$. Let $q \in K\langle X\rangle$ be obtained from p by substituting w by s.

Then we have

$$\begin{array}{ccc} & r_{u\bar{\rho}v}(p) & \\ \nearrow & & \searrow \\ p & & q \\ \searrow & & \nearrow \\ & r_{x\bar{\sigma}y}(p) & \end{array} \tag{3.4}$$

By induction, q is uniquely reduced to a terminal t, and therefore one has $r_{u\rho v}(p) \to t$ and $r_{x\sigma y}(p) \to t$.

In the second case, note that $r_{x\sigma y} r_{u\rho v}(p) = r_{u\rho v} r_{x\sigma y}(p)$. Denote this polynomial by q. Then relations (3.4) still hold, and we finish by the same argument as in the first case. ∎

3.4. *Basis of a Free Vertex Algebra*

Return to the setup of Section 3.1. From now on we take the locality function $N(a,b)$ to be constant: $N(a,b) \equiv N$. Let $C = C(N)$ be the free Lie conformal algebra and let $L = \operatorname{Coeff} C$ be its Lie algebra of coefficients; see Proposition 3.1. In this section we build a basis of the universal enveloping algebra $U(L)$ of L and a basis of the free vertex algebra $V = V(C)$.

We start by endowing $\mathscr{B}$ with an arbitrary linear order. Then we define a linear order on the set X of generators of L, given by (3.1), in the following way:

$$a(m) < b(n) \Leftrightarrow m < n \text{ or } (m = n \text{ and } a < b). \tag{3.5}$$

On the set X^* of words in X introduce the standard lexicographical order: For $u, v \in X^*$ if $|u| < |v|$, set $u < v$; if $|u| = |v|$, then set $u < v$ whenever there is some $1 \le i \le |v|$ such that $u(i) < v(i)$ and $u(j) = v(j)$ for all $1 \le j < i$.

In a defining relation from (3.2) the biggest term has form $b(n)a(m)$ such that

$$n - m > N \text{ or } \big(n - m = N \text{ and } \big(b > a \text{ or } (b = a \text{ and } N \text{ is odd})\big)\big). \tag{3.6}$$

Taking it as a principal part, we get a rule on $\Bbbk\langle X\rangle$

$$\rho(b(n), a(m)) = \left(b(n)a(m), a(m)b(n) - \sum_{s=1}^{N} (-1)^s \binom{N}{s} [b(n-s), a(m+s)] \right), \tag{3.7a}$$

and in the case when $a = b$, $n - m = N$, and N is odd,

$$\rho(a(m+N), a(m)) = \Bigg(a(m+N)a(m), a(m)a(m+N) - \frac{1}{2} \sum_{s=1}^{(N-1)/2} (-1)^s \binom{N}{s} [a(n-s), a(m+s)] \Bigg). \quad (3.7b)$$

Denote the set of all such rules by $\mathscr{R}$:

$$\mathscr{R} = \{\rho(b(n), a(m)) \,|\, (3.6) \text{ holds}\}. \quad (3.8)$$

LEMMA 3.3. *The set of rules $\mathscr{R}$ is a confluent rewriting system on $\Bbbk\langle X\rangle$.*

We prove this lemma in Section 3.5. Here we derive from it and from Lemma 3.2 the following theorem.

THEOREM 3.1. (a) *Let $C = C(N)$ be the free Lie conformal algebra generated by a linearly ordered set $\mathscr{B}$ corresponding to a constant locality function N. Let $L = \operatorname{Coeff} C$ be the Lie algebra of coefficients and let $U = U(L)$ be its universal enveloping algebra. Then a basis of U is given by all monomials*

$$a_1(n_1)a_2(n_2)\cdots a_k(n_k), \qquad a_i \in \mathscr{B}, n_i \in \mathbb{Z}, \quad (3.9)$$

such that for any $1 \le i < k$ one has

$$n_i - n_{i+1} \le \begin{cases} N-1 & \text{if } a_i > a_{i+1} \text{ or } (a_i = a_{i+1} \text{ and } N \text{ is odd}), \\ N & \text{otherwise}. \end{cases} \quad (3.10)$$

(b) *A basis of the algebra $U(L_+)$ is given by all monomials* (3.9) *satisfying the condition* (3.10) *and such that all $n_i \ge 0$.*

(c) *Let $V = V(C)$ be the corresponding free vertex algebra. Then a basis of V consists of elements*

$$a_1(n_1)a_2(n_2)\cdots a_k(n_k k)\mathbb{1}, \qquad a_i \in \mathscr{B}, n_i \in \mathbb{Z}, \quad (3.11)$$

such that the condition (3.10) *holds and, in addition, $n_k < 0$.*

Proof. Statement (a) is a direct corollary of Lemmas 3.3 and 3.2, because (3.9) is precisely the set of all terminal monomials with respect to $\mathscr{R}$.

Statement (b) follows immediately from Lemma 3.1, since any subset of rules $\mathscr{R}$ is also a confluent rewriting system. Note also that for a rule ρ given by (3.7) if the principal term $\bar{\rho}$ contains only elements from X_0 then so does the whole rule ρ.

For the proof of (c) recall that $V \cong U/UL_+$ as linear spaces (and even as L-modules), where UL_+ is the left ideal generated by L_+; see Section 2.4. By Lemma 3.1, this ideal is the linear span of all monomials $a_1(n_1)a_2(n_2)\cdots a_k(n_k)$ such that $n_k \geq 0$. But under the action of the rewriting system $\mathscr{R}$ the index of the rightmost symbol in a word can only increase; hence the linear span of these monomials in $\Bbbk\langle X\rangle$ is stable under $\mathscr{R}$. It follows that the terminal monomials with a non-negative rightmost index form a basis of UL_+. This proves (b). ∎

3.5. *Proof of Lemma* 3.3

First we prove that the set of rules $\mathscr{R}$, given by (3.8), is a rewriting system on $\Bbbk\langle X\rangle$. Take a word $u = a_1(m_1)\cdots a_k(m_k) \in X^*$. Let $p \in \Bbbk\langle X\rangle$ be such that $u \to p$. Then any word v that appears in p lies in the finite set

$$W_u = \left\{ b_1(n_1)\cdots b_k(n_k) \in X^* \middle| n_i \geq \min_{1\leq j\leq k}\{m_j\} \text{ and } \sum n_i = \sum m_i \right\}. \tag{3.12}$$

Therefore condition (a) of Definition 3.1 holds.

Thus we are left to prove that $\mathscr{R}$ is confluent. According to Lemma 3.2, it is enough to check that it is confluent on a composition $w = c(k)b(j)a(i)$ of principal parts of a pair of rules $\rho(b(j), a(i)), \rho(c(k), b(j)) \in \mathscr{R}$. Thus it is sufficient to prove the following claim.

LEMMA 3.4. *Let $u = c(k)b(j)a(i) \in X^*$ be a word of length* 3. *Then $\mathscr{R}$ is confluent on u; i.e., there is a unique terminal $r(w) \in \Bbbk\langle X\rangle$ such that $u \to r(w)$.*

Proof. Assume for simplicity that the three rules $\rho(b(n), a(m))$, $\rho(c(p), b(n))$, and $\rho(c(p), a(m))$ are of the form (3.7a). The general case is essentially the same, but requires some additional calculations.

Consider the set W_u, given by (3.12). We prove that the lemma holds for all $w \in W_u$ by induction on w. If w is sufficiently small then it is a terminal itself. By induction, it is enough to consider $w = c(p)b(n)a(m) \in W_u$ such that $\mathscr{R}$ is applicable to both $b(n)a(m)$ and $c(p)b(n)$. Apply

$\rho(b(n), a(m))$ and $\rho(c(p), b(n))$ to w and take the difference of the results:

$$v = b(n)c(p)a(m) - \sum_{s=1}^{N} (-1)^s \binom{N}{s} [c(p-s), b(n+s)] a(m)$$
$$- c(p)a(m)b(n) + \sum_{s=1}^{N} (-1)^s \binom{N}{s} c(p) [b(n-s), a(m+s)].$$

By induction, v is reduced uniquely to a terminal t and we only have to show that $t = 0$. First we apply the rules $\rho(b(n), a(m))$, $\rho(c(p), b(n))$, and $\rho(c(p), a(m))$ to v several times and get

$$\begin{aligned} v \to & - \sum_{s=1}^{N} (-1)^s \binom{N}{s} b(n) [c(p-s), a(m+s)] + b(n)a(m)c(p) \\ & + \sum_{s=1}^{N} (-1)^s \binom{N}{s} [c(p-s), a(m+s)] b(n) - a(m)c(p)b(n) \\ & - \sum_{s=1}^{N} (-1)^s \binom{N}{s} [c(p-s), b(n+s)] a(m) \\ & + \sum_{s=1}^{N} (-1)^s \binom{N}{s} c(p) [b(n-s), a(m+s)] \\ & \to \sum_{s=1}^{N} (-1)^s \binom{N}{s} [a(m), [c(p-s), b(n+s)]] \\ & + \sum_{s=1}^{N} (-1)^s \binom{N}{s} [[c(p-s), a(m+s)], b(n)] \\ & + \sum_{s=1}^{N} (-1)^s \binom{N}{s} [c(p), [b(n-s), a(m+s)]]. \end{aligned} \tag{3.13}$$

Next we introduce two rules acting on the linear combinations of (formal) commutators: For any $a(m), b(n), c(p) \in X$ let

$$\kappa = \big([a(m), [b(n), c(p)]], [[a(m), b(n)], c(p)] + [b(n), [a(m), c(p)]]\big),$$

$$\lambda = \left([b(n), a(m)], -\sum_{s=1}^{N} (-1)^s \binom{N}{s} [b(n-s), a(m+s)]\right).$$

The rule λ is the locality relation, and κ is nothing else but the Jacoby identity. The lemma will be proved after we show two things:

(1) *There always exists a finite sequence of applications of the rules* κ *and* λ *that reduces* (3.13) *to* 0.

(2) *All words which appear in the process of reduction in* (1) *are smaller than the initial word* $u = c(p)b(n)a(m)$ *with respect to the order* (3.5).

Indeed, assume (1) and (2) hold. Denote the polynomial in (3.13) by p_0. Let

$$p_0 \to p_1 \to \cdots \to 0$$

be the reduction, guaranteed by (1). By (2) and by the induction hypothesis, any two neighboring polynomials $p_i \to p_{i+1}$ from this sequence are uniquely $\mathscr{R}$-reduced to a terminal, and this terminal must be the same, since either $p_i \overset{\mathscr{R}}{\to} p_{i+1}$ or $p_{i+1} \overset{\mathscr{R}}{\to} p_i$.

Denote the three last terms in (3.13) by $\boxed{\text{a}}$, $\boxed{\text{b}}$, and $\boxed{\text{c}}$. In Fig. 1 we present a scheme of how κ and λ should be applied in order to reduce (3.13) to 0.

Each box in Fig. 1 stands for a sum of commutators:

$$\boxed{\text{j}} = -\boxed{\text{r}}$$
$$= \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [[c(p-s-t), a(m+t)], b(n+s)],$$

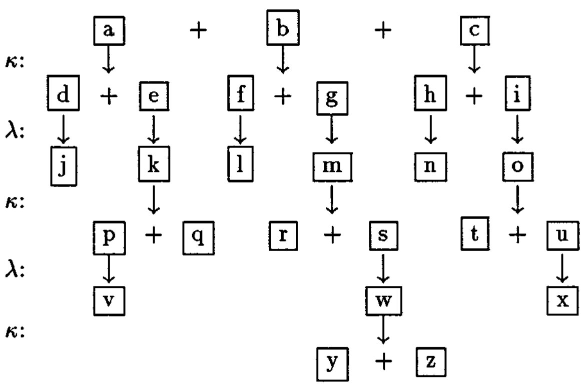


FIG. 1. Application of rules κ and λ.

$$\boxed{\text{k}} = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [c(p-s), [b(n+s-t), a(m+t)]],$$

$$\boxed{\text{l}} = -\boxed{\text{t}} = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t}$$
$$\times [c(p-s), [b(n-t), a(m+s+t)]],$$

$$\boxed{\text{m}} = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [[b(n+t), c(p-s-t)], a(m+s)],$$

$$\boxed{\text{n}} = -\boxed{\text{q}} = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t}$$
$$\times [[b(n-s+t), c(p-t)], a(m+s)],$$

$$\boxed{\text{o}} = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [b(n-s), [a(m+s+t), c(p-t)]],$$

$$\boxed{\text{v}} = -\boxed{\text{y}} = \sum_{s,t,r=1}^{N} (-1)^{s+t+r} \binom{N}{s} \binom{N}{t} \binom{N}{r}$$
$$[b(n+s-t), [a(m+t+r), c(p-s-r)]],$$

$$\boxed{\text{w}} = \sum_{s,t,r=1}^{N} (-1)^{s+t+r} \binom{N}{s} \binom{N}{t} \binom{N}{r}$$
$$\times [[a(m+s+r), b(n+t-r)], c(p-s-t)],$$

$$\boxed{\text{x}} = -\boxed{\text{z}} = \sum_{s,t,r=1}^{N} (-1)^{s+t+r} \binom{N}{s} \binom{N}{t} \binom{N}{r}$$
$$\times [a(m+s+t), [c(p-t-r), b(n-s+r)]].$$

One can see that all terminal boxes in the above scheme cancel, so that $\boxed{\text{a}} + \boxed{\text{b}} + \boxed{\text{c}} \to 0$. Claim (2) also holds, since every symbol in every box in Fig. 1 is less than $c(p)$. ∎

3.6. *Digression on Hall Bases*

Let again $\mathscr{B}$ be some linearly ordered alphabet, $N \in \mathbb{Z}_+$, $C = C(N)$ the free Lie conformal algebra generated by $\mathscr{B}$ with respect to the constant locality N, and $L = \operatorname{Coeff} C(N)$. A basis of the Lie algebra L could be obtained by modifying the construction of a Hall basis of a free Lie algebra; see [12, 23, 24]. Here we review the latter construction. We closely follow [22], except that all the order relations are reversed.

As in Section 3.3, take an alphabet X and a commutative ring K. Let $T(X)$ be the set of all binary trees with leaves from X. For typographical reasons we will write the tree $\widehat{xy}$ as $\langle x, y\rangle$. Assume that $T(X)$ is endowed with a linear order such that $\langle x, y\rangle > \min\{x, y\}$ for any $x, y \in T(X)$.

DEFINITION 3.2. A *Hall set* $\mathscr{H} \subset T(X)$ is a subset of all trees $h \in T(X)$ satisfying the following (recursive) properties:

1. If $h = \langle x, y\rangle$ then $y, x \in \mathscr{H}$ and $x > y$.
2 If $h = \langle\langle x, y\rangle, z\rangle$ then $z \geq y$, so that $\langle x, y\rangle > z \geq y$.

In particular, $X \subset \mathscr{H}$.

Introduce two maps α: $T(X) \to X^*$ and λ: $T(X) \to K\langle X\rangle$ in the following recursive way: For $a \in X$ set $\alpha(a) = \lambda(a) = a$ and $\alpha(\langle x, y\rangle) = \alpha(x)\alpha(y)$, $\lambda(\langle x, y\rangle) = [\lambda(x), \lambda(y)]$.

It is a well-known fact (see, e.g., [22]) that

(a) *$\lambda(\mathscr{H})$ is a basis of the free Lie algebra generated by X and*

(b) *$\alpha|_{\mathscr{H}}$ is injective.*

A word $w \in \alpha(\mathscr{H})$ is called a *Hall word*.

On the set X^* of words in X introduce a (lexicographic) order as follows: If u is a prefix of v then $u > v$; otherwise $u > v$ whenever for some index i one has $u_i > v_i$ and $u_j = v_j$ for all $j < i$.

DEFINITION 3.3 [25, 7]. A word $v \in X^*$ is called *Lyndon–Shirshov* if it is bigger than all its proper suffices.

PROPOSITION 3.2. (a) *There is a Hall set $\mathscr{H}_{\mathrm{LS}}$ such that $\alpha(\mathscr{H}_{\mathrm{LS}})$ is the set of all Lyndon–Shirshov words and α: $T(X) \to X^*$ preserves the order.*

(b) *For any tree $h \in \mathscr{H}_{\mathrm{LS}}$ the biggest term in $\lambda(h)$ is $\alpha(h)$.*

3.7. *Basis of the Algebra of Coefficients of a Free Lie Conformal Algebra*

Here we apply general results from Section 3.6 to the situation of Section 3.1.

Recall that starting from a set of symbols $\mathscr{B}$ and a number $N > 0$, we build the free conformal algebra $C = C(N)$ generated by $\mathscr{B}$ such that $a \textcircled{n} b = 0$ for any two $a, b \in \mathscr{B}$ and $n \geq N$. Let $L = \operatorname{Coeff} C$ be the corresponding Lie algebra of coefficients. It is generated by the set $X = \{a(n) \mid a \in \mathscr{B}, n \in \mathbb{Z}\}$ subject to relations (3.2).

The set of generators X is equipped with the linear order defined by (3.5). W define the order on X^* as in Section 3.6. Consider the set of all Lyndon words in X^* and let $\mathscr{H} = \mathscr{H}_{\mathrm{LS}} \subset T(X)$ be the corresponding Hall set. Recall that there is a rewriting system $\mathscr{R}$ on $\Bbbk\langle X\rangle$, given by (3.8). Define

$$\mathscr{H}_{\text{term}} = \{h \in \mathscr{H} \mid \alpha(h) \text{ is terminal}\}.$$

LEMMA 3.5. (a) *Let $v_1 \leq \cdots \leq v_n$ be a non-decreasing sequence of terminal Lyndon–Shirshov words. Then their concatenation $w = v_1 \cdots v_n \in X^*$ is a terminal word.*

(b) *Each terminal word $w \in X^*$ can be uniquely represented as a concatenation $w = v_1 \cdots v_n$, where $v_1 \leq \cdots \leq v_n$ is a non-decreasing sequence of terminal Lyndon–Shirshov words.*

Proof. (a) Take two terminal Lyndon–Shirshov words $v_1 \leq v_2$. Let $x \in X$ be the last symbol of v_1 and let $y \in X$ be the first symbol of v_2. Then, since a word is less than its prefix and since v_1 is a Lyndon–Shirshov word, we get

$$x < v_1 \leq v_2 < y.$$

Therefore, xy is a terminal, and hence $v_1 v_2$ is a terminal, too.

(b) Take a terminal word $w \in X^*$. Assume it is not Lyndon–Shirshov. Let v be the maximal among all proper suffices of w. Then v is Lyndon–Shirshov, $v > w$, and $w = uv$ for some word u. By induction, $u = v_1 \cdots v_{n-1}$ for a non-decreasing sequence of Lyndon–Shirshov words $v_1 \leq \cdots \leq v_{n-1}$. We are left to show that $v \geq v_{n-1}$.

Assume on the contrary that $v < v_{n-1}$. Then, since $v > v_{n-1}v$, v_{n-1} must be a prefix of v so that $v = v_{n-1}v'$. But then $v' > v$ which contradicts the Lyndon–Shirshov property of v.

The uniqueness is obvious. ∎

Let φ: $\Bbbk\langle X\rangle \to U(L)$ be the canonical projection with the kernel $I(\mathscr{R})$.

THEOREM 3.2. *The set $\varphi(\lambda(\mathscr{H}_{\text{term}}))$ is a basis of L.*

Proof. Let $s = \{h_1, \dots, h_n\} \subset \mathscr{H}_{\text{term}}$ be a non-decreasing sequence of terminal Hall trees. Let $\lambda(s) = \lambda(h_1) \cdots \lambda(h_n) \in \Bbbk\langle X\rangle$ and $\alpha(s) = \alpha(h_1) \cdots \alpha(h_n) \in X^*$.

By the Poincaré–Birkhoff–Witt theorem it is sufficient to prove that the set $\{\varphi(\lambda(s))\}$, when s ranges over all non-decreasing sequences s of terminal Hall trees, is a basis of $U(L)$.

By Proposition 3.2 (b), $\lambda(s) = \alpha(s) + O(\alpha(s))$, where $O(v)$ stands for a sum of terms which are less than v. Now let $t(s) \in \Bbbk\langle X\rangle$ be a terminal such that $\lambda(s) \to t(s)$. One can view $t(s)$ as the decomposition of $\varphi(\lambda(s))$ in basis (3.9). By Lemma 3.5, $\alpha(s)$ is a terminal monomial; hence $t(s)$ has form $t(s) = \alpha(s) + f(s)$ where $f(s)$ is a sum of terms $v \in X^*$ satisfying the following properties:

1. v is terminal and $v < \alpha(s)$.

2. If v contains a symbol $a(n) \in X$ then a appears in $\alpha(s)$ and $n_{\min} \leq n \leq n_{\max}$, where $n_{\min}$ and $n_{\max}$ are, respectively, minimum and maximum of all indices that appear in $\alpha(s)$.

Indeed, due to Proposition 3.2(b) properties 1 and 2 are satisfied by all the terms in $\lambda(s) - \alpha(s)$, and they cannot be broken by an application of the rules $\mathscr{R}$.

Property 1 implies that all $t(s)$ and, therefore, $\varphi(\lambda(s))$ are linearly independent. So we are left to show that they span $U(L)$. For that purpose we show that any terminal word $w \in X^*$ can be represented as a linear combination of $t(s)$.

By Lemma 3.5(b) any terminal word w could be written as $w = \alpha(s)$ for some non-decreasing sequence s of terminal Hall trees. So we can write $w = t(s) - f(s)$. Now do the same with any term v that appears in $f(s)$, and so on. This process should terminate, because every term v that appears during this process must satisfy properties 1 and 2 and there are only finitely many such terms. ∎

Remark. Alternatively we could use the theorem of Bokut' and Malcolmson [5].

As in Theorem 3.1(b), we deduce that all the elements of $\varphi(\lambda(\mathscr{H}_{\text{term}}))$ containing only symbols from X_0 form a basis of L_+.

Note that we have an algorithm for building a basis of the free Lie conformal algebra $C = C(N)$. Let $L = \operatorname{Coeff} C$, $V = V(C)$, and $U = U(L)$. Recall that the image if C in V under the canonical embedding $\psi: C \to V$ is $\psi(C) = L_- \mathbb{1} = L\mathbb{1} \subset V$. So, the algorithm goes as follows: Take the basis of L provided by Theorem 3.2. Decompose its element in basis (3.9) of the universal enveloping algebra $U(L)$, and then cancel all terms of the form $a_1(n_1) \cdots a_k(n_k)$ where $n_k \geq 0$. What remains, being interpreted as elements of the vertex algebra V, form a basis of $\psi(C) \subset V$.

3.8. *Basis of the Algebra of Coefficients of a Free Associative Conformal Algebra*

Let again $\mathscr{B}$ be some alphabet, and let $N: \mathscr{B} \times \mathscr{B} \to \mathbb{Z}_+$ be a locality function, not necessarily constant and not necessarily symmetric. By Proposition 3.1, the coefficient algebra $A = \operatorname{Coeff} C(N)$ of the free associative conformal algebra $C(N)$ corresponding to the locality function N is presented in terms of generators and relations by the set of generators $X = \{b(n) \mid b \in \mathscr{B}, n \in \mathbb{Z}\}$ and relations (3.2).

THEOREM 3.3. (a) *A basis of the algebra A is given by all monomials of the form*

$$a_1(n_1)\cdots a_{l-1}(n_{l-1})a_l(n_l), \tag{3.14}$$

where $a_i \in \mathscr{B}$ and

$$-\left\lceil \frac{N_i - 1}{2} \right\rceil \le n_i \le \left\lfloor \frac{N_i - 1}{2} \right\rfloor,$$
$$N_i = N(a_i, a_{i+1}) \qquad \text{for } i = 1, \ldots, l-1.$$

(b) *A basis of the algebra A_+ is given by all monomials of the form*

$$a_1(n_1)\cdots a_{l-1}(n_{l-1})a_l(n_l), \tag{3.15}$$

where $a_i \in \mathscr{B}$ and

$$0 \le n_i \le N_i - 1, \qquad N_i = N(a_i, a_{i+1}) \qquad \text{for } i = 1, \ldots, l-1.$$

COROLLARY 3.1. *Assume that the locality function N is constant. Consider the homogeneous component $A_{k,l}$ of A, spanned by all the words of length l and of the sum of indexes k. Then* $\dim A_{k,l} = N^{l-1}$.

Proof of Theorem 3.3. (a) Introduce a linear order on $\mathscr{B}$ and define an order on the set of generators X by the rule

$$a(m) > b(n) \Leftrightarrow |m| > |n| \text{ or } m = -n > 0 \text{ or } (m = n \text{ and } a > b).$$

In particular, for some $a \in \mathscr{B}$ we have

$$a(0) < a(-1) < a(1) < a(-2) < a(2) < \cdots .$$

For any relation r from (3.2) take the biggest term $\bar{r}$ and consider the rule $(\bar{r}, r - \bar{r})$. This way we get a collection of rules

$$\mathscr{R} = \left\{ \rho_1(b(n), a(m)) \,\middle|\, a, b \in \mathscr{B},\, n > \left\lfloor \frac{N(b,a) - 1}{2} \right\rfloor \right\}$$
$$\cup \left\{ \rho_2(b(n), a(m)) \,\middle|\, n < -\left\lceil \frac{N(b,a) - 1}{2} \right\rceil \right\},$$

where

$$\rho_1(b(n), a(m)) = \left(b(n)a(m), \sum_{s=1}^{N(b,a)} (-1)^{s+1} \binom{N(b,a)}{s} b(n-s)a(m+s) \right),$$

$$\rho_2(b(n), a(m)) = \left(b(n)a(m), \sum_{s=1}^{N(b,a)} (-1)^{s+1} \binom{N(b,a)}{s} b(n+s)a(m-s) \right).$$

By Lemma 3.2, we have to prove that these rules form a confluent rewriting system on $\Bbbk\langle X \rangle$. Clearly $\mathscr{R}$ is a rewriting system, since it decreases the order, and each subset of $\Bbbk\langle X \rangle$, containing only finitely many different letters from $\mathscr{B}$, has the minimal element, in contrast to the situation of Section 3.5.

As before, it is enough to check that $\mathscr{R}$ is confluent on any composition $w = c(p)b(n)a(m)$, of the principal parts of rules from $\mathscr{R}$. Consider the set $W = \{c(k)b(j)a(i) \mid k, j, i \in \mathbb{Z}\} \subset X^*$. We prove by induction on $w \in W$ that $\mathscr{R}$ is confluent on w. If w is sufficiently small, then it is terminal. Assume that $w = c(k)b(j)a(i)$ is an ambiguity, for example, that $\rho_1(c(p), b(n))$ and $\rho_2(b(n), a(m))$ are both applicable to w. Other cases are done in the same way. Let

$$w_1 = \rho_1(c(p), b(n))(w) = \sum_{s=1}^{N(c,b)} (-1)^s \binom{N(c,b)}{s} c(p-s)b(n+s)a(m),$$

$$w_2 = \rho_2(b(n), a(m))(w) = \sum_{t=1}^{N(b,a)} (-1)^t \binom{N(b,a)}{t} c(p)b(n+t)a(m-t).$$

Applying $\rho_2(b(n+s), a(m))$ for $s = 1, \ldots, N(b,a)$ to w_1 gives the same result as we get from applying $\rho_1(c(p), b(n+t))$ for $t = 1, \ldots, N(c,b)$ to w_2, namely,

$$\sum_{s,t \geq 1} (-1)^{s+t} \binom{N(c,b)}{s} \binom{N(b,a)}{t} c(p-s)b(n+s+t)a(m-t). \tag{3.16}$$

By the induction assumption, $w_1 - w_2$ is uniquely reduced to a terminal, and since all monomials in (3.16) are smaller than w, we conclude that this terminal must be 0.

(b) Follows at once from Lemma 3.1. ∎

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