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Algebraic shifting of cyclic polytopes and stacked polytopes

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Abstract

Gil Kalai introduced the shifting-theoretic upper bound relation as a method to generalize the g-theorem for simplicial spheres by using algebraic shifting. We will study the connection between the shifting-theoretic upper bound relation and combinatorial shifting. Also, we will compute the exterior algebraic shifted complex of the boundary complex of the cyclic d-polytope as well as of a stacked d-polytope. It will turn out that, in both cases, the exterior algebraic shifted complex coincides with the symmetric algebraic shifted complex.

Keywords: Algebraic shifting; Simplicial polytopes

0. Introduction

Let K be a simplicial complex on $[n]=\{1,\ldots,n\}$, $K_i=\{S\in K : |K|=i+1\}$ the *i-skeleton* of K and $f_i(K)=|K_i|$ the numbers of i-faces of K. The *dimension* of K is the integer $\dim K=\max\{i : |K_i|\neq 0\}$ and the *f-vector* of K is the vector $f(K)=(f_0(K), f_1(K),\ldots,f_{\dim K}(K))$.

If $S=\{s_1,s_2,\ldots,s_r\}$ and $T=\{t_1,t_2,\ldots,t_r\}$ are r-subsets of $[n]$ with $s_j<s_{j+1}$ and $t_j<t_{j+1}$ for $j=1,2,\ldots,r-1$, write $S\prec_p T$ if $s_j\leqslant t_j$ for all $1\leqslant j\leqslant r$. A simplicial complex K is called *shifted* if $T\in K$ and $S\prec_p T$ implies $S\in K$.

Algebraic shifting is an operation which associates to a simplicial complex K another shifted simplicial complex $\varDelta(K)$. Two types of algebraic shifting were introduced by Kalai. The first one is called exterior algebraic shifting, denoted by $K\to\varDelta^{\mathrm{e}}(K)$, and the second one is called symmetric algebraic shifting, denoted by $K\to\varDelta^{\mathrm{s}}(K)$. (See [14].)

Algebraic shifting is particularly useful to study the f-vector of simplicial complexes, since shifted simplicial complexes have a quite simple structure and algebraic shifting preserve f-vectors together with some combinatorial and algebraic properties, such as Betti numbers of reduced homology groups and Cohen–Macaulay property. For example, Björner and Kalai [3] characterize the f-vector of all simplicial complexes with prescribed Betti numbers by using algebraic shifting.

One of the major open problems in f-vector theory would be the generalization of the g-theorem for simplicial spheres. A *simplicial d-sphere* is a d-dimensional simplicial complex whose geometric realization is isomorphic to a d-dimensional sphere. In 1980, Stanley and Billera–Lee characterized the f-vectors of the boundary complex of simplicial

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d-polytopes. This characterization is called the g-theorem [19, III Theorem 1.1]. Although not every simplicial sphere is the boundary complex of a simplicial polytope, it has been conjectured that the f-vector of any simplicial $(d-1)$-sphere is equal to the f-vector of the boundary complex of some simplicial d-polytope.

In [13], Kalai introduced the shifting-theoretic upper bound relation (S-UBR) to prove this conjecture. Let $C(n,d)$ be the boundary complex of the cyclic d-polytope on n vertices. Kalai determined the structure of $\Delta^{\mathrm{s}}(C(n,d))$ and proved that if K is the boundary complex of a simplicial d-polytope on n vertices then K satisfies $\Delta^{\mathrm{s}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$ [13, p. 405]. Note that this inclusion follows from the Lefschetz property of the boundary complex of simplicial polytopes, which Stanley used for the proof of the necessity part of the g-theorem. (We refer the reader to [13, p. 394] for the Lefschetz property. Note that the Lefschetz property is called the ω-Hypothesis in [13].) Although the same inclusion has not been proved for exterior algebraic shifting, Kalai noticed that if K is a simplicial $(d-1)$-sphere on $[n]$ and $\Delta^{\mathrm{e}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$ is satisfied then the f-vector of K is equal to that of the boundary complex of some simplicial d-polytope, and conjectured that every simplicial $(d-1)$-sphere K on $[n]$ satisfies $\Delta^{\mathrm{e}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$.

We say that a $(d-1)$-dimensional simplicial complex K satisfies the S-UBR if K satisfies $\Delta^{\mathrm{e}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$. In this paper, we will study the relation between the S-UBR and combinatorial shifting.

Combinatorial shifting, which was introduced by Erdös et al. [6], is also an operation which associates to a simplicial complex K another shifted simplicial complex $\Delta^{\mathrm{c}}(K)$. Although combinatorial shifting may not be uniquely determined, it is easily computed by a simple combinatorial method. We will show that if there exists a combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ of K which satisfies $\Delta^{\mathrm{c}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$ then K satisfies the S-UBR (Proposition 2.4). Moreover, in case that K is a simplicial $(d-1)$-sphere, we will prove that if there is a combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ of K with $\Delta^{\mathrm{c}}(K)_{d-1} \subset \Delta^{\mathrm{s}}(C(n,d))$, then one has $\Delta^{\mathrm{e}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$ (Theorem 2.5). The benefit of the later result is that we need to compute combinatorial shifting only for the set of facets of K.

Thus, we can use combinatorial shifting for deducing the S-UBR and, therefore, we may use it for a generalization of the g-theorem. Also, since combinatorial shifting is entirely a combinatorial operation, proving $\Delta^{\mathrm{c}}(P) \subset \Delta^{\mathrm{s}}(C(n,d))$ for the boundary complex P of a simplicial d-polytope without using the Lefschetz property of P would be interesting.

By using this relation between the S-UBR and combinatorial shifting, we will show that $C(n,d)$ satisfies the S-UBR, that is, we will show that $\Delta^{\mathrm{e}}(C(n,d)) = \Delta^{\mathrm{s}}(C(n,d))$ for all $1 < d < n$ (Theorem 3.9).

We also compute algebraic shifting of the boundary complex of stacked polytopes. Let $P(n,d)$ be the boundary complex of a stacked d-polytope on n vertices. Although the combinatorial type of a stacked d-polytope may not be uniquely determined, it turns out that exterior algebraic shifting and symmetric algebraic shifting of the boundary complex of a stacked polytope does not depend on its combinatorial type. Also, we will show that $\Delta^{\mathrm{e}}(P(n,d)) = \Delta^{\mathrm{s}}(P(n,d))$. Our computation together with [13, p. 405 (10.1)] implies that if K is the boundary complex of a simplicial d-polytope on n vertices, then K satisfies the beautiful relation

$$\Delta^{\mathrm{s}}(P(n,d)) \subset \Delta^{\mathrm{s}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$$

which includes the classical upper bound theorem [13, p. 318] and the lower bound theorem [2] for the boundary complex of simplicial polytopes.

This paper is organized as follows: In Section 1, we recall the definition of algebraic shifting and combinatorial shifting. In Section 2, we will study the relation between combinatorial shifting and the S-UBR. In Section 3, we will construct $\Delta^{\mathrm{c}}(C(n,d))$ with $\Delta^{\mathrm{c}}(C(n,d)) = \Delta^{\mathrm{s}}(C(n,d))$, and prove $\Delta^{\mathrm{e}}(C(n,d)) = \Delta^{\mathrm{s}}(C(n,d))$. In Section 4, we will compute the exterior algebraic shifted complex of the boundary complex of a stacked d-polytope.

1. Algebraic shifting and combinatorial shifting

In this section, we recall the definitions and basic properties of algebraic shifting and combinatorial shifting. Let $\mathbf{k}$ be an infinite field and $R = \mathbf{k}[x_1, x_2, \ldots, x_n]$ the polynomial ring over $\mathbf{k}$ with $\deg(x_i) = 1$ for all i. For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$, we write $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Let $<_{\mathrm{rev}}$ denote the degree reverse lexicographic order induced by $x_1 < x_2 < \cdots < x_n$, namely, $x^a <_{\mathrm{rev}} x^b$ if $\deg(x^a) < \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and the first nonzero entry of $b - a$ is negative.

Let $<_\sigma$ be a term order. For any polynomial $f = \sum \alpha_a x^a \in R$, we write $\mathrm{in}_\sigma(f) = \max_{<_\sigma}\{x^a : \alpha_a \neq 0\}$. The *initial ideal* $\mathrm{in}_\sigma(I)$ of an ideal I in R w.r.t. the term order $<_\sigma$ is the monomial ideal generated by the set of monomials $\{\mathrm{in}_\sigma(f) : f \in I\}$.

Let $GL_n(\mathbf{k})$ denote the general linear group with coefficients in $\mathbf{k}$. Any $\varphi=(a_{ij}) \in GL_n(\mathbf{k})$ induce an automorphism of the graded ring R defined by

$$\varphi(f(x_1, x_2, \ldots, x_n)) = f\left(\sum_{i=1}^{n} a_{i1}x_i, \sum_{i=1}^{n} a_{i2}x_i, \ldots, \sum_{i=1}^{n} a_{in}x_i\right).$$

A monomial ideal I is called *strongly stable* if $ux_q \in I$ implies $ux_p \in I$ for any integers $n \geqslant p > q \geqslant 1$. The following fact is known.

Lemma 1.1 (Green [8, Theorem 1.27]). *Fix a term order* $<_\sigma$ *with* $x_1 <_\sigma x_2 <_\sigma \cdots <_\sigma x_n$. *For each graded ideal* $I \subset R$, *there is a nonempty Zariski open subset* $U \subset GL_n(\mathbf{k})$ *such that* $\mathrm{in}_\sigma(\varphi(I))$ *is constant for all* $\varphi \in U$. *Furthermore, if* $\mathbf{k}$ *is a field of characteristic* 0, *then this* $\mathrm{in}_\sigma(\varphi(I))$ *is strongly stable.*

The above monomial ideal $\mathrm{in}_\sigma(\varphi(I))$ with $\varphi \in U$ is called the *generic initial ideal of I* w.r.t. the term order $<_\sigma$, and will be denoted $\mathrm{gin}_\sigma(I)$. In particular, we write $\mathrm{gin}_{\mathrm{rev}}(I) = \mathrm{gin}(I)$.

Next, we recall shifting operations. A *shifting operation* on $[n]$ is an operation which associates with each simplicial complex K on $[n]$ a simplicial complex $\Delta(K)$ on $[n]$ and which satisfies the following conditions:

(S_1) $\Delta(K)$ is shifted;
(S_2) $\Delta(K) = K$ if K is shifted;
(S_3) $f(K) = f(\Delta(K))$;
(S_4) $\Delta(K') \subset \Delta(K)$ if $K' \subset K$.

Symmetric algebraic shifting: Assume that $\mathbf{k}$ is a field of characteristic 0. Let K be a simplicial complex on $[n]$. Thus K is a collection of subsets of $[n]$ such that (i) $\{j\} \in K$ for all $j \in [n]$ and (ii) if $S \subset [n]$ and $T \in K$ with $S \subset T$, then $S \in K$. The *Stanley–Reisner ideal* I_K *of K* is the monomial ideal generated by all squarefree monomials $x^S = x_{s_1}x_{s_2}\cdots x_{s_r}$ with $S = \{s_1, s_2, \ldots, s_r\} \notin K$ and $S \subset [n]$. Let

$$\mathrm{GIN}(K) = \{x^a \in R : x^a \notin \mathrm{gin}(I_K)\}.$$

For every monomial $x^a \in R$, we write $\min(x^a)$ for the minimal integer i such that x_i divides x^a. If $m \in \mathrm{GIN}(K)$ is a monomial with $\deg(m) = r \leqslant n$ and with $\min(m) \geqslant r$, write $m = x_{i_1}x_{i_2}\cdots x_{i_r}$ with $r \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$, and define

$$S(m) = \{i_1 - r + 1, i_2 - r + 2, \ldots, i_{r-1} - 1, i_r\}.$$

The *symmetric algebraic shifted complex* $\Delta^{\mathrm{s}}(K)$ of K is defined by

$$\Delta^{\mathrm{s}}(K) = \{S(m) : m \in \mathrm{GIN}(K), \deg(m) = r \leqslant n \text{ and } \min(m) \geqslant r\}.$$

Kalai [13] introduced the operation $K \to \Delta^{\mathrm{s}}(K)$, and proved that this operation is in fact a shifting operation. The shifting operation $K \to \Delta^{\mathrm{s}}(K)$ is called *symmetric algebraic shifting*.

Exterior algebraic shifting: Let $\mathbf{k}$ be an infinite field and V a $\mathbf{k}$-vector space with basis $e_1, e_2 \ldots, e_n$ and $E = \bigoplus_{i=0}^{n} \bigwedge^i V$ the exterior algebra of V. If $S = \{s_1, s_2, \ldots, s_r\} \subset [n]$ with $s_1 < s_2 < \cdots < s_r$, then $e_S = e_{s_1} \wedge e_{s_2} \wedge \cdots \wedge e_{s_r} \in E$ is called a monomial of E of degree r. We can define term orders, strongly stable monomial ideals and generic initial ideals $\mathrm{Gin}(J)$ of a graded ideal J in E in the same way as for the polynomial ring R. We refer the reader to [1] for foundations.

Let K be a simplicial complex on $[n]$. The *exterior face ideal* J_K *of K* is the monomial ideal in E generated by all monomials $e_S \in E$ with $S \notin K$. If $\varphi \in GL_n(\mathbf{k})$ and K is a simplicial complex on $[n]$, the simplicial complex $\Delta_\varphi(K)$ is defined by

$$J_{\Delta_\varphi(K)} = \mathrm{in}_{\mathrm{rev}}(\varphi(J_K)).$$

The *exterior algebraic shifted complex* $\Delta^{\mathrm{e}}(K)$ of a simplicial complex K on $[n]$ is the simplicial complex on $[n]$ defined by

$$J_{\Delta^{\mathrm{e}}(K)} = \mathrm{Gin}(J_K).$$

The operation $K \to \Delta^{\mathrm{e}}(K)$, which was also introduced by Kalai, is called *exterior algebraic shifting*, and this operation is in fact a shifting operation (see [14, Section 2]). In case of exterior algebraic shifting, we need not assume that the characteristic of the field is zero. However, it may depend on the characteristic of the field $\mathbf{k}$. (See Remark 1.7 at the end of this section.)

We recall some useful properties of algebraic shifting. Let K be a simplicial complex on $[n]$. If S is a face in K, then $\mathrm{lk}(S,K)=\{T\backslash S : T \in K \text{ and } S \subset T\}$ is called the *link of S in K*. Let $\tilde{H}_i(K)$ be the ith *reduced homology group of K*. Then $\beta_i(K)=\dim_k \tilde{H}_i(K)$ is called the ith *reduced Betti number* of K. A pure simplicial complex K is called *Cohen–Macaulay* if, for every face S of K (including the empty face), one has $\tilde{H}_i(\mathrm{lk}(S,K))=0$ for all $i<\dim(\mathrm{lk}(S,K))$. Note that the boundary complex of every simplicial polytope is Cohen–Macaulay.

Lemma 1.2 (*Kalai*). *Let K be a simplicial complex on* $[n]$. *The following conditions are equivalent*:

(i) *K is Cohen–Macaulay*;
(ii) $\Delta^{\mathrm{e}}(K)$ *is pure*;
(iii) $\Delta^{\mathrm{s}}(K)$ *is pure*.

See [10, Theorem 8.13] for the proof of Lemma 1.2. Also, the following fact is known (See [14, Theorem 3.2]).

Lemma 1.3. *Let K be a simplicial complex. Then* $\beta_i(K)=\beta_i(\Delta^{\mathrm{e}}(K))=\beta_i(\Delta^{\mathrm{s}}(K))$ *for all i. Furthermore, if K is shifted, then*

$$\beta_i(K)=|\{S\in K : |S|=i+1, 1\notin S \text{ and } S\cup\{1\}\notin K\}|.$$

Combinatorial shifting: Erdös et al. [6] introduced combinatorial shifting. Let K be a collection of r-subsets of $[n]$, where $r\leqslant n$. For $1\leqslant i<j\leqslant n$, write $\mathrm{Shift}_{ij}(K)$ for the collection of r-subsets of $[n]$ whose elements are $C_{ij}(S)\subset[n]$, where $S\in K$ and where

$$C_{ij}(S)=\begin{cases}(S\backslash\{j\})\cup\{i\} & \text{if } j\in S,\ i\notin S \text{ and } (S\backslash\{j\})\cup\{i\}\notin K,\\ S & \text{otherwise.}\end{cases}$$

We can define $\mathrm{Shift}_{ij}(K)$ for a simplicial complex K by the same way. It follows from, e.g., [10, Corollary 8.6] that there exists a finite sequence of pairs of integers $(i_1,j_1),(i_2,j_2),\ldots,(i_q,j_q)$ with each $1\leqslant i_k<j_k\leqslant n$ such that

$$\mathrm{Shift}_{i_q j_q}(\mathrm{Shift}_{i_{q-1} j_{q-1}}(\cdots(\mathrm{Shift}_{i_1 j_1}(K))\cdots))$$

is shifted. Such a shifted complex is called a *combinatorial shifted complex* of K and will be denoted by $\Delta^{\mathrm{c}}(K)$. A combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ of K is, however, not necessarily unique. The operation $K\to\Delta^{\mathrm{c}}(K)$, which is in fact a shifting operation [10, Lemma 8.4], is called *combinatorial shifting*.

We recall the following fact.

Lemma 1.4 (*Herzog [10, Lemma 8.3]*). *Let K be a simplicial complex on* $[n]$. *For any integers* $1\leqslant i<j\leqslant n$, *let* $\varphi_{ij}\in GL_n(\mathbf{k})$ *be the matrix defined by* $\varphi_{ij}(x_k)=x_k$ *for* $k\neq i$ *and* $\varphi_{ij}(x_i)=x_i+x_j$. *Then one has*

$$\Delta_{\varphi_{ij}}(K)=\mathrm{Shift}_{ij}(K).$$

For any simplicial complex K on $[n]$ and any subset $S\subset[n]$, define

$$m_{\leqslant S}(K)=|\{T\in K : T\leqslant_{\mathrm{rev}} S \text{ and } |T|=|S|\}|.$$

The next property is known.

Lemma 1.5 (*Murai [15, Theorem 3.1]*). *Let K be a simplicial complex on* $[n]$ *and* $\varphi\in GL_n(\mathbf{k})$. *Then, for any* $S\subset[n]$, *one has*

$$m_{\leqslant S}(\Delta^{\mathrm{e}}(K))\geqslant m_{\leqslant S}(\Delta^{\mathrm{e}}(\Delta_{\varphi}(K))).$$

Then we have the following relation between $\Delta^{\mathrm{c}}(K)$ and $\Delta^{\mathrm{e}}(K)$.

Lemma 1.6. *Let K be a simplicial complex on $[n]$. Then, for any combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ and for any subset $S \subset [n]$, one has*

$$m_{\leqslant S}(\Delta^{\mathrm{e}}(K)) \geqslant m_{\leqslant S}(\Delta^{\mathrm{c}}(K)).$$

Proof. By the definition of combinatorial shifting, there exists a finite sequence of pairs of integers $(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)$ with each $1 \leqslant i_k < j_k \leqslant n$ such that $\Delta^{\mathrm{c}}(K) = \Delta_{\varphi_{i_q j_q}}(\Delta_{\varphi_{i_{q-1} j_{q-1}}}(\cdots(\Delta_{\varphi_{i_1 j_1}}(K))\cdots))$. Also, condition ($\mathrm{S}_2$) of shifting operations says that $\Delta^{\mathrm{e}}(\Delta^{\mathrm{c}}(K)) = \Delta^{\mathrm{c}}(K)$ since $\Delta^{\mathrm{c}}(K)$ is shifted. Then, by Lemma 1.5, we have

$$\begin{aligned} m_{\leqslant S}(\Delta^{\mathrm{e}}(K)) &\geqslant m_{\leqslant S}(\Delta^{\mathrm{e}}(\Delta_{\varphi_{i_1 j_1}}(K))) \\ &\geqslant m_{\leqslant S}(\Delta^{\mathrm{e}}(\Delta_{\varphi_{i_2 j_2}}(\Delta_{\varphi_{i_1 j_1}}(K)))) \\ &\;\;\vdots \\ &\geqslant m_{\leqslant S}(\Delta^{\mathrm{e}}(\Delta^{\mathrm{c}}(K))) = m_{\leqslant S}(\Delta^{\mathrm{c}}(K)), \end{aligned}$$

for every $S \subset [n]$, as desired. $\square$

Remark 1.7. It is not always possible to realize $\Delta^{\mathrm{e}}(K)$ by $\Delta^{\mathrm{c}}(K)$, even if K is a 2-dimensional simplicial complex. One reason why we cannot realize $\Delta^{\mathrm{e}}(K)$ by combinatorial shifting is the fact that $\Delta^{\mathrm{e}}(K)$ depends on the characteristics of the field $\mathbf{k}$. Let $\Delta^{\mathrm{e}}_{(p)}(K)$ denote the exterior algebraic shifted complex of K w.r.t. the field $\mathbf{k}$ of characteristic p. Then $m_{\leqslant S}(\Delta^{\mathrm{e}}_{(p)}(K))$ is equal to the rank of a certain matrix whose entries can be seen as polynomials of $\mathbb{Z}[x_{ij}]_{1 \leqslant i,j \leqslant n}$. (See [15, Lemma 1.5] or [16, Section 2].) The rank of this matrix where $\mathrm{char}(\mathbf{k}) = 0$ is equal to or larger than the rank of this matrix where $\mathrm{char}(\mathbf{k}) = p > 0$. Then, for all p and for all $S \subset [n]$, we have

$$m_{\leqslant S}(\Delta^{\mathrm{e}}_{(0)}(K)) \geqslant m_{\leqslant S}(\Delta^{\mathrm{e}}_{(p)}(K)) \geqslant m_{\leqslant S}(\Delta^{\mathrm{c}}(K)).$$

This implies that if $\Delta^{\mathrm{e}}_{(p)}(K) \neq \Delta^{\mathrm{e}}_{(0)}(K)$ for some p, then we cannot realize $\Delta^{\mathrm{e}}_{(0)}(K)$ by $\Delta^{\mathrm{c}}(K)$. One such example is a triangulation A of the projective space with 6 vertices and with 10 triangles (see [14, Section 6]), that is, A is the simplicial complex generated by

$$\left\{ \begin{array}{l} \{1,2,4\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,5,6\}, \\ \{2,3,5\}, \{2,3,6\}, \{2,4,5\}, \{3,4,6\}, \{4,5,6\} \end{array} \right\}.$$

It is known that (see [5, p. 236]) $\beta_2(A) = 1$ if $\mathrm{char}(\mathbf{k}) = 2$ and $\beta_2(A) = 0$ if $\mathrm{char}(\mathbf{k}) \neq 2$. Then Lemma 1.3 says that $\{2,3,4\} \in \Delta^{\mathrm{e}}_{(2)}(A)$ but $\{2,3,4\} \notin \Delta^{\mathrm{e}}_{(0)}(A)$. Set $S_0 = \{1,2,6\}$. Since $S_0 <_{\mathrm{rev}} \{2,3,4\}$, we have $m_{\leqslant S_0}(\Delta^{\mathrm{e}}_{(2)}(A)) < f_2(A)$. Then Lemma 1.6 says that $f_2(A) > m_{\leqslant S_0}(\Delta^{\mathrm{c}}(A))$. This fact implies that $\{2,3,4\} \in \Delta^{\mathrm{c}}(A)$ and $\Delta^{\mathrm{c}}(A) \neq \Delta^{\mathrm{e}}_{(0)}(A)$ for any combinatorial shifted complex $\Delta^{\mathrm{c}}(A)$ of A.

Note that, in [16], we introduce a simplicial complex K whose exterior algebraic shifted complex $\Delta^{\mathrm{e}}(K)$ does not depend on the characteristic of the field but satisfies $\Delta^{\mathrm{e}}(K) \neq \Delta^{\mathrm{c}}(K)$ for all $\Delta^{\mathrm{c}}(K)$.

2. The shifting-theoretic upper bound relation

First, we recall h-vectors introduced by McMullen and Walkup. Let K be a $(d-1)$-dimensional simplicial complex and $(f_0, f_1, \ldots, f_{d-1})$ the f-vector of K. The *h-vector* $h(K) = (h_0, h_1, \ldots, h_d)$ of K is defined by the relation

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i},$$

where $f_{-1} = 1$. This is equivalent to

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1} \quad \text{and to} \quad f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{d-i} h_i \quad \text{for } 0 \leqslant i \leqslant d.$$

For positive integers $i<j$ and $d>0$, define $[i,j]=\{i,i+1,\ldots,j-1,j\}$, $[j]=[1,j]$ and $\binom{[i,j]}{d}=\{S\subset[i,j]: |S|=d\}$. Let K be a shifted $(d-1)$-dimensional simplicial complex. Set

$$W_i=\left\{S\in\binom{[n]}{d}:[d-i]\subset S \text{ and } d-i+1\notin S\right\} \quad \text{for } i=0,1,\ldots,d.$$

Then, for every $0\leqslant i\leqslant d-1$, $S\in W_i$ and $T\in W_{i+1}$ clearly implies $S<_{\mathrm{rev}}T$. Also $\biguplus_{i=0}^{d}W_i=\binom{[n]}{d}$. Now, we define $W_i(K)=W_i\cap K$ for $i=0,1,\ldots,d$.

Lemma 2.1 (*Kalai [13, Lemma 7.1]*). *Let K be a pure shifted $(d-1)$-dimensional simplicial complex. Then one has $h_i(K)=|W_i(K)|$.*

Kalai found the following nice relation by using the Lefschetz property. Let $C(n,d)$ be the boundary complex of the cyclic d-polytope on n vertices.

Lemma 2.2 (*Kalai [14, Theorem 5.1]*). *If K is the boundary complex of a simplicial d-polytope on n vertices, then one has*

$$\Delta^{\mathrm{s}}(K)\subset\Delta^{\mathrm{s}}(C(n,d)).$$

Although the same property has not been proved for exterior algebraic shifting, Kalai and Sarkaria conjectured the following.

Conjecture 2.3 (*Kalai [14, Conjecture 27]*). Every simplicial $(d-1)$-sphere K on $[n]$ satisfies $\Delta^{\mathrm{e}}(K)\subset\Delta^{\mathrm{s}}(C(n,d))$.

As written in the introduction, it was proved in [13] that Conjecture 2.3 implies the g-theorem for all simplicial spheres. We say that a $(d-1)$-dimensional simplicial complex K satisfies the S-UBR if $\Delta^{\mathrm{e}}(K)\subset\Delta^{\mathrm{s}}(C(n,d))$. In the rest of this section, we will study the relation between combinatorial shifting and the S-UBR.

We recall the structure of $\Delta^{\mathrm{s}}(C(n,d))$. A d-subset $S\subset[n]$ is called *admissible* if $j\notin S$ implies $[j+1,d-j+2]\subset S$. Kalai proved that $\Delta^{\mathrm{s}}(C(n,d))$ is pure and $\Delta^{\mathrm{s}}(C(n,d))_{d-1}$ consists of all admissible d-subsets of $[n]$ (see [14, Proposition 5.2]), in other words, $\Delta^{\mathrm{s}}(C(n,d))$ is the simplicial complex generated by

$$\left\{\left[1,\left\lfloor\frac{d+1}{2}\right\rfloor\right]\cup S: S\subset\left[\left\lfloor\frac{d+1}{2}\right\rfloor+1,n\right],\ |S|=d-\left\lfloor\frac{d+1}{2}\right\rfloor\right\}$$
$$\bigcup_{1\leqslant j\leqslant\lfloor\frac{d+1}{2}\rfloor}\{([1,d-j+2]\backslash\{j\})\cup S: S\subset[d-j+3,n],\ |S|=j-1\}, \tag{1}$$

where $\lfloor\frac{d+1}{2}\rfloor$ is the integer part of $\frac{d+1}{2}$.

Proposition 2.4. *Let K be a $(d-1)$-dimensional simplicial complex on $[n]$. If $\Delta^{\mathrm{c}}(K)\subset\Delta^{\mathrm{s}}(C(n,d))$ for some combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ of K, then one has $\Delta^{\mathrm{e}}(K)\subset\Delta^{\mathrm{s}}(C(n,d))$.*

Proof. Formula (1) says that, for $0\leqslant i\leqslant\frac{d}{2}$, we have

$$W_i(\Delta^{\mathrm{s}}(C(n,d)))=\{[d-i]\cup S: S\subset[d-i+2,n],|S|=i\} \tag{2}$$

and, for $\frac{d}{2}<i\leqslant d$, we have

$$W_i(\Delta^{\mathrm{s}}(C(n,d)))=\{([i+1]\backslash\{d-i+1\})\cup S: S\subset[i+2,n],|S|=d-i\}. \tag{3}$$

The above equations say that $W_i(\Delta^{\mathrm{s}}(C(n,d)))$ is the set of first $|W_i(\Delta^{\mathrm{s}}(C(n,d)))|$ d-subsets w.r.t. $<_{\mathrm{rev}}$ which contain $[d-i]$ and which do not contain $d-i+1$.

We will show that $\Delta^{\mathrm{e}}(K)\not\subset\Delta^{\mathrm{s}}(C(n,d))$ implies $\Delta^{\mathrm{c}}(K)\not\subset\Delta^{\mathrm{s}}(C(n,d))$ for any combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ of K.

Assume that there is an $S \in \Delta^{\mathrm{e}}(K)$ with $S \notin \Delta^{\mathrm{s}}(C(n,d))$. Let T be the set of first $d-|S|$ integers in $[n]\backslash S$ and $p=\min\{t : t \notin S \cup T\}-1$. Then we have $|S \cup T|=d$, $[p] \subset S \cup T$ and $p+1 \notin S \cup T$. Set $S_0=(S \cup T)\backslash[p]$. Note that $S_0 \subset S$ and $S_0 \in \Delta^{\mathrm{e}}(K)$ by the construction.

First, we will show $S_0 \notin \Delta^{\mathrm{s}}(C(n,d))$. If $S_0 \in \Delta^{\mathrm{s}}(C(n,d))$, then there exists $T_0 \subset [n]\backslash S_0$ with $|T_0|=p$ such that $S_0 \cup T_0 \in \Delta^{\mathrm{s}}(C(n,d))$ since $\Delta^{\mathrm{s}}(C(n,d))$ is pure. However, since $\Delta^{\mathrm{s}}(C(n,d))$ is shifted, we have $S \cup T=[p] \cup S_0 \in \Delta^{\mathrm{s}}(C(n,d))$. This contradicts the assumption $S \notin \Delta^{\mathrm{s}}(C(n,d))$. Thus we have $S_0 \notin \Delta^{\mathrm{s}}(C(n,d))$.

Second, we will show $S_0 >_{\mathrm{rev}} F_0=\max_{<_{\mathrm{rev}}}\{F \in \Delta^{\mathrm{s}}(C(n,d)) : |F|=d-p\}$. Since $W_{d-p}(\Delta^{\mathrm{s}}(C(n,d))) \neq \emptyset$, there exists an $F' \in \Delta^{\mathrm{s}}(C(n,d))$ such that $|F'|=d-p$, $F' \cap [p+1]=\emptyset$ and $[p] \cup F' \in W_{d-p}(\Delta^{\mathrm{s}}(C(n,d)))$. Then, since $F_0 >_{\mathrm{rev}} F'$, we have $[p+1] \cap F_0=\emptyset$. Since $\Delta^{\mathrm{s}}(C(n,d))$ is pure and shifted, we have $[p] \cup F_0 \in W_{d-p}(\Delta^{\mathrm{s}}(C(n,d)))$. Recall that $[p] \cup S_0=S \cup T$ satisfies that $S \cup T \notin \Delta^{\mathrm{s}}(C(n,d))$, $[p] \subset S \cup T$ and $p+1 \notin S \cup T$. Since $W_{d-p}(\Delta^{\mathrm{s}}(C(n,d)))$ is the set of first $|W_{d-p}(\Delta^{\mathrm{s}}(C(n,d)))|$ d-subsets w.r.t. $<_{\mathrm{rev}}$ which contain $[p]$ and which do not contain $p+1$, we have $[p] \cup S_0=S \cup T >_{\mathrm{rev}} [p] \cup F_0$. Hence we have $S_0 >_{\mathrm{rev}} F_0$.

Fix a combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$. Let $C_0=\max_{<_{\mathrm{rev}}}\{S \in \Delta^{\mathrm{c}}(K) : |S|=d-p\}$. Then, since $m_{\leqslant C_0}(\Delta^{\mathrm{c}}(K))=f_{d-p-1}(\Delta^{\mathrm{c}}(K))=f_{d-p-1}(\Delta^{\mathrm{e}}(K))$, we have $m_{\leqslant C_0}(\Delta^{\mathrm{e}}(K))=f_{d-p-1}(\Delta^{\mathrm{e}}(K))$ by Lemma 1.6. This fact says that $C_0 \geqslant_{\mathrm{rev}} S_0$. Thus we have $C_0 \geqslant_{\mathrm{rev}} S_0 >_{\mathrm{rev}} F_0$ and $C_0 \notin \Delta^{\mathrm{s}}(C(n,d))$ by the definition of F_0. Hence we have $\Delta^{\mathrm{c}}(K) \not\subset \Delta^{\mathrm{s}}(C(n,d))$. $\square$

A $(d-1)$-dimensional simplicial complex K is called *Gorenstein** if, for every face $S \in K$ (including the empty face), one has $\tilde{H}_i(\mathrm{lk}(S,K))=0$ for $i<\dim(\mathrm{lk}(S,K))$ and $\tilde{H}_i(\mathrm{lk}(S,K))=\mathbf{k}$ for $i=\dim(\mathrm{lk}(S,K))$. For example, simplicial spheres are Gorenstein* [19, II Corollary 5.2]. If K is a $(d-1)$-dimensional Gorenstein* simplicial complex, then the h-vector $h(K)=(h_0(K),h_1(K),\ldots,h_d(K))$ of K satisfies

$$h_i(K)=h_{d-i}(K) \quad \text{for } 0 \leqslant i \leqslant d. \tag{4}$$

Relation (4) is called the *Dehn–Sommerville equations*. (See [19, p. 67].)

Theorem 2.5. *Let K be a $(d-1)$-dimensional Gorenstein* complex on $[n]$.*

(i) *If $\Delta^{\mathrm{c}}(K)_{d-1} \subset \Delta^{\mathrm{s}}(C(n,d))_{d-1}$, then this $\Delta^{\mathrm{c}}(K)$ is pure.*
(ii) *If there is a combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ of K with $\Delta^{\mathrm{c}}(K)_{d-1} \subset \Delta^{\mathrm{s}}(C(n,d))_{d-1}$, then one has $\Delta^{\mathrm{e}}(K) \subset \Delta^{\mathrm{s}}(C(n,d))$.*

Proof. Statement (ii) immediately follows from statement (i) together with Proposition 2.4. Thus we will show statement (i).

Fix a combinatorial shifted complex $\Delta^{\mathrm{c}}(K)$ with $\Delta^{\mathrm{c}}(K)_{d-1} \subset \Delta^{\mathrm{s}}(C(n,d))_{d-1}$. Since every Gorenstein* simplicial complex is Cohen–Macaulay, Lemma 1.2 says that $\Delta^{\mathrm{e}}(K)$ is pure. Also, Lemma 2.1 together with the Dehn–Sommerville equations says that $|W_i(\Delta^{\mathrm{e}}(K))|=|W_{d-i}(\Delta^{\mathrm{e}}(K))|=h_i(K)$ for $i=0,1,\ldots,d$.

Since $\Delta^{\mathrm{c}}(K)_{d-1} \subset \Delta^{\mathrm{s}}(C(n,d))$, Eq. (3) say that any $F \in W_{d-i}(\Delta^{\mathrm{c}}(K))$ with $0 \leqslant i \leqslant \frac{d}{2}$ can be written of the form $F=([d-i+1]\backslash\{i+1\}) \cup F_0$ with $F_0 \subset [d-i+2,n]$. Since $\Delta^{\mathrm{c}}(K)$ is shifted, we have $[d-i] \cup F_0 \in W_i(\Delta^{\mathrm{c}}(K))$. By using this fact, for $0 \leqslant i \leqslant \frac{d}{2}$, we can define the injection from $W_{d-i}(\Delta^{\mathrm{c}}(K))$ to $W_i(\Delta^{\mathrm{c}}(K))$ by

$$([d-i+1]\backslash\{i+1\}) \cup F_0 \to [d-i] \cup F_0.$$

Thus we have $|W_{d-i}(\Delta^{\mathrm{c}}(K))| \leqslant |W_i(\Delta^{\mathrm{c}}(K))|$ for all $0 \leqslant i \leqslant \frac{d}{2}$.

Let $W_i=\left\{S \in \binom{[n]}{d} : [d-i] \subset S, d-i+1 \notin S\right\}$ for $i=0,1,\ldots,d$. Recall that if $S \in W_i$ and $T \in W_j$ with $i<j$ then we have $S <_{\mathrm{rev}} T$. Set $S_i=\max_{<_{\mathrm{rev}}}(W_i)$ for $i=0,1,\ldots,d$. Then, Lemma 1.6 says that, for any $0 \leqslant i \leqslant d$, we have

$$\sum_{j=0}^{i}|W_j(\Delta^{\mathrm{e}}(K))|=m_{\leqslant S_i}(\Delta^{\mathrm{e}}(K)) \geqslant m_{\leqslant S_i}(\Delta^{\mathrm{c}}(K))=\sum_{j=0}^{i}|W_j(\Delta^{\mathrm{c}}(K))|. \tag{5}$$

Since $\sum_{j=0}^{i}|W_{d-j}(L)| = f_{d-1}(L) - \sum_{j=0}^{d-i-1}|W_j(L)|$ for any $0 \leqslant i \leqslant \frac{d}{2}$ and for any $(d-1)$-dimensional shifted simplicial complex L, Eq. (5) together with the property (S_3) of shifting operations say that

$$\sum_{j=0}^{i}|W_{d-j}(\Delta^{\mathrm{e}}(K))| \leqslant \sum_{j=0}^{i}|W_{d-j}(\Delta^{\mathrm{c}}(K))| \quad \text{for } i=0,1,\ldots,d. \tag{6}$$

Recall that we already proved that $|W_i(\Delta^{\mathrm{e}}(K))| = |W_{d-i}(\Delta^{\mathrm{e}}(K))| = h_i(K)$ for $i=0,1,\ldots,d$, that is, the left-hand-side of (5) is equal to that of (6). We also proved that $\sum_{j=0}^{i}|W_j(\Delta^{\mathrm{c}}(K))| \geqslant \sum_{j=0}^{i}|W_{d-j}(\Delta^{\mathrm{c}}(K))|$ for $i=0,1,\ldots,\lfloor \frac{d}{2} \rfloor$. Then (5) and (6) say that, for all $0 \leqslant i \leqslant \frac{d}{2}$, we have

$$\sum_{j=0}^{i}|W_i(\Delta^{\mathrm{e}}(K))| = \sum_{j=0}^{i}|W_i(\Delta^{\mathrm{c}}(K))| = \sum_{j=0}^{i}|W_{d-i}(\Delta^{\mathrm{c}}(K))| = \sum_{j=0}^{i}|W_{d-i}(\Delta^{\mathrm{e}}(K))|.$$

Inductively, we have $|W_i(\Delta^{\mathrm{c}}(K))| = |W_i(\Delta^{\mathrm{e}}(K))| = h_i(K)$ for all $0 \leqslant i \leqslant d$.

Let L be the pure simplicial complex generated by $\Delta^{\mathrm{c}}(K)_{d-1}$. Then Lemma 2.1 says that $h_i(L) = |W_i(L)| = |W_i(\Delta^{\mathrm{c}}(K))| = h_i(K)$ for all i. Thus L and K have the same h-vector, that is, they have the same f-vector. Since $\Delta^{\mathrm{c}}(K) \supset L$, the property ($S_3$) of shifting operations says that $\Delta^{\mathrm{c}}(K) = L$. Hence $\Delta^{\mathrm{c}}(K)$ is pure. □

3. Cyclic polytopes

We recall the definition of cyclic polytopes. We refer the reader to [4] for the basic theory of convex polytopes.

Let $\mathbb{R}$ denote the field of real numbers. For any subset M of the d-dimensional Euclidean space $\mathbb{R}^d$, there is the smallest convex set containing M. This convex set is called the *convex hull of* M and will be denoted by $\mathrm{conv}(M)$. For $d \geqslant 2$, the *moment curve* in $\mathbb{R}^d$ is the curve parameterized by

$$t \to x(t) = (t, t^2, \ldots, t^d) \in \mathbb{R}^d.$$

The *cyclic d-polytope* with n vertices is the convex hull P of the form

$$P = \mathrm{conv}(\{x(t_1), x(t_2), \ldots, x(t_n)\}),$$

where $t_1, t_2, \ldots, t_n$ are distinct real numbers. The boundary complex of P is denoted by $C(n,d)$ where $x(t_i) \in P$ corresponds to $i \in C(n,d)$.

For $S \subset [i,j]$, we say that S satisfies the *evenness condition on* $[i,j]$ if $|S \cap [p,q]|$ is even for all $i \leqslant p < q \leqslant j$ with $p \notin S$ and $q \notin S$. Let $C([i,j],d)$ be the collection of d-subsets of $[i,j]$ which satisfies the evenness condition on $[i,j]$, where $C([i,j],0) = \emptyset$. The following property is called Gale's evenness condition (See e.g., [4, Theorem 13.6]).

Lemma 3.1. *Let $1 < d < n$. Then $C(n,d)_{d-1}$ is combinatorially isomorphic to $C([n],d)$, where $C(n,d)_{d-1} = \{S \in C(n,d) : |S| = d\}$.*

Let $d \leqslant n$ be positive integers. If K is a collection of subsets of $[n]$, define

$$\mathrm{Shift}_{i\downarrow n}(K) = \mathrm{Shift}_{ii+1}(\cdots(\mathrm{Shift}_{in-1}(\mathrm{Shift}_{in}(K)))\cdots)$$

and

$$\mathrm{Shift}_{i\uparrow n}(K) = \mathrm{Shift}_{in}(\cdots(\mathrm{Shift}_{ii+2}(\mathrm{Shift}_{ii+1}(K)))\cdots).$$

If $S \subset [n]$ and K is a collection of d-subsets of $[n]\backslash S$, then we write

$$S * K = \{S \cup T : T \in K\}.$$

In this section, our aim is finding pairs of integers $(p_1,q_1), \ldots, (p_r,q_r)$ which satisfies

$$\mathrm{Shift}_{p_1q_1}(\cdots(\mathrm{Shift}_{p_rq_r}(C([n],d)))) = \Delta^{\mathrm{s}}(C(n,d))_{d-1}.$$

The case when d is even will be proved in Lemma 3.4, and the case when d is odd will be proved in Lemma 3.8. Throughout this section, we assume that all subsets of the form $\{i_1, i_2, \ldots, i_t\} \subset [n]$ satisfy $i_1 < i_2 < \cdots < i_t$.

Even case: First, we consider the even case. Fix integers $1 < 2d < n$. Let

$$G(m) = \bigcup_{j=1}^{m} \{([1, 2d - j + 2] \backslash \{j\}) \cup S : S \subset [2d - j + 3, n], |S| = j - 1\}$$

and

$$H(m) = \bigcup_{\{j_1, j_2, \ldots, j_m\} \subset [2d-m+1, n]} ([m] \cup \{j_1, j_2, \ldots, j_m\}) * C([m+1, j_1 - 1], 2d - 2m)$$

for $m = 1, 2, \ldots, d$. Then it is easily verified that

$$\Delta^{\mathrm{s}}(C(n, 2d))_{2d-1} = G(d) \cup H(d).$$

Lemma 3.2. *Let* $1 < 2d < n$. *Then, one has*

$$\mathrm{Shift}_{1\downarrow n}(C([n], 2d)) \supset H(1) \cup G(1).$$

Proof. By the evenness condition, $C([n], 2d)$ contains all $2d$-subsets $S \subset [n]$ of the form $S = \{1, i_1, i_1 + 1, i_2, i_2 + 1, \ldots, i_{d-1}, i_{d-1} + 1, n\}$ and these of the form $S = \{i_1, i_1 + 1, i_2, i_2 + 1, \ldots, i_d, i_d + 1\}$. Let

$$L = \mathrm{Shift}_{1,2d+2}(\mathrm{Shift}_{1,2d+3} \cdots (\mathrm{Shift}_{1n}(C([n], 2d))) \cdots).$$

Then we have $[2, 2d + 1] \in L$ since $[2, 2d + 1] \in C([n], 2d)$. Let $S \in C([n], d)$. If $1 \in S$, then S does not change by Shift_{1k} for all $k > 1$, and therefore we have $S \in L$. Otherwise, $(S \backslash \{\max(S)\}) \cup \{1\}$ must be contained in L. Let

$$\begin{aligned} \tilde{L} = &\{\{1, i_1, i_1 + 1, i_2, i_2 + 1, \ldots, i_{d-1}, i_{d-1} + 1, n\} : 2 \leqslant i_1 \text{ and } i_{d-1} + 1 \leqslant n - 1\} & (7)\\ &\cup \{\{1, 2, i_2, i_2 + 1, \ldots, i_d, i_d + 1\} : 3 \leqslant i_2 \text{ and } i_d + 1 \leqslant n\} & (8)\\ &\cup \{\{1, i_1, i_1 + 1, \ldots, i_{d-1}, i_{d-1} + 1, i_d\} : 2 \leqslant i_1 \text{ and } i_d \leqslant n - 1\} & (9)\\ &\cup \{[2, 2d + 1]\}. \end{aligned}$$

Then we have $L \supset \tilde{L}$. Now, $\tilde{L}$ contains all $2d$-subsets S with $S \prec_p [2, 2d + 1]$. Hence $[2, 2d + 1] \in \mathrm{Shift}_{1k}(\tilde{L})$ for all $k > 1$. Since other elements in $\tilde{L}$ contain 1, we have $\mathrm{Shift}_{1k}(\tilde{L}) = \tilde{L}$ for all $k > 1$. Thus we have $\mathrm{Shift}_{1\downarrow n}(C([n], 2d)) \supset \tilde{L}$. We will show $\tilde{L} \supset H(1) \cup G(1)$.

If $S \in H(1)$, then $S \in \{1, j_1\} * C([2, j_1 - 1], 2d - 2)$ for some $j_1 \in [2d, n]$. Thus S is a $2d$-subset of the form either

$$S = \{1, j_1\} \cup \{2, i_1, i_1 + 1, \ldots, i_{d-2}, i_{d-2} + 1, j_1 - 1\}$$

or

$$S = \{1, j_1\} \cup \{i_1, i_1 + 1, \ldots, i_{d-2}, i_{d-2} + 1, i_{d-1}, i_{d-1} + 1\},$$

where $i_{d-1} + 1 < j_1$. In each case, S is a $2d$-subset of the form either (7), (8) or (9). Thus we have

$$\mathrm{Shift}_{1\downarrow n}(C([n], 2d)) \supset \bigcup_{j_1=2d}^{n} \{\{1, j_1\} * C([2, j_1 - 1], 2d - 2)\} \cup \{[2, 2d + 1]\} = H(1) \cup G(1),$$

as desired. □

Remark that $\mathrm{Shift}_{1\downarrow n}(C([n], 2d))$, $H(1) \cup G(1)$ and $\tilde{L}$ are all same. (This fact follows from Lemma 3.4 which will be proved later.) Next, we recall the following simple fact which immediately follows from the definition of combinatorial shifting.

Lemma 3.3. *Let K and L be collections of subsets of* $[n]$. *For all* $1 \leqslant i < j \leqslant n$, *one has* $\mathrm{Shift}_{ij}(K \cup L) \supset \mathrm{Shift}_{ij}(K) \cup \mathrm{Shift}_{ij}(L)$.

Lemma 3.4. *Let $1 < 2d < n$ and $m \leqslant d$. Then one has*

$$\mathrm{Shift}_{m\downarrow n}(\mathrm{Shift}_{m-1\downarrow n}(\cdots(\mathrm{Shift}_{1\downarrow n}(C([n],2d)))\cdots)) \supset H(m) \cup G(m).$$

In particular, one has

$$\mathrm{Shift}_{d\downarrow n}(\mathrm{Shift}_{d-1\downarrow n}(\cdots(\mathrm{Shift}_{1\downarrow n}(C([n],2d)))\cdots)) = H(d) \cup G(d) = \Delta^{\mathrm{s}}(C(n,2d))_{2d-1}.$$

Proof. The case $m=1$ is Lemma 3.2. We will show $\mathrm{Shift}_{m+1\downarrow n}(H(m)\cup G(m)) \supset H(m+1)\cup G(m+1)$ for all $m<d$ by induction on m.

The evenness condition says that if $S \in H(m)$ with $m+1 \notin S$ then S is a $2d$-subset of the form

$$S = [m] \cup \{j_1,\ldots,j_m\} \cup \{i_1, i_1+1, \ldots, i_{d-m}, i_{d-m}+1\}$$

with $m+1 < i_1$ and with $i_{d-m}+1 < j_1$. Note that the above union is disjoint. Then, for every $j_t \in \{j_1,\ldots,j_m\}$, we have

$$\begin{aligned}(S\backslash\{j_t\}) \cup \{m+1\} = [m] &\cup (\{i_{d-m}+1, j_1,\ldots,j_m\}\backslash\{j_t\})\\ &\cup \{m+1, i_1, i_1+1, \ldots, i_{d-m-1}, i_{d-m-1}+1, i_{d-m}\}.\end{aligned}$$

Since $\{m+1, i_1, i_1+1, \ldots, i_{d-m-1}, i_{d-m-1}+1, i_{d-m}\}$ satisfies the evenness condition on $[m+1, i_{d-m}]$, we have $(S\backslash\{j_t\}) \cup \{m+1\} \in H(m)$. Then, since any $2d$-subset T with $m+1 \in T$ does not change by Shift_{m+1k} for all $k > m+1$, any $S \in H(m)$ with $m+1 \notin S$ does not change by Shift_{m+1k} for $k \geqslant j_1$. Thus we have

$$\begin{aligned}&\mathrm{Shift}_{m+1\downarrow n}(H(m)\cup G(m))\\ &\quad\supset \mathrm{Shift}_{m+1\downarrow n}(H(m)) \cup \mathrm{Shift}_{m+1\downarrow n}(G(m))\\ &\quad\supset \bigcup_{\{j_1,\ldots,j_m\}\subset[2d-m+1,n]} ([m]\cup\{j_1,\ldots,j_m\}) * \mathrm{Shift}_{m+1\downarrow n}(C([m+1, j_1-1], 2d-2m))\\ &\quad\cup \mathrm{Shift}_{m+1\downarrow n}(G(m)),\end{aligned}$$

where the first inclusion follows from Lemma 3.3. Since every $S \in G(m)$ contains $m+1$, we have $\mathrm{Shift}_{m+1\downarrow n}(G(m)) = G(m)$. Also, Lemma 3.2 says (shift the induces by m)

$$\begin{aligned}&\mathrm{Shift}_{m+1\downarrow n}(C([m+1, j_1-1], 2d-2m))\\ &\quad\supset \left(\bigcup_{p=2d-m}^{j_1-1} \{\{m+1, p\} * C([m+2, p-1], 2d-2m-2)\}\right) \cup \{[m+2, 2d-m+1]\}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\mathrm{Shift}_{m+1\downarrow n}(H(m))\\ &\quad\supset [m+1] * \left(\bigcup_{\substack{\{j_1,\ldots,j_m\}\subset[2d-m+1,n]\\ p\in[2d-m, j_1-1]}} \{p, j_1,\ldots,j_m\} * C([m+2, p-1], 2d-2m-2)\right)\\ &\quad\cup \left(\bigcup_{\{j_1,\ldots,j_m\}\subset[2d-m+2,n]} \{[m]\cup[m+2, 2d-m+1]\cup\{j_1,\ldots,j_m\}\}\right)\\ &\quad= \bigcup_{\{p,j_1,\ldots,j_m\}\subset[2d-m,n]} \{([m+1]\cup\{p, j_1,\ldots,j_m\}) * C([m+2, p-1], 2d-2m-2)\}\\ &\qquad \cup \{([2d-m+1]\backslash\{m+1\}) \cup S : S \subset [2d-m+2, n] \text{ and } |S| = m\}.\end{aligned}$$

Hence we have $\mathrm{Shift}_{m+1\downarrow n}(H(m)) \cup G(m) \supset H(m+1) \cup G(m+1)$.

Now, we have $\mathrm{Shift}_{d\downarrow n}(\mathrm{Shift}_{d-1\downarrow n}(\cdots(\mathrm{Shift}_{1\downarrow n}(C([n],2d)))\cdots)) \supset H(d)\cup G(d)$. However, since $H(d)\cup G(d) = \Delta^{\mathrm{s}}(C(n,2d))_{2d-1}$, the cardinalities of both sides of the inclusion are equal. Hence we have the desired equality. □

Odd case: Next, we consider the odd case. Fix integers $1 < 2d + 1 < n$. Let

$$U(m) = \bigcup_{\{j_1,\ldots,j_m\}\subset[m+1,n-2d-1+2m]} ([m] \cup \{j_1, \ldots, j_m\}) * C([j_m + 1, n], 2d + 1 - 2m),$$

for $m = 1, 2, \ldots, d$, and let

$$U(d + 1) = \{[d + 1] \cup S : S \subset [d + 2, n], |S| = d\}.$$

For $1 \leqslant j \leqslant d + 1$ and $1 \leqslant m \leqslant 2d - j + 3$, let

$$D(j, m) = \{([m]\backslash\{j\}) \cup [n - 2d + m + j - 2, n]\} * \{S \subset [m + 1, n - 2d + m + j - 3] : |S| = j - 1\}.$$

Note that

$$D(j, 2d - j + 3) = \{([2d - j + 3]\backslash\{j\}) \cup S : S \subset [2d - j + 4, n], |S| = j - 1\}$$

is a subcollection of $\Delta^{\mathrm{s}}(C(n, 2d + 1))_{2d}$. Indeed, a routine computation implies

$$\Delta^{\mathrm{s}}(C(n, 2d + 1))_{2d} = U(d + 1) \cup \bigcup_{j=1}^{d+1} D(j, 2d - j + 3).$$

Lemma 3.5. *Let* $1 < 2d + 1 < n$. *Then one has*

$$\mathrm{Shift}_{1\uparrow n}(C([n], 2d + 1)) \supset U(1) \cup D(1, 1).$$

Proof. The evenness condition says that $C([n], 2d+1)$ contains all $(2d+1)$-subsets $S \subset [n]$ of the form $S=\{1, i_1, i_1+1, \ldots, i_d, i_d + 1\}$ and $S = \{i_1, i_1 + 1, \ldots, i_d, i_d + 1, n\}$.

Let $L = \mathrm{Shift}_{1n-2d-1}(\cdots(\mathrm{Shift}_{12}(C([n], 2d + 1)))\cdots)$. Then L must contain all elements $(S\backslash\{\min(S)\}) \cup \{1\}$ with $\min(S) < n - 2d$ and $S \in C([n], 2d + 1)$. Thus

$$\begin{aligned} L \supset\ & \{\{1, i_1, i_1 + 1, \ldots, i_d, i_d + 1\} : 2\leqslant i_1 \text{ and } i_d + 1\leqslant n\} \\ & \cup \{\{1, i_1 + 1, i_2, i_2 + 1, \ldots, i_d, i_d + 1, n\} : 1\leqslant i_1 < n - 2d \text{ and } i_d + 1\leqslant n - 1\} \\ & \cup \{[n - 2d, n]\}. \end{aligned}$$

The right-hand-side is exactly $U(1) \cup D(1, 1)$. Also, for every integer $t \in [n - 2d, n]$, $([n - 2d, n]\backslash\{t\}) \cup \{1\}$ is contained in $U(1)$. Then, since any element $S \in U(1) \cup D(1, 1)$ with $S \neq [n - 2d, n]$ contains 1, we have $\mathrm{Shift}_{1k}(U(1) \cup D(1, 1)) = U(1) \cup D(1, 1)$ for all $k > 1$. Thus $\mathrm{Shift}_{1\uparrow n}(C(2d + 1, [n]))$ contains $U(1) \cup D(1, 1)$. $\square$

Lemma 3.6. *Let* $1 < 2d + 1 < n$ *and* $1 \leqslant j \leqslant d$. *One has*

(i) $\mathrm{Shift}_{m+1\uparrow n}(D(j, m)) = D(j, m + 1)$, *if* $m \leqslant 2d - j + 2$;
(ii) $\mathrm{Shift}_{m\uparrow n}(D(j, 2d - j + 3)) = D(j, 2d - j + 3)$, *for all* $m > 2d - j + 3$.

Proof. (i) Assume that $m \leqslant 2d - j + 2$. Let $S \in (D(j, m))$. Then S is a $(2d + 1)$-subset of the form

$$S = ([m]\backslash\{j\}) \cup [n - 2d + m + j - 2, n] \cup T,$$

with $T \subset [m + 1, n - 2d + m + j - 3]$. If $m + 1 \in T$, then S does not change by $\mathrm{Shift}_{m+1\uparrow n}$. Moreover, in this case, S can be seen as

$$\begin{aligned} S = & ([m + 1]\backslash\{j\}) \cup [n - 2d + m + j - 1, n] \\ & \cup ((T\backslash\{m + 1\}) \cup \{n - 2d + m + j - 2\}) \in D(j, m + 1). \end{aligned} \tag{10}$$

On the other hand, if $m + 1 \notin S$, then, for any $t \in T$,

$$(S\backslash\{t\}) \cup \{m + 1\} = [m] \cup [n - 2d + m + j - 2, n] \cup ((T\backslash\{t\}) \cup \{m + 1\}) \tag{11}$$

is contained in $D(j,m)$. Also, $(2d+1)$-subsets of form (11) do not change by Shift_{m+1k} for any $k>m+1$ since they contain $m+1$. Thus S does not change by Shift_{m+1t} for all $t\in T$. Then, since $\min(S\backslash([m]\cup T))=n-2d+m+j-2$,

$$(S\backslash\{n-2d+m+j-2\})\cup\{m+1\}=[m+1]\cup[n-2d+m+j-1,n]\cup T \tag{12}$$

is contained in $\mathrm{Shift}_{m+1\uparrow n}(D(j,m))$, where $T\subset[m+2,n-2d+m+j-3]$.

Then $\mathrm{Shift}_{m+1\uparrow n}(D(j,m))$ contains all subsets S of form (10) and (12). In particular, the set of subsets of form (10) and (12) is $D(j,m+1)$. Thus we have $\mathrm{Shift}_{m+1\uparrow n}(D(j,m))\supset D(j,m+1)$. On the other hand, it follows from the definition of $D(j,m)$ that the cardinality of $D(j,m)$ is $\binom{n-2d+m+j-3-(m+1)+1}{j-1}=\binom{n-2d+j-3}{j-1}$ and that of $D(j,m+1)$ is also $\binom{n-2d+j-3}{j-1}$, that is, $|D(j,m+1)|=|D(j,m)|$. Since $|\mathrm{Shift}_{m+1\uparrow n}(D(j,m))|=|D(j,m)|$, we have $\mathrm{Shift}_{m+1\uparrow n}(D(j,m))=D(j,m+1)$.

(ii) Recall that $D(j,2d-j+3)=([2d-j+3]\backslash\{j\})*\binom{[2d-j+4,n]}{j-1}$. Then, for all $2d-j+3<p<q\leqslant n$, we have $\mathrm{Shift}_{pq}(D(j,2d-j+3))=D(j,2d-j+3)$ since $\mathrm{Shift}_{pq}\left(\binom{[2d-j+4,n]}{j-1}\right)=\binom{[2d-j+4,n]}{j-1}$. Hence we have equality (ii). □

Lemma 3.7. *Let $1\leqslant m\leqslant d$. Then one has*

$$\mathrm{Shift}_{m+1\uparrow n}(U(m))\supset U(m+1)\cup D(m+1,m+1).$$

Proof. First, we consider the case $1\leqslant m<d$. If

$$S\in([m]\cup\{j_1,\ldots,j_m\})*C([j_m+1,n],2d+1-2m),$$

where $m+1\leqslant j_1<j_2<\cdots<j_m\leqslant n-2d-1+2m$, then the evenness condition says that S has one of the following four patterns:

(a) $S=[m]\cup\{m+1,j_2,\ldots,j_m\}\cup T$, where $T\in C([j_m+1,n],2d+1-2m)$,
(b) $S=[m]\cup\{j_1,j_2,\ldots,j_m\}\cup\{j_m+1,i_1,i_1+1,\ldots,i_{d-m},i_{d-m}+1\}$, where $j_m+1<n-2d+2m$ and $j_1>m+1$,
(c) $S=[m]\cup\{j_1,j_2,\ldots,j_m\}\cup\{i_1,i_1+1,\ldots,i_{d-m},i_{d-m}+1,n\}$, where $j_m<i_1<n-2d+2m$ and $j_1>m+1$, or
(d) $S=[m]\cup\{j_1,\ldots,j_m\}\cup[n-2d+2m,n]$ with $j_1>m+1$.

Thus $U(m)$ is the set of all $(2d+1)$-subsets of form (a), (b), (c) or (d). We will consider how these subsets change by $\mathrm{Shift}_{m+1\uparrow n}$ in each cases.

Case A: Let S be a subset of form (a). In this case, S does not change by $\mathrm{Shift}_{m+1\uparrow n}$. Then S can be seen as either

$$S=[m+1]\cup\{j_2,\ldots,j_m,j_m+1,i_1\}\cup\{i_1+1,i_2,i_2+1,\ldots,i_{d-m},i_{d-m}+1\} \tag{13}$$

or

$$S=[m+1]\cup\{j_2,\ldots,j_m,i_1,i_1+1\}\cup\{i_2,i_2+1,\ldots,i_{d-m},i_{d-m}+1,n\}, \tag{14}$$

where $m+1<j_2$. Thus all $(2d+1)$-subsets of form (13) and (14) are contained in $\mathrm{Shift}_{m+1\uparrow n}(U(m))$. Note that these subsets are contained in $U(m+1)$.

Case B: Let S be a subset of form (b). Then $(S\backslash\{j_t\})\cup\{m+1\}$ is the subset of form (13) for $t<m$. Since any subset T of form (13) satisfies $T\in U(m)$ and T does not change by Shift_{m+1k} for all $m+1<k$, the subset S does not change by Shift_{m+1k} for $k<j_m$. On the other hand, we have

$$\begin{aligned}&(S\backslash\{j_m\})\cup\{m+1\}\\&\quad=[m+1]\cup\{j_1,\ldots,j_{m-1},j_m+1,i_1\}\cup\{i_1+1,i_2,i_2+1,\ldots,i_{d-m},i_{d-m}+1\}.\end{aligned} \tag{15}$$

Thus $\mathrm{Shift}_{m+1\uparrow n}(U(m))$ contains all $(2d+1)$-subsets of form (15). Also, it is easily verified that these subsets of form (15) are contained in $U(m+1)$.

Case C: Let S be a subset of form (c). Then $(S\backslash\{j_t\})\cup\{m+1\}$ is the subset of form (14) for $t\leqslant m$. Thus, by the same way as Case B, S does not change by Shift_{m+1k} for $k\leqslant j_m$. On the other hand, we have

$$(S\backslash\{i_1\})\cup\{m+1\}=[m+1]\cup\{j_1,\ldots,j_{m-1},j_m,i_1+1\}\cup\{i_2,i_2+1,\ldots,i_{d-m},i_{d-m}+1,n\}. \tag{16}$$

Thus $\mathrm{Shift}_{m+1\uparrow n}(U(m))$ contains all $(2d+1)$-subsets of form (16). Note that these subsets of form (16) are also contained in $U(m+1)$.

Case D: Let S be a subset of form (d). Then $(S\backslash\{j_t\})\cup\{m+1\}$ coincides with the right-hand-side of (14) for $t\leqslant m$. Thus, by the same way as Case B, S does not changes by Shift_{m+1k} for $k<n-2d+2m$. On the other hand, for every $t\in[n-2d+2m,n]$,

$$(S\backslash\{t\})\cup\{m+1\}=[m+1]\cup\{j_1,\ldots,j_m\}\cup([n-2d+2m,n]\backslash\{t\})$$

is the same as right-hand-side of either (13), (14), (15) or (16). Since Case A, Case B and Case C say that $\mathrm{Shift}_{m+1n-2d+2m-1}(\cdots(\mathrm{Shift}_{m+1m+2}(U(m))\cdots))$ contains all $(2d+1)$-subsets of form (13), (14), (15) or (16), the subset S does not change by $\mathrm{Shift}_{m+1\uparrow n}$. Thus all subsets of form (d) is contained in $\mathrm{Shift}_{m+1\uparrow n}(U(m))$. Note that these subsets of form (d) are contained in $D(m+1,m+1)$.

By the definition of $D(m+1,m+1)$, the set of all $(2d+1)$-subsets of form (d) is equal to $D(m+1,m+1)$. Thus Case D says that $\mathrm{Shift}_{m+1\uparrow n}(U(m))\supset D(m+1,m+1)$. It remains to show that $\mathrm{Shift}_{m+1\uparrow n}(U(m))\supset U(m+1)$. However, any element $S\in U(m+1)$ is a $(2d+1)$-subset of the form either (13), (14), (15) or (16), and we already proved that $\mathrm{Shift}_{m+1\uparrow n}(U(m))$ contains all these subsets. Thus we have $\mathrm{Shift}_{m+1\uparrow n}(U(m))\supset U(m+1)$ for $1\leqslant m<d$.

Finally, we discuss the case $m=d$. The proof for the case $m=d$ can be done by the same way as the case $m<d$. Indeed, in case of $m=d$, Case A becomes

$$S=[d+1]\cup\{j_2,\ldots,j_d,j_d+1\}\quad\text{or}\quad S=[d+1]\cup\{j_2,\ldots,j_d,n\},$$

and Case B becomes

$$(S\backslash\{j_d\})\cup\{d+1\}=[d+1]\cup\{j_1,\ldots,j_{d-1},j_d+1\}\quad\text{with } j_d+1<n.$$

These three forms say that $\mathrm{Shift}_{d+1\uparrow n}(U(d))$ contains all $(2d+1)$-subsets which contain $[d+1]$. Thus $\mathrm{Shift}_{d+1\uparrow n}(U(d))$ contains $U(d+1)$. Next, we do not need to consider the Case C since form (c) coincides with form (d) when $m=d$. Also, Case D says that $\mathrm{Shift}_{d+1\uparrow n}(U(d))\supset D(d+1,d+1)$ by the same way as the case $m<d$. $\square$

Lemma 3.8. *Let $1<2d+1<n$. Then one has*

$$\mathrm{Shift}_{n-1\uparrow n}(\cdots\mathrm{Shift}_{1\uparrow n}(C([n],2d+1))\cdots)=\Delta^{\mathrm{s}}(C(n,2d+1))_{2d}.$$

Proof. By Lemma 3.5, we have $\mathrm{Shift}_{1\uparrow n}(C([n],2d+1))\supset U(1)\cup D(1,1)$. Next, for $m\leqslant d$, Lemmas 3.6 and 3.7 say that

$$\begin{aligned}\mathrm{Shift}_{m+1\uparrow n}\left(U(m)\cup\bigcup_{j=1}^{m}D(j,m)\right)&\supset\mathrm{Shift}_{m+1\uparrow n}(U(m))\cup\bigcup_{j=1}^{m}\mathrm{Shift}_{m+1\uparrow n}(D(j,m))\\&\supset U(m+1)\cup\bigcup_{j=1}^{m+1}D(j,m+1).\end{aligned}$$

Hence, we have

$$\mathrm{Shift}_{d+1\uparrow n}(\cdots(\mathrm{Shift}_{1\uparrow n}(C([n],2d+1)))\cdots)\supset U(d+1)\cup\bigcup_{j=1}^{d+1}D(j,d+1).$$

We denote $\mathrm{Shift}_* = \mathrm{Shift}_{n-1\uparrow n} \cdots \mathrm{Shift}_{d+2\uparrow n}$. Then $\mathrm{Shift}_*(U(d+1)) = U(d+1)$ since $U(d+1)$ is shifted. Hence, by Lemma 3.6, we have

$$\begin{aligned}\mathrm{Shift}_*\left(U(d+1) \cup \bigcup_{j=1}^{d+1}(D(j,d+1))\right) &\supset \mathrm{Shift}_*(U(d+1)) \cup \bigcup_{j=1}^{d+1}\mathrm{Shift}_*(D(j,d+1)) \\ &= U(d+1) \cup \bigcup_{j=1}^{d+1}(D(j,2d+3-j)) \\ &= \Delta^{\mathrm{s}}(C(n,2d+1))_{2d}.\end{aligned}$$

Thus we conclude $\mathrm{Shift}_{n-1\uparrow n}(\cdots \mathrm{Shift}_{1\uparrow n}(C([n],2d+1))\cdots) \supset \Delta^{\mathrm{s}}(C(n,2d+1))_{2d}$. Since cardinalities of both sides are equal, we have the desired equality. $\square$

Now, Lemmas 3.4 and 3.8 guarantee the existence of combinatorial shifted complex $\Delta^{\mathrm{c}}(C(n,d))$ with $\Delta^{\mathrm{c}}(C(n,d))_{d-1} = \Delta^{\mathrm{s}}(C(n,d))_{d-1}$ for all $1<d<n$. Then, by virtue of Theorem 2.5, we have the following result.

Theorem 3.9. *Let $C(n,d)$ be the boundary complex of the cyclic d-polytope on n vertices. Then there is a combinatorial shifted complex $\Delta^{\mathrm{c}}(C(n,d))$ such that*

$$\Delta^{\mathrm{c}}(C(n,d)) = \Delta^{\mathrm{e}}(C(n,d)) = \Delta^{\mathrm{s}}(C(n,d)).$$

4. Algebraic shifting of stacked polytopes

In this section, we compute the algebraic shifted complex of the boundary complex of stacked polytopes. First, we recall the construction of stacked polytopes. Let $d>1$ be an integer. Starting with a d-simplex, one can add new vertices by building a shallow pyramids over facets to obtain a simplicial convex d-polytope with n vertices. This convex polytopes are called *stacked d-polytopes*. Let $P(n,d)$ be the boundary complex of a stacked d-polytope on n vertices. Note that the combinatorial type of $P(n,d)$ is not unique.

A 1-dimensional simplicial complex G on $[n]$ is said to be k-acyclic if $\{k+1,k+2\} \notin \Delta^{\mathrm{e}}(G)$. We recall the next lemma.

Lemma 4.1 (*Kalai [11, Lemma 4.3]*). *Let G be a 1-dimensional simplicial complex on $[n]$ and $S \subset [n]$. If G is k-acyclic and $|S| \leqslant k$, then $G \cup \{\{n+1, v\} : v \in S\} \cup \{n+1\}$ is k-acyclic.*

Lemma 4.2. *Let $1<d<n$ and $P(n,d)$ the boundary complex of a stacked d-polytope on n vertices. Then $\{d+1, d+2\} \notin \Delta^{\mathrm{e}}(P(n,d))$.*

Proof. We use induction on n. If $n=d+1$, then the assertion is obvious.

Let P and Q be two simplicial d-polytopes such that P is obtained from Q by adding a pyramid over a facet T of Q. Write $\partial(P)$ (resp. $\partial(Q)$) for the boundary complex of P (resp. Q), and $G(P)$ (resp. $G(Q)$) for the 1-dimensional simplicial complex generated by $\partial(P)_1$ (resp. $\partial(Q)_1$). Assume that $G(P)$ is on $[n]$ and $G(Q)$ is on $[n-1]$, where $n>d+1$. Then $G(P)$ is obtained from $G(Q)$ by adding a new vertex n together with d edges $\{\{n,t\} : t \in T\}$. Then Lemma 4.1 says that $\{d+1,d+2\} \notin \Delta^{\mathrm{e}}(G(Q))$ implies $\{d+1,d+2\} \notin \Delta^{\mathrm{e}}(G(P))$.

Since $\Delta^{\mathrm{e}}(\partial(P))_1 = \Delta^{\mathrm{e}}(G(P))_1$ and $\Delta^{\mathrm{e}}(\partial(Q))_1 = \Delta^{\mathrm{e}}(G(Q))_1$, the assertion follows from the construction of stacked polytopes. $\square$

Let $L(n,d)$ be the collection of d-subsets of $[n]$ defined by

$$L(n,d) = \{[2,d+1]\} \cup \{([d]\backslash\{i\}) \cup \{j\} : 1<i \leqslant d,\ j>d \text{ or } j=i\}.$$

Lemma 4.3. *Let K be a $(d-1)$-dimensional shifted simplicial complex which satisfies $\beta_{d-1}(K)=1$. If $\{d+1, d+2\} \notin K$, then $K_{d-1} \subset L(n,d)$.*

Proof. Since $\beta_{d-1}(K) = 1$, Lemma 1.3 says that the only $(d-1)$-face S of K with $1 \notin S$ is $[2, d+1]$. Assume that there is an $S \in K_{d-1}$ with $1 \in S$ and $S \notin L(n, d)$. Then $|S \cap [d]| \leqslant d-2$. Write $S = (S \cap [d]) \cup S'$, where $S' \cap [d] = \emptyset$. Since K is shifted, we have $(S \cap [d]) \cup \{d+1, d+2, \ldots, 2d - |S \cap [d]|\} \in K$. Thus we have $\{d+1, d+2\} \in K$. However, this contradicts the assumption $\{d+1, d+2\} \notin K$. Thus we have $K_{d-1} \subset L(n, d)$. □

Theorem 4.4. *Let* $1 < d < n$. *Then*

(i) $\Delta^{\mathrm{e}}(P(n, d))$ *is the simplicial complex generated by* $L(n, d)$.
(ii) $\Delta^{\mathrm{e}}(P(n, d)) = \Delta^{\mathrm{s}}(P(n, d))$. *Moreover, if K is the boundary complex of simplicial d-polytope on n vertices, then one has*

$$\Delta^{\mathrm{s}}(P(n, d)) \subset \Delta^{\mathrm{s}}(K) \subset \Delta^{\mathrm{s}}(C(n, d)).$$

Proof. Let K be the boundary complex of a simplicial d-polytope on n vertices. First, we will show that $\Delta^{\mathrm{s}}(K) \supset L(n, d)$. Since $\beta_{d-1}(K) = 1$, Lemma 1.3 says that $[2, d+1] \in \Delta^{\mathrm{s}}(K)$. On the other hand, by Dehn–Sommerville equations, we have $h_{d-1}(K) = h_1(K) = n - d$. Also, we have $\Delta^{\mathrm{s}}(K) \subset \Delta^{\mathrm{s}}(C(n, d))$ by Lemma 2.2. Then Lemma 2.1 says that

$$W_{d-1}(\Delta^{\mathrm{s}}(K)) = W_{d-1}(\Delta^{\mathrm{s}}(C(n, d))) = \{([d]\backslash\{2\}) \cup \{t\} : t = d+1, d+2, \ldots, n\}.$$

On the other hand, any d-subset $S \in L(n, d)$ with $S \neq [2, d+1]$ satisfies $S \prec_p ([d]\backslash\{2\}) \cup \{n\}$. Since $\Delta^{\mathrm{s}}(K)$ is shifted, we have $\Delta^{\mathrm{s}}(K) \supset L(n, d)$.

Recall that shifting operations do not change the *f*-vector of simplicial complexes. Then we have $f_{d-1}(P(n, d)) \geqslant |L(n, d)|$ since $\Delta^{\mathrm{s}}(P(n, d)) \supset L(n, d)$. However, by Lemmas 4.2 and 4.3, we have $\Delta^{\mathrm{e}}(P(n, d))_{d-1} \subset L(n, d)$. Hence $f_{d-1}(P(n, d))$ is smaller than or equal to $|L(n, d)|$. Thus we have $f_{d-1}(P(n, d)) = |L(n, d)|$. These facts say that $\Delta^{\mathrm{s}}(P(n, d))_{d-1} = \Delta^{\mathrm{e}}(P(n, d))_{d-1} = L(n, d)$. Since $\Delta^{\mathrm{e}}(P(n, d))$ and $\Delta^{\mathrm{s}}(P(n, d))$ are pure by Lemma 1.2, it follows that $\Delta^{\mathrm{e}}(P(n, d)) = \Delta^{\mathrm{s}}(P(n, d))$ and $\Delta^{\mathrm{e}}(P(n, d))$ is the simplicial complex generated by $L(n, d)$.

It remains to show $\Delta^{\mathrm{s}}(P(n, d)) \subset \Delta^{\mathrm{s}}(K) \subset \Delta^{\mathrm{s}}(C(n, d))$. However, $\Delta^{\mathrm{s}}(P(n, d)) \subset \Delta^{\mathrm{s}}(K)$ follows from the inclusion $\Delta^{\mathrm{s}}(K) \supset L(n, d)$. Also, $\Delta^{\mathrm{s}}(K) \subset \Delta^{\mathrm{s}}(C(n, d))$ is Lemma 2.2. □

To prove the above theorem, we used Lemma 2.2, which follows from the Lefschetz property. However, it was pointed out by a referee that Theorem 4.4 (i) follows from [17, Theorem 4.6] without using Lemma 2.2.

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