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Weak Clarke epiderivative in set-valued optimization

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Abstract

A new notion of weak Clarke epiderivative for a set-valued map is introduced using the concept of Clarke tangent cone. The existence, characterization and properties of weak Clarke epiderivative are then studied. Finally optimality criteria are established for a constrained set-valued optimization problem in terms of weak Clarke epiderivative.

Keywords: Set-valued optimization; Epiderivative; Weak minimizers; Tangent cone; Semilocal convexlikeness

1. Introduction

Several notions of derivatives of single valued maps have been studied in literature. Using the concept of tangency for any set it is possible to have the concept of derivative of set-valued maps as well. In this regard using the concept of contingent cone Aubin [1] introduced the notion of contingent derivative of a set-valued map. It is defined as the set-valued map whose graph coincides with the contingent cone to the graph of the set-valued map. It can be seen that in case of contingent derivative, necessary and sufficient optimality conditions do not coincide under the standard assumptions [5]. Therefore, while characterizing optimality conditions, it is useful to consider derivatives involving epigraph of set-valued maps rather than their graph [8,10,13].

In order to obtain optimality conditions, generalizing the known classical conditions, another notion of differentiability of set-valued maps known as contingent epiderivative was introduced by Jahn and Rauh [8] where they related epigraph of the derivative with the contingent cone. Since contingent cones are not necessarily convex, Sach and Craven [13] introduced a derivative by relating Clarke tangent cone to the epigraph where the derivative is a set-valued map. Later on Lalitha, Dutta and Govil [10] introduced the notion of Clarke epiderivative of a set-valued map in terms of Clarke tangent cone where epiderivative is a single valued map.

Chen and Jahn [4] introduced the concept of generalized contingent epiderivative in terms of minimizers of projection of the contingent cone to epigraph of a set-valued map. Refer the excellent book of Jahn [7] for further discussions

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and commentaries on the generalized contingent epiderivative. This concept was improved by Chen [3] by considering Clarke tangent cone instead of contingent cone. Jahn and Khan [7] introduced the notion of weak contingent epiderivative and proper contingent epiderivative where derivatives are in terms of weak minimizers and proper minimizers of projection sets, respectively.

Since Clarke tangent cone is convex and provides a complete characterization of optimality therefore it is relevant to introduce the notion of weak Clarke epiderivative. The existence, characterization and properties of these epiderivatives are studied in this paper and their relations with the existing generalized notions of epiderivatives are observed. Finally optimality criteria for a constrained set-valued optimization problem are established in terms of weak Clarke epiderivative.

The paper is organized as follows. Section 2 presents some basic definitions and results used in the paper. In Section 3 the notion of weak Clarke epiderivative is introduced and its existence criteria is studied. Section 4 deals with the characterization and properties of weak Clarke epiderivative. Also its relation with some of the existing notions of epiderivatives is examined. Section 5 deals with both Fritz John and Kuhn Tucker type necessary optimality criteria and concludes with sufficient optimality criteria using a weakened form of cone convexity assumption. Finally some concluding remarks are made at the end.

2. Preliminaries

Throughout the paper we assume that $X = R^n$, $Y = R^m$ and $K \subseteq Y$ is a convex pointed cone with nonempty interior. For a nonempty set B in Y we denote the interior of B by $\operatorname{int} B$. For a set-valued map $F : X \to 2^Y$ we denote by $\operatorname{dom} F$ the set $\{x \in X \mid F(x) \neq \emptyset\}$.

An element $\bar{y} \in B$ is said to be a *minimizer* of B if $(B - \bar{y}) \cap (-K) = \{0\}$ and a *weak minimizer* of B if $(B - \bar{y}) \cap (-\operatorname{int} K) = \emptyset$. The set of all minimizers and weak minimizers of B with respect to the cone K is denoted by $\operatorname{Min}(B, K)$ and $\operatorname{WMin}(B, K)$, respectively. It is obvious that $\operatorname{Min}(B, K) \subseteq \operatorname{WMin}(B, K)$.

The *Clarke tangent cone* to B at $\bar{y} \in B$ is defined as

$$T(B, \bar{y}) := \left\{ y \in Y \;\middle|\; \forall \bar{y}_n \to \bar{y},\ \bar{y}_n \in B,\ t_n \to \infty,\ t_n > 0,\ \exists y_n \in B \text{ such that } y_n \to \bar{y} \text{ and } t_n(y_n - \bar{y}_n) \to y \right\}.$$

Equivalently it is also defined as

$$T(B, \bar{y}) := \left\{ y \in Y \;\middle|\; \forall \bar{y}_n \to \bar{y},\ \bar{y}_n \in B,\ t_n \downarrow 0,\ \exists y_n \to y \text{ with } \bar{y}_n + t_n y_n \in B\ \forall n \right\}.$$

It can be seen that $T(B, \bar{y})$ is a closed convex cone.

The following notion of Clarke epiderivative for set-valued maps was introduced by Lalitha, Dutta and Govil [10] where the epiderivative is a single valued map.

Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$. A single valued map $D_e F(\bar{x}, \bar{y}) : X \to Y$ whose epigraph equals the Clarke tangent cone to $\operatorname{epi} F$ at $(\bar{x}, \bar{y})$, that is,

$$\operatorname{epi} D_e F(\bar{x}, \bar{y}) = T\big(\operatorname{epi} F, (\bar{x}, \bar{y})\big)$$

is called the *Clarke epiderivative* of F at $(\bar{x}, \bar{y})$ where

$$\operatorname{epi} F := \left\{ (x, y) \in X \times Y \;\middle|\; x \in \operatorname{dom} F,\ y \in F(x) + K \right\}.$$

In practice there are numerous set-valued maps for which Clarke epiderivative does not exist. The following example is one such simple case where the ordinary Clarke epiderivative does not exist. In this example $\operatorname{epi} F$ is convex, however similar situation may occur for the set-valued maps with nonconvex epigraph, which would be discussed in other examples given later.

Example 2.1. Let $F : R \to 2^{R^2}$ be a set-valued map defined by

$$F(x) = \begin{cases} \{(y_1, y_2) \in R^2 \mid y_1^2 + 2y_2^2 \leqslant x^2\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0, \end{cases}$$

and $K = R^2_+$ where $R^2_+ = \{(y_1, y_2) \mid y_1 \geqslant 0,\ y_2 \geqslant 0\}$. For $\bar{x} = 0$, $\bar{y} = (0, 0)$ it can be seen that $T(\operatorname{epi} F, (\bar{x}, \bar{y})) = \operatorname{epi} F$ and that the Clarke epiderivative at $(\bar{x}, \bar{y})$ does not exist for any x in $\operatorname{dom} F$.

Later Chen [3] introduced the concept of generalized Clarke epiderivative. Here instead of considering the Clarke tangent cone to the epigraph of F at $(\bar{x}, \bar{y})$, its projection on the image space is taken at a point and minimizers of this projection set is the value of the generalized Clarke epiderivative at that point. A set-valued map $D_g F(\bar{x}, \bar{y}) : X \to 2^Y$ is said to be the *generalized Clarke epiderivative* of F at $(\bar{x}, \bar{y})$ if

$$D_g(\bar{x}, \bar{y})(x) = \mathrm{Min}\big(G(x), K\big)$$

where

$$G(x) := \big\{ y \in Y \,\big|\, (x, y) \in T\big(\operatorname{epi} F, (\bar{x}, \bar{y})\big)\big\}.$$

It can be easily observed that $G(x)$ is a closed convex set and $G(x) = G(x) + K$.

However there are set-valued maps for which even the generalized Clarke epiderivative may not exist. The following example highlights one such set-valued map.

Example 2.2. Let $F : R \to 2^{R^2}$ be a set-valued map defined by

$$F(x) = \begin{cases} \{(y, -y^2) \in R^2 \mid 0 \leqslant y \leqslant x\} \cup \{(-1, 0)\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0, \end{cases}$$

and $K = R^2_+$. Clearly epi F is a nonconvex set and for $\bar{x} = 0$, $\bar{y} = (0, 0)$, $T(\operatorname{epi} F, (\bar{x}, \bar{y})) = \{(x, (y_1, y_2)) \in R^3 \mid x \geqslant 0,$ $y_2 \geqslant 0,\ y_1 \in R\}$. The set-valued map $G : R \to 2^{R^2}$ is given by

$$G(x) = \begin{cases} \{(y_1, y_2) \in R^2 \mid y_2 \geqslant 0,\ y_1 \in R\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0, \end{cases}$$

and hence $D_g F(\bar{x}, \bar{y})(x) = \emptyset$ for all $x \in R$. Observe that for this set-valued map even $D_e F(\bar{x}, \bar{y})(x) = \emptyset$ for all $x \in R$.

To deal with the situation where the minimizers do not exist, we are motivated to consider the notion in terms of the weak minimizers of the projection set $G(x)$. In the following section we introduce this weaker notion and study its existence criteria.

3. Existence criteria for weak Clarke epiderivative

We first introduce the notion of weak Clarke epiderivative.

Definition 3.1. Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$. A set-valued map $D_w F(\bar{x}, \bar{y}) : X \to 2^Y$ is said to be the *weak Clarke epiderivative* of F at $(\bar{x}, \bar{y})$ if

$$D_w F(\bar{x}, \bar{y})(x) = \mathrm{WMin}\big(G(x), K\big).$$

As the set of minimizers is contained in the set of weak minimizers, it is obvious that

$$D_g F(\bar{x}, \bar{y})(x) \subseteq D_w F(\bar{x}, \bar{y})(x) \tag{1}$$

for all x in X.

In Example 2.2 considered above it can be seen that at $\bar{x} = 0$, $\bar{y} = (0, 0)$,

$$D_w F(\bar{x}, \bar{y})(x) = \begin{cases} \{(y, 0) \in R^2 \mid y \in R\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

We now provide an example where the generalized Clarke epiderivative exists and is a proper subset of weak Clarke epiderivative.

Example 3.1. Let $F : R \to 2^{R^2}$ be a set-valued map defined by

$$F(x) = \big\{(y_1, y_2) \in R^2 \,\big|\, y_1 y_2 \leqslant 0\big\}$$

and $K = R^2_+$. Here $\operatorname{epi} F = R \times (R^2 \setminus \operatorname{int} R^2_-)$ is a nonconvex set where $R^2_- = \{(y_1, y_2) \mid y_1 \leqslant 0,\ y_2 \leqslant 0\}$. For $\bar{x} = 0$, $\bar{y} = (0, 0)$, $T(\operatorname{epi} F, (\bar{x}, \bar{y})) = R \times R^2_+$ and $G : R \to 2^{R^2}$ is $G(x) = R^2_+$. Here for all x in R

$$D_g F(\bar{x}, \bar{y})(x) = \{(0, 0)\}$$

which is a proper subset of

$$D_w F(\bar{x}, \bar{y})(x) = \{(y_1, y_2) \in R^2_+ \mid y_1 y_2 = 0\}.$$

We now give an example of a set-valued map where the Clarke epiderivative does not exist and the associated cone is not closed but both $D_w F(\bar{x}, \bar{y})(x)$ and $D_g F(\bar{x}, \bar{y})(x)$ exist and coincide for all x in $\operatorname{dom} F$.

Example 3.2. Let $F : R \to 2^{R^2}$ be a set-valued map defined by

$$F(x) = \begin{cases} \{(y_1, y_2) \in R^2 \mid y_1 + y_2 = 0\} \cup \{(y_1, y_2) \in R^2 \mid 0 \leqslant y_1 \leqslant x,\ y_2 \leqslant 0\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0, \end{cases}$$

and $K = \operatorname{int} R^2_+ \cup \{(0, 0)\}$. Here $\operatorname{epi} F$ is a nonconvex set. For $\bar{x} = 0$, $\bar{y} = (0, 0)$, we have $T(\operatorname{epi} F, (\bar{x}, \bar{y})) = \{(x, y) \mid x \geqslant 0,\ y = (y_1, y_2) \in R^2,\ y_1 + y_2 \geqslant 0,\ y_1 \geqslant 0\}$ and $G : R \to 2^{R^2}$ is given by

$$G(x) = \begin{cases} \{(y_1, y_2) \in R^2 \mid y_1 + y_2 \geqslant 0,\ y_1 \geqslant 0\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

Here $D_e F(\bar{x}, \bar{y})(x)$ does not exist for any x in $\operatorname{dom} F$ whereas $D_g F(\bar{x}, \bar{y})(x)$ and $D_w F(\bar{x}, \bar{y})(x)$ both exist and coincide for all x in $\operatorname{dom} G$ and are given by

$$D_g F(\bar{x}, \bar{y})(x) = \begin{cases} \{(y_1, y_2) \in R^2 \mid y_1 + y_2 = 0,\ y_1 \geqslant 0\} \cup \{(0, y_2), y_2 \geqslant 0\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

We now discuss the existence of the weak Clarke epiderivative for which we recall the following definitions.

Definition 3.2. (See [7,11].) Let $B \subseteq Y$ and $\bar{y} \in Y$.

(i) The set $B \cap (\bar{y} - K)$ is said to be a K*-lower section* of B at $\bar{y}$.
(ii) The set B is said to be *minorized* if there exists $y \in Y$ such that $B \subseteq \{y\} + K$.
(iii) The cone K is said to be *Daniell* if any decreasing sequence in Y having a lower bound converges to its infimum.
(iv) The set B is said to satisfy *weak domination property* if $B \subseteq \operatorname{WMin}(B, K) + \operatorname{int} K \cup \{0_Y\}$.

Since $\operatorname{Min}(B, K) \subseteq \operatorname{WMin}(B, K)$ it may be noted that the weak domination property is a weaker condition in comparison with the *domination property* namely, $B \subseteq \operatorname{Min}(B, K) + K$ (see [3]).

In view of existence theorems for the efficient points given by Borwein [2], the following conclusions can be made.

Lemma 3.1. *If the convex pointed cone K in Y is closed and B is nonempty in Y, then the following implications hold*:

(i) *If K is Daniell and the set B has a nonempty minorized closed K-lower section, then* $\operatorname{WMin}(B, K) \neq \emptyset$ *and weak domination property holds.*
(ii) *If the set B has a nonempty compact K-lower section, then* $\operatorname{WMin}(B, K) \neq \emptyset$ *and weak domination property holds.*
(iii) *Let B be closed and convex and K has a compact base. If* $\operatorname{WMin}(B, K) \neq \emptyset$*, then weak domination property holds.*

We now give the existence theorems for weak Clarke epiderivative.

Theorem 3.1. *Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$. If the cone K is closed and Daniell and if $G(x)$ has a nonempty minorized K-lower section for every $x \in \operatorname{dom} G$, then $D_w F(\bar{x}, \bar{y})(x)$ exists for every $x \in \operatorname{dom} G$ and*

$$\operatorname{epi} D_w F(\bar{x}, \bar{y}) = T\big(\operatorname{epi} F, (\bar{x}, \bar{y})\big). \tag{2}$$

Moreover, if $D_e F(\bar{x}, \bar{y})$ exists, then $\operatorname{epi} D_w F(\bar{x}, \bar{y}) = \operatorname{epi} D_e F(\bar{x}, \bar{y})$.

Proof. The existence of $D_w F(\bar{x}, \bar{y})(x)$ follows directly from Lemma 3.1(i) which also assures that $G(x) \subseteq D_w F(\bar{x}, \bar{y})(x) + K$. As $D_w F(\bar{x}, \bar{y})(x) + K \subseteq G(x) + K$ and $G(x) = G(x) + K$ it follows that $D_w F(\bar{x}, \bar{y})(x) + K = G(x)$. Hence $(x, y) \in \operatorname{epi} D_w F(\bar{x}, \bar{y})$ if and only if $y \in G(x)$, which is equivalent to the fact that $(x, y) \in T(\operatorname{epi} F, (\bar{x}, \bar{y}))$. □

Theorem 3.2. *Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$ and the cone K be closed. If $G(x)$ has a nonempty bounded K-lower section for every $x \in \operatorname{dom} G$, then $D_w F(\bar{x}, \bar{y})(x)$ exists for all $x \in \operatorname{dom} G$ and (2) holds.*

Proof. Proof follows on using Lemma 3.1(ii) as the set $G(x)$ is a closed set. □

Remark 3.1. If $D_w F(\bar{x}, \bar{y})(x)$ and $D_e F(\bar{x}, \bar{y})(x)$ both exist for every $x \in \operatorname{dom} G$, then the inclusion $\operatorname{epi}(D_e F(\bar{x}, \bar{y})) \subseteq \operatorname{epi} D_w F(\bar{x}, \bar{y})$ holds trivially, whereas equality holds if weak domination property holds for $G(x)$.

Remark 3.2. The conditions for the existence in the above theorems are necessary but not sufficient. For instance in Example 2.2, $D_w F(\bar{x}, \bar{y})(x)$ exists for every $x \in \operatorname{dom} G$ but none of the assumptions stated above hold true as the *K-lower section* of $G(x)$ are neither minorized nor bounded. Also in Example 3.2 it may be noted that $D_w F(\bar{x}, \bar{y})(x)$ exists for all $x \in \operatorname{dom} G$ despite the fact that the underlying cone is not closed.

The following theorem states the condition under which relation (2) holds, assuming the existence of $D_w F(\bar{x}, \bar{y})$.

Theorem 3.3. *Let $F : X \to 2^Y$ be a set-valued map, $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$ and the cone K be closed. If $D_w F(\bar{x}, \bar{y})(x)$ exists for any $x \in \operatorname{dom} G$, then (2) holds.*

Proof. The proof follows on using Lemma 3.1(iii) since $G(x)$ is a closed convex set for any $x \in \operatorname{dom} G$ and K has a compact base as Y is a finite dimensional space. □

Theorem 3.4. *If $D_w F(\bar{x}, \bar{y})(x)$ exists for every $x \in \operatorname{dom} G$, and if $D_w F(\bar{x}, \bar{y})(X) := \bigcup\{D_w F(\bar{x}, \bar{y})(x) \mid x \in \operatorname{dom} G\}$ and $F(X) := \bigcup\{F(x) | x \in \operatorname{dom} F\}$, then the following hold:*

(i) $D_w F(\bar{x}, \bar{y})(X) \subseteq T(F(X) + K, \bar{y})$;
(ii) $\operatorname{epi} D_w F(\bar{x}, \bar{y}) \subseteq T(\operatorname{dom} F, \bar{x}) \times Y$.

Proof. (i) If $T(F(X) + K, \bar{y}) = Y$, there is nothing to prove. For $T(F(X) + K, \bar{y}) \neq Y$, let $x \in \operatorname{dom} G$ and $y \in D_w F(\bar{x}, \bar{y})(x) \subseteq G(x)$. Clearly $(x, y) \in T(\operatorname{epi} F, (\bar{x}, \bar{y}))$ and hence by the definition of Clarke tangent cone, for every $(\bar{x}_n, \bar{y}_n) \to (\bar{x}, \bar{y})$ with $(\bar{x}_n, \bar{y}_n) \in \operatorname{epi} F$ and $t_n \downarrow 0$ there exists $(x_n, y_n) \to (x, y)$ such that $(\bar{x}_n, \bar{y}_n) + t_n(x_n, y_n) \in \operatorname{epi} F$. This implies $\bar{y}_n + t_n y_n \in F(\bar{x}_n + t_n x_n) + K \subseteq F(X) + K$ for every positive integer n or equivalently $y \in T(F(X) + K, \bar{y})$.

(ii) It can be seen that for every $\bar{x}_n \to \bar{x}$ and $t_n \downarrow 0$ there exists $x_n \to x$ such that $\bar{x}_n + t_n x_n \in \operatorname{dom} F$, that is $x \in T(\operatorname{dom} F, \bar{x})$ which implies $(x, y) \in T(\operatorname{dom} F, \bar{x}) \times Y$. □

4. Nature of weak Clarke epiderivative

As mathematical methodology on the comparison between sets is not so popular for practical researches therefore like any other vector optimization problem, one of the most suitable approaches to obtain optimal solution to a set-valued optimization problem is to study characterizations of set-valued maps via scalarization. By means of some scalarization methods an equivalent scalar problem can be formulated whose optimal solution is comparatively easier to obtain as the target space is one-dimensional and is a total ordering space.

In the following section the projection set $G(x)$ is scalarized and weak Clarke epiderivative is characterized in terms of a set related to scalarized form of $G(x)$.

For a cone $K \subseteq Y$, K^* denotes the dual cone K and is given by

$$K^* := \{k \in Y \mid \langle k, c\rangle \geqslant 0,\ \forall c \in K\}.$$

The scalarization of the set $G(x)$ by an element k in Y is defined as $\langle k, G(x)\rangle := \{\langle k, y\rangle \mid y \in G(x)\}$ and

$$\langle k, G(x)\rangle^- := \{y \in G(x) \mid \langle k, y\rangle \leqslant \langle k, G(x)\rangle\}.$$

We now characterize weak Clarke epiderivative in terms of $\langle k, G(x)\rangle^-$ for $k \in K^* \setminus \{0_Y\}$. Similar characterization has been made by Jahn and Khan [7] for weak contingent epiderivative where the contingent cone to the epigraph of F at $(\bar{x}, \bar{y})$ is assumed to be convex whereas no such assumption is required in case of weak Clarke epiderivative.

Theorem 4.1. *Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$. If $D_w F(\bar{x}, \bar{y})$ exists, then $D_w F(\bar{x}, \bar{y})(x) = \bigcup\{\langle k, G(x)\rangle^- \mid k \in K^* \setminus \{0\}\}$ for each $x \in \operatorname{dom} G$.*

Proof. Let $y^* \in D_w F(\bar{x}, \bar{y})(x)$ for $x \in \operatorname{dom} G$. This implies $(G(x) - y^*) \cap (-\operatorname{int} K) = \emptyset$. Since $(G(x) - y^*)$ and $(-\operatorname{int} K)$ are both convex, by standard separation theorem [6] there exists $k^* \in K^* \setminus \{0\}$ such that $\langle k^*, y - y^*\rangle \geqslant 0$, that is $\langle k^*, y\rangle \geqslant \langle k^*, y^*\rangle$, for every $y \in G(x)$. Hence $y^* \in \langle k^*, G(x)\rangle^- \subseteq \bigcup\{\langle k, G(x)\rangle^- \mid k \in K^* \setminus \{0\}\}$.

Conversely, let $k^* \in K^* \setminus \{0\}$ be arbitrary such that $y^* \in \langle k^*, G(x)\rangle^-$ for some $x \in \operatorname{dom} G$. This implies for each $y \in G(x)$, $\langle k^*, y - y^*\rangle \geqslant 0$. On the contrary suppose that $y^* \notin \operatorname{WMin}(G(x), K)$ and let $z \in (G(x) - y^*) \cap (-\operatorname{int} K)$. As $z \in -\operatorname{int} K$ we have $\langle k^*, z\rangle < 0$ contradicting the given hypothesis for $z \in G(x) - y^*$. Thus $y^* \in D_w F(\bar{x}, \bar{y})(x)$ and hence the result holds. □

The above theorem is now illustrated for a set-valued map. For the function F considered in Example 3.2 and for any $k = (k_1, k_2) \in K^* \setminus \{(0, 0)\}$ where $K^* = R^2_+$ we have

$$\langle k, G(x)\rangle = \{\langle k, y\rangle \mid y \in G(x)\} = \{k_1 y_1 + k_2 y_2 \mid (y_1, y_2) \in R^2,\ y_1 + y_2 \geqslant 0,\ y_1 \geqslant 0\}$$

and hence

$$\langle k, G(x)\rangle^- = \{y \in G(x) \mid \langle k, y\rangle \leqslant \langle k, G(x)\rangle\} = \begin{cases} \{(y_1, y_2) \mid y_1 = 0,\ y_2 \geqslant 0\} & \text{if } 0 = k_2 < k_1, \\ \{(0,0)\} & \text{if } 0 < k_2 < k_1, \\ \{(y_1, y_2) \mid y_1 + y_2 = 0,\ y_1 \geqslant 0\} & \text{if } k_1 = k_2, \\ \emptyset & \text{if } k_2 > k_1. \end{cases}$$

We thus have $D_w F(\bar{x}, \bar{y})(x) = \bigcup\{\langle k, G(x)\rangle^- \mid k \in K^* \setminus \{0\}\}$.

We now discuss the nature of weak Clarke epiderivative by means of certain properties. For this purpose we recall the following definitions given in literature [7,8].

Definition 4.1. A set-valued map $F : X \to 2^Y$ is said to be

(i) *strictly positive homogeneous* if for all $x \in X$ and $\alpha > 0$,

$$F(\alpha x) = \alpha F(x);$$

(ii) *subadditive* if for all $x_1, x_2 \in X$,

$$F(x_1) + F(x_2) \subseteq F(x_1 + x_2) + K;$$

(iii) *K-lower semicontinuous* in X if for all $y \in Y$, the set

$$\{x \in X \mid F(x) \cap (y - K) \neq \emptyset\}$$

is closed.

Theorem 4.2. *Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$ and the cone K be closed. If $D_w F(\bar{x}, \bar{y})(x)$ exists for all x in* $\operatorname{dom} G$*, then it is strictly positive homogeneous. Moreover, $D_w F(\bar{x}, \bar{y})$ is subadditive if $G(x)$ fulfills the weak domination property for all x in X.*

Proof. As $T(\operatorname{epi} F, (\bar{x}, \bar{y}))$ is convex, the proof follows on the lines of Theorem 1 in [4]. □

Cone-lower semicontinuity of weak contingent epiderivative has been proved by Jahn and Khan (Corollary 3.5 of [7]). We now have the following theorem for cone-lower semicontinuity of weak Clarke epiderivative.

Theorem 4.3. *Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$ and the cone K be closed. If $D_w F(\bar{x}, \bar{y})(x)$ exists for all x in* $\operatorname{dom} G$*, then $D_w F(\bar{x}, \bar{y})(x)$ is K-lower semicontinuous.*

Proof. Observe that

$$\bigl\{x \in X \bigm| D_w F(\bar{x}, \bar{y})(x) \cap (y - K) \neq \emptyset\bigr\} \times \{y\} = \operatorname{epi} D_w F(\bar{x}, \bar{y}) \cap \bigl(X \times \{y\}\bigr) = T\bigl(\operatorname{epi} F, (\bar{x}, \bar{y})\bigr) \cap \bigl(X \times \{y\}\bigr),$$

for all $y \in Y$. Since the Clarke tangent cone is closed it follows that, for all $y \in Y$ the set $\{x \in X \mid D_w F(\bar{x}, \bar{y})(x) \cap (y - K) \neq \emptyset\}$ is closed, that is, $D_w F(\bar{x}, \bar{y})(x)$ is K-lower semicontinuous. □

We now study the relations between the notions of Clarke epiderivatives and the contingent epiderivatives. The *contingent* (*Bouligand tangent*) *cone* to B at $\bar{y} \in B$ is defined as

$$T_c(B, \bar{y}) := \bigl\{y \in Y \bigm| \exists y_n \to \bar{y},\ y_n \in B,\ t_n > 0 \text{ such that } t_n(y_n - \bar{y}) \to y\bigr\}.$$

It can be seen that $T_c(B, \bar{y})$ is a closed cone, $T(B, \bar{y}) \subseteq T_c(B, \bar{y})$ and $T(B, \bar{y}) = T_c(B, \bar{y})$ if B is a convex set.

The epiderivatives in terms of contingent cone are defined as the respective minimizers of the projection set $G_c(x)$ of the contingent cone on the image space given by the set

$$G_c(x) := \bigl\{y \in Y \bigm| (x, y) \in T_c\bigl(\operatorname{epi} F, (\bar{x}, \bar{y})\bigr),\ \forall x \in X\bigr\}.$$

Based on the nature of the contingent cone, it can be seen that $G_c(x)$ is a closed set. Moreover $G(x) \subseteq G_c(x)$, for every $x \in \operatorname{dom} G$ and $G(x) = G_c(x)$ if $\operatorname{epi} F$ is a convex set. Also $G_c(x) = G_c(x) + K$.

We now recall the definitions of the generalized contingent epiderivative and weak contingent epiderivative given by Jahn and Khan [7].

Definition 4.2. Let $F : X \to 2^Y$ be a set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$.

(i) A set-valued map $D_g^c F(\bar{x}, \bar{y}) : X \to 2^Y$ defined by

$$D_g^c F(\bar{x}, \bar{y}) = \operatorname{Min}\bigl(G_c(x), K\bigr)$$

is said to be the *generalized contingent epiderivative* of F at $(\bar{x}, \bar{y})$.

(ii) A set-valued map $D_w^c F(\bar{x}, \bar{y}) : X \to 2^Y$ defined by

$$D_w^c F(\bar{x}, \bar{y}) = \operatorname{WMin}\bigl(G_c(x), K\bigr)$$

is said to be the *weak contingent epiderivative* of F at $(\bar{x}, \bar{y})$.

The weak (generalized) Clarke epiderivative defined as the set of weak minimizers (minimizers) of the underlying set $G(x)$ is always convex even if $\operatorname{epi} F$ is nonconvex which is not the case for contingent epiderivatives.

Apart from the convexity feature lacking in case of contingent epiderivative, another overriding characteristic of Clarke epiderivative over contingent epiderivative is based on their existence. We now have a set-valued map for which the generalized contingent epiderivative does not exist whereas the generalized Clarke epiderivative exists.

For the set-valued map considered in Example 3.1, it can be seen that for $\bar{x} = 0$, $\bar{y} = (0, 0)$, $T_c(\operatorname{epi} F, (\bar{x}, \bar{y})) = R \times (R^2 \setminus \operatorname{int} R^2_-)$. Therefore $G_c(x) : R \to R^2$ is given by $G_c(x) = (R^2 \setminus \operatorname{int} R^2_-)$ for all $x \in R$. Hence

$D_g^c F(\bar{x},\bar{y})(x) = \emptyset$ whereas as seen earlier $D_g F(\bar{x},\bar{y})(x) = \{(0,0)\}$, for every $x \in R$. Here we observe that weak contingent epiderivative of F at $(\bar{x},\bar{y})$ is given by $D_w^c F(\bar{x},\bar{y})(x) = \{(y_1,y_2) \in R_-^2 \mid y_1 y_2 = 0\}$ which is different from the weak Clarke epiderivative given in Example 3.1.

Based on the relationship between the set of minimizers of two sets, one being a subset of the other, as in (1), the following relations hold true. If the weak and generalized contingent epiderivatives and corresponding Clarke epiderivatives exist, then for every $x \in \operatorname{dom} G$,

(i) $D_g^c F(\bar{x},\bar{y})(x) \cap G(x) \subseteq D_g F(\bar{x},\bar{y})(x) \subseteq D_g^c F(\bar{x},\bar{y})(x) + K$;
(ii) $D_w^c F(\bar{x},\bar{y})(x) \cap G(x) \subseteq D_w F(\bar{x},\bar{y})(x) \subseteq D_w^c F(\bar{x},\bar{y})(x) + K$;
(iii) $D_g F(\bar{x},\bar{y})(x) \subseteq D_w F(\bar{x},\bar{y})(x) \subseteq D_w^c F(\bar{x},\bar{y})(x) + K$.

Next theorem claims the coincidence of the weak Clarke epiderivative with the weak contingent epiderivative for a cone convex set-valued map. We recall that a set-valued map $F : X \to 2^Y$ defined on a convex set X is said to be *K-convex* if for all $x, \bar{x} \in X$, $y \in F(x)$ and $\bar{y} \in F(\bar{x})$ we have $(1-\lambda)\bar{y} + \lambda y \in F((1-\lambda)\bar{x} + \lambda x) + K$ for $0 \leqslant \lambda \leqslant 1$.

Theorem 4.4. *Let $F : X \to 2^Y$ be a K-convex set-valued map with $\bar{y} \in F(\bar{x})$ for $\bar{x} \in X$. Then $D_w F(\bar{x},\bar{y})(x) = D_w^c F(\bar{x},\bar{y})(x)$ for every $x \in \operatorname{dom} G$.*

Proof. The result follows trivially for all x in $\operatorname{dom} G$ as $G(x) = G_c(x)$ for a K-convex set-valued map as epi F of a K-convex map is a convex set. □

Remark 4.1. In the above theorem, the condition of cone convexity of the set-valued map F cannot be relaxed. For the set-valued map F considered in Example 3.1, F is not a K-convex set-valued map. Also it can be seen that

$$D_w^c F(\bar{x},\bar{y})(x) = \begin{cases} \{(y_1,y_2) \in R_-^2 \mid y_1 y_2 = 0\} & \text{if } x \geqslant 0, \\ \emptyset & \text{if } x < 0, \end{cases}$$

which does not coincide with $D_w F(\bar{x},\bar{y})(x)$ evaluated earlier.

5. Optimality conditions

Throughout this section we assume K and D to be closed convex pointed cones with nonempty interiors. We first recall the notion of cone semilocal convexlikeness (see [9]) generalizing the well-known notion of cone convexity.

A set A is said to be a *locally star shaped* at a point $\bar{x} \in A$ if for any $x \in A$, there exists a positive real number $a(x,\bar{x}) \leqslant 1$ such that $(1-\lambda)\bar{x} + \lambda x \in A$ for $0 \leqslant \lambda \leqslant a(x,\bar{x})$. If A is locally star shaped at each $x \in A$, then A is said to a locally star shaped set. For example union of open sets is a locally star shaped set. The set $A =]0,1[\cup]2,3[$ is a locally star shaped set in R but is not a convex set in R.

If X is a locally star shaped set, then a set-valued map $F : X \to 2^Y$ is said to be *K-semilocally convexlike* at a point $\bar{x} \in X$ if for all $x \in X$, $y \in F(x)$ and $\bar{y} \in F(\bar{x})$ there exists a positive real number $d((x,y),(\bar{x},\bar{y})) \leqslant a(x,\bar{x})$ such that $(1-\lambda)\bar{y} + \lambda y \in F(X) + K$ for $0 \leqslant \lambda \leqslant d((x,y),(\bar{x},\bar{y}))$. F is said to be K-semilocally convexlike on X if F is K-semilocally convexlike at each $x \in X$.

If F is K-semilocally convexlike on X with $a(x,\bar{x}) = 1$ and $d((x,y),(\bar{x},\bar{y})) = 1$ for all $\bar{x}, x \in X$, $y \in F(x)$ and $\bar{y} \in F(\bar{x})$, then it is K-convex on X. It has also been observed in [9] that F is K-semilocally convexlike on X if and only if $F(X) + K$ is a locally star shaped set.

Remark 5.1. The class of cone semilocally convexlike set-valued maps is larger than the class of cone convex set-valued maps. We illustrate this fact by giving an example of a K-semilocally convexlike set-valued map, which is not a K-convex set-valued map. If $K = R_+^2$, then the set-valued map $F : X \to 2^Y$ is K-semilocally convexlike on X where $X = [0,1]$ and

$$F(x) = \begin{cases} \{(y_1,y_2) \in R_+^2 \mid y_1 y_2 > 1\} & \text{if } x \in [0,1[, \\ \{(y_1,y_2) \in R^2 \mid y_2 > 2\} & \text{if } x = 1. \end{cases}$$

However F is not K-convex because for $x = 0$, $\bar{x} = 1$, $y = (4/3, 4/3) \in F(x)$, $\bar{y} = (-4/3, 7/3) \in F(\bar{x})$ and $\lambda = \frac{1}{2}$,

$$(1-\lambda)\bar{y} + \lambda y = (0, 11/6) \notin F(X) + K.$$

The following theorem gives necessary condition for K-semilocal convexlikeness of a set-valued map in terms of weak Clarke epiderivative.

Theorem 5.1. *If $F : X \to 2^Y$ is a K-semilocally convexlike set-valued map defined on a locally star shaped set X and $D_w F(\bar{x}, \bar{y})(x)$ exists for each $x \in X$ where $\bar{y} \in F(\bar{x})$ for some $\bar{x} \in X$, then*

$$F(x) - \{\bar{y}\} \subseteq D_w F(\bar{x}, \bar{y})(x - \bar{x}) + K$$

for every $x \in X$.

Proof. For $x \in X$ and $y \in F(x)$ we need to prove that $y - \bar{y} \in D_w F(\bar{x}, \bar{y})(x - \bar{x}) + K$, that is $(x - \bar{x}, y - \bar{y}) \in \operatorname{epi} D_w F(\bar{x}, \bar{y})$. Since $D_w F(\bar{x}, \bar{y})(x)$ exists for each $x \in X$ by Theorem 3.3, condition (2) holds and hence it is enough to prove that $(x - \bar{x}, y - \bar{y}) \in T(\operatorname{epi} F, (\bar{x}, \bar{y}))$. Let $(\bar{x}_n, \bar{y}_n) \to (\bar{x}, \bar{y})$ with $(\bar{x}_n, \bar{y}_n) \in \operatorname{epi} F$ and $t_n \to \infty$ with $t_n > 0$. Since X is a locally star shaped set there exists a positive real number $a(x, \bar{x}_n) \leqslant 1$ such that $(1-\lambda)\bar{x}_n + \lambda x \in X$ for $0 \leqslant \lambda \leqslant a(x, \bar{x}_n)$. Clearly $y \in F(x) \subseteq F(X) + K$ and $\bar{y}_n \in F(\bar{x}_n) + K \subseteq F(X) + K$ as $(\bar{x}_n, \bar{y}_n) \in \operatorname{epi} F$. We know that $F(X) + K$ is a locally star shaped set as F is a K-semilocally convexlike set-valued map and hence there exists a positive real number $b(y, \bar{y}_n) \leqslant 1$ such that $(1-\lambda)\bar{y}_n + \lambda y \in F(X) + K$ for $0 \leqslant \lambda \leqslant b(y, \bar{y}_n)$. For each n define $c(x, y, \bar{x}_n, \bar{y}_n) = \min\{a(x, \bar{x}_n), b(y, \bar{y}_n)\}$. Without loss of generality we can assume that $0 < 1/t_n \leqslant c(x, y, \bar{x}_n, \bar{y}_n)$ for each n. Define $x_n = (1 - 1/t_n)\bar{x}_n + (1/t_n)x$ and $y_n = (1 - 1/t_n)\bar{y}_n + (1/t_n)y$ for $0 < 1/t_n < c(x, y, \bar{x}_n, \bar{y}_n)$. Hence for each n we have $x_n \in X$, $y_n \in F(x_n) + K$, $\{x_n\} \to \bar{x}$ and $\{y_n\} \to \bar{y}$. This implies $(x_n, y_n) \in \operatorname{epi} F$, $(x_n, y_n) \to (\bar{x}, \bar{y})$ and $t_n\{(x_n, y_n) - (\bar{x}_n, \bar{y}_n)\} \to (x - \bar{x}, y - \bar{y})$. □

We now consider the following *set-valued optimization problem*.

$$\text{(VP)} \quad \operatorname{Min} F(x) \quad \text{subject to } H(x) \cap (-D) \neq \emptyset,$$

where $H : X \to 2^Z$ is a set-valued map.

The *feasible region* is given by the set $S := \{x \in X \mid H(x) \cap (-D) \neq \emptyset\}$. The *image set* of S under F is given by $F(S) = \bigcup\{F(x) \mid x \in S\}$.

Definition 5.1. A point $(\bar{x}, \bar{y}, \bar{z})$ is said to be a *weak minimizer* of (VP) if $\bar{x} \in S$, $\bar{y} \in F(\bar{x}) \cap \operatorname{WMin}(F(S), K)$ and $\bar{z} \in H(\bar{x}) \cap (-D)$.

Our main aim in this section is to establish the optimality conditions for (VP). In the following theorems we use the set-valued map $(F, H) : X \to 2^{Y \times Z}$ defined as $(F, H)(x) := F(x) \times H(x)$, for every $x \in X$.

We first establish the Fritz John type necessary optimality criteria for (VP).

Theorem 5.2. *If $G(x)$ satisfies the weak domination property for every $x \in X$, $G(0)$ is pointed and $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of* (VP)*, then there exists $(\varphi, \psi) \in K^* \times D^* \setminus \{(0_Y, 0_Z)\}$ such that for all $(y, z) \in D_w(F, H)(\bar{x}, \bar{y}, \bar{z})(X)$,*

$$\text{(i)}\ \ \varphi(y) + \psi(z) \geqslant 0; \tag{3}$$

$$\text{(ii)}\ \ \psi(\bar{z}) = 0.$$

Proof. The proof follows on the lines of Theorem 3.1 in [3] by replacing generalized Clarke epiderivative by weak Clarke epiderivative and domination property by weak domination property for each x in X. □

Next we have the Kuhn Tucker type necessary optimality criteria for (VP) under Slater's type constraint qualification and cone-semilocal convexlikeness conditions. We say the problem (VP) satisfies the *generalized Slater's constraint qualification* if there exists $x' \in X$ such that $H(x') \cap (-\operatorname{int} D) \neq \emptyset$.

Remark 5.2. For nonsmooth optimization problem while dealing with Clarke subdifferential there is no need to assume any convexity assumption as the Clarke subdifferential is a closed convex set. (See Mordukhovich [12] for more details.) However for the set-valued optimization problem (VP) we need to express the optimality condition (3) in terms of the set-valued map H if Slater's constraint qualification is to be applied. This is achieved with the help of Theorem 5.1 if the set-valued maps are cone semilocally convexlike.

Here we establish the optimality criteria by relaxing the convexity assumptions taken by Chen [3] for generalized Clarke epiderivative.

Theorem 5.3. *Suppose that F is K-semilocally convexlike on X and H is D-semilocally convexlike on X and* (VP) *satisfies the generalized Slater's constraint qualification. If $G(x)$ satisfies the weak domination property for every $x \in X$ and $G(0)$ is pointed and $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of* (VP)*, then there exist $\varphi \in K^* \setminus \{0_Y\}$ and $\psi \in D^*$ such that for all $(y, z) \in D_w(F, H)(\bar{x}, \bar{y}, \bar{z})(X)$,*

(i) $\varphi(y) + \psi(z) \geqslant 0$,
(ii) $\psi(\bar{z}) = 0$.

Proof. As $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of (VP) therefore by Theorem 5.3, there exists $(\varphi, \psi) \in K^* \times D^* \setminus \{(0_Y \times 0_Z)\}$ such that $\psi(\bar{z}) = 0$ and $\varphi(y) + \psi(z) \geqslant 0$ for each (y, z) in $D_w(F, H)(\bar{x}, \bar{y}, \bar{z})(X)$. In view of Theorem 5.1 for any (y, z) in $\bigcup\{F(x) \times H(x) \mid x \in S\}$ we have $(y, z) - (\bar{y}, \bar{z}) \in D_w(F, H)(\bar{x}, \bar{y}, \bar{z})(x - \bar{x}) + K \times D$. Since $\varphi(k) \geqslant 0$ for every $k \in K$ and $\psi(d) \geqslant 0$ for every $d \in D$, it follows that $\varphi(y - \bar{y}) + \psi(z - \bar{z}) \geqslant 0$, that is $\varphi(y - \bar{y}) + \psi(z) \geqslant 0$. If $\varphi = 0_Y$, then $\psi \neq 0_Z$ and hence for all $z \in \bigcup\{H(x) \mid x \in S\}$, we get $\psi(z) \geqslant 0$. As the generalized Slater's constraint qualification is satisfied there exists $x' \in X$ such that $H(x') \cap (-\operatorname{int} D) \neq \emptyset$ which implies that there exists $z' \in H(x') \cap (-\operatorname{int} D)$. Since $z' \in -\operatorname{int} D$ it follows that $\psi(z') < 0$ which is a contradiction. □

In order to establish the sufficiency optimality criteria some convexity assumption is required on the objective function and the constraints. For example, dealing with the generalized tangent epiderivatives, Chen [3] established the sufficient optimality criteria by assuming the cone convexity of the objective function and the set-valued map involved in the constraints. As remarked above we impose convexlike assumptions as we need to express the optimality condition (3) in terms of the set-valued maps F and H. The following sufficiency theorem involving weak Clarke epiderivative uses the cone semilocal convexlikeness assumption.

Theorem 5.4. *Suppose that F is K-semilocally convexlike on X and H is D-semilocally convexlike on X and $G(x)$ satisfies the weak domination property for all x in X. For $\bar{x} \in S$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in H(\bar{x}) \cap (-D)$ if there exist $\varphi \in K^* \setminus \{0\}$ and $\psi \in D^*$ such that for all $(y, z) \in D_w(F, H)(\bar{x}, \bar{y}, \bar{z})(X)$,*

(i) $\varphi(y) + \psi(z) \geqslant 0$,
(ii) $\psi(\bar{z}) = 0$,

then $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of (VP).

Proof. In view of Theorem 5.1 for any (y, z) in $\bigcup\{F(x) \times H(x) \mid x \in S\}$ we have

$$(y, z) - (\bar{y}, \bar{z}) \in D_w(F, H)(\bar{x}, \bar{y}, \bar{z})(x - \bar{x}) + K \times D.$$

Using the hypothesis we thus have $\varphi(y - \bar{y}) + \psi(z) \geqslant 0$, for every (y, z) in $\bigcup\{F(x) \times H(x) \mid x \in S\}$. If $\bar{y} \notin \operatorname{WMin}(F(S), K)$ there exists $x^* \in S$, $y^* \in F(x^*)$ such that $\varphi(y^* - \bar{y}) < 0$. Since $x^* \in S$ there exists $z^* \in G(x^*) \cap (-D)$ which satisfies the relation $\psi(z^*) \leqslant 0$. Hence $\varphi(y^* - \bar{y}) + \psi(z^*) < 0$ which is a contradiction. □

6. Conclusions

A new notion of weak Clarke epiderivative has been introduced in this paper. This concept is studied by means of its existence, characterizations and properties. The study is significant as weak Clarke epiderivative is one of the most generalized versions of epiderivatives. One of the advantages of using the weak Clarke epiderivative is that while deriving the Fritz John type necessary optimality criteria no convexity assumption is needed as in the case of epiderivatives involving contingent cone. While dealing with the weak Clarke epiderivative the convexity assumption has been weakened and a weaker condition namely weak domination property has been used to obtain the sufficiency criteria. These facts signify the introduction and application of the concept of the weak Clarke epiderivative in the field of set-valued optimization.

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