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Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces

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Abstract

The aim of this work is to propose implicit and explicit viscosity-like methods for finding specific common fixed points of infinite countable families of nonexpansive self-mappings in Hilbert spaces. Two numerical approaches to solving this problem are considered: an implicit *anchor*-like algorithm and a non-implicit one. The considered methods appear to be of practical interests from the numerical point of view and strong convergence results are proved.

Keywords: Viscosity method; Nonexpansive mapping; Common fixed point; Convex optimization; Nearest point projection

1. Introduction

Let H be a real Hilbert space endowed with inner product $\langle\cdot,\cdot\rangle$ and induced norm $|\cdot|$. Let $(T_i)_{i\geqslant 0}$ be an infinite countable family of nonexpansive self-mappings defined on a closed convex subset D of H, such that $S := \bigcap_{i\geqslant 0}\mathrm{Fix}(T_i) \neq \emptyset$, where $\mathrm{Fix}(T_i) := \{x \in D \mid T_i x = x\}$ is the set of fixed points of T_i. It is well known that S is a closed convex set of D (see for instance [14]). Let us recall that a mapping $T : D \to D$ is called *nonexpansive* if $|Tx - Ty| \leqslant |x - y|$ for all $x, y \in D$.

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In this paper our attention will be focused on the problem of finding a common fixed point of $(T_i)_{i\geqslant 0}$:

$$\text{find } \bar{x} \in H \quad \text{such that } T_i\bar{x} = \bar{x} \text{ for all } i \geqslant 0. \tag{1.1}$$

More precisely, we propose and study implicit and nonimplicit algorithms for computing a specific point in S. Throughout, we denote

$$\mathcal{N}_I := \{i \in \mathbb{N} \mid T_i \neq I\} \quad (I \text{ being the identity mapping on } H), \tag{1.2}$$

and by $C : D \to D$ a given contraction with constant $\varrho \in [0, 1)$, that is

$$|Cx - Cy| \leqslant \varrho|x - y| \quad \text{for all } x, y \in D. \tag{1.3}$$

Two numerical approaches to solving (1.1) are considered; an implicit regularization-like algorithm, a nonimplicit one:

(1) The first one consists in the solution x_t (as $t \to 0$) of the fixed point equation

$$x_t = tCx_t + \sum_{i\geqslant 0} w_{i,t}T_ix_t, \tag{1.4}$$

where $t \in (0, 1)$, $w_{i,t} \geqslant 0$ for all $i \geqslant 0$ and $\sum_{i\geqslant 0} w_{i,t} = 1 - t$. Moreover, when $i \in \mathcal{N}_I$, we assume $w_{i,t} \neq 0$ for t small enough.

Consider the map $W(\cdot,\cdot)$ defined on $(0, 1) \times D$ by $W(t, x) := tCx + \sum_{i\geqslant 0} w_{i,t}T_ix$ for $(t, x) \in (0, 1) \times D$. Since $t \in (0, 1)$, it is clear that $W(t,\cdot)$ is a self-mapping on D. For all x, y in D, we also have $|W(t, x) - W(t, y)| \leqslant (1 - (1 - \varrho)t)|x - y|$, so that $W(t,\cdot)$ is a contraction on D. As a straightforward consequence, Banach's theorem ensures existence and uniqueness of x_t as fixed point of $W(t,\cdot)$.

As an interesting special case of (1.4), we also investigate the implicit method

$$x_n = \alpha_nCx_n + \frac{(1 - \alpha_n)}{\sum_{k=1}^{n} \gamma_k} \sum_{i=1}^{n} \gamma_iT_ix_n, \tag{1.5}$$

for all $n \geqslant 0$, where $(\alpha_n) \subset (0, 1)$ and $(\gamma_n) \subset (0, +\infty)$.

(2) The second one is the sequence (x_n) generated by a given initial point x_0 in D and the iterative process

$$x_{n+1} := \alpha_nCx_n + \sum_{i\geqslant 0} w_{i,n}T_ix_n, \tag{1.6}$$

for all $n \geqslant 0$, where $(\alpha_n) \subset (0, 1)$, $w_{i,n} \geqslant 0$ for all $i \geqslant 0$ and $\sum_{i\geqslant 0} w_{i,n} = 1 - \alpha_n$. When $i \in \mathcal{N}_I$, we assume $w_{i,n} \neq 0$ for n sufficiently large.

As a practical special case of (1.6), setting $\alpha_{-1} := 1$, we also investigate the iteration

$$x_{n+1} := \alpha_nCx_n + \sum_{i=0}^{n} (\alpha_{i-1} - \alpha_i)T_ix_n, \tag{1.7}$$

for all $n \geqslant 0$, where (α_n) is any decreasing sequence in $(0, 1)$.

There are already several viscosity-like methods for finding common fixed points of nonexpansive operators. Most of them are iterative processes for approximating common fixed points of finite families of nonexpansive mappings (even for more general operators such as *asymptotically-nonexpansive* or *quasi-nonexpansive* mappings) in Hilbert or Banach spaces. These implicit or nonimplicit algorithms have been investigated by several authors, e.g.: see, for instance, Browder [2], Halpern [6], Lions [8], Wittman [12], Bauschke [1], O'Hara et al. [10], Kimura et al. [7], Cirik et al. [4], Yamada et al. [14], Xu et al. [13], Zhou et al. [15], Sun [11]. Some of the existing methods cover the special cases of (1.4) and (1.6) when only one map T occurs (that is, $T_i = T$ for all $i \geqslant 0$) and C is either a constant operator (see Lions [8], Wittman [12], Bauschke [1], O'Hara et al. [10], Xu et al. [13]) or a more general contraction (see Moudafi [9]). In this latter framework, it turns out that the corresponding solution x_t (as $t \to 0$) of (1.4) converges strongly to the unique fixed point of the map $P_{\mathrm{Fix}(T)} \circ C$ (where $P_{\mathrm{Fix}(T)}$ is the metric projection from H onto $\mathrm{Fix}(T)$). Again in this setting, the same convergence result is obtained for (x_n) generated by (1.6) under the following conditions (P1) and (P2):

$$\text{(P1)}\quad \alpha_n \to 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

$$\text{(P2)}\quad \frac{\alpha_{n+1}}{\alpha_n} \to 1 \quad \text{or} \quad \sum_{n\geqslant 0} |\alpha_{n+1} - \alpha_n| < \infty.$$

For details on metric projection in Hilbert spaces the reader is referred to Goebel and Kirk [5].

It is worth noting that our considered problem (1.1) can certainly be solved by all the existing algorithms which involves only one operator. Indeed, if we denote $T := \sum_{i\geqslant 0} w_i T_i$, where $(w_i)_{i\geqslant 0} \subset (0, +\infty)$ and $\sum_{i\geqslant 0} w_i = 1$, then under the same hypotheses on (T_i), T is also a nonexpansive mapping such that $\mathrm{Fix}(T) = S := \bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$. Nevertheless this strategy does not seem really realistic from the computational point of view, because of the infinite sum. To the best of our knowledge, the most significant attempt to solve the proposed problem is due to Combettes [3]. This author suggested a Mann-like iteration process which is applicable to infinite countable families of firmly nonexpansive mappings $(T_i)_{i\in K}$ (where $K \subset \mathbb{Z}$). The proposed method has the following form:

$$x_{n+1} := x_n + \lambda_n \left(\sum_{i\in K_n} w_{i,n} T_i x_n - x_n \right), \tag{1.8}$$

for all $n \geqslant 0$, where K_n is a bounded block included in K, $(\alpha_n) \subset (0, 1]$, $w_{i,n} \in [0, 1]$ ($\forall i \in K_n$), $\sum_{i\in K_n} w_{i,n} = 1$. At each step n, the parameters λ_n and $w_{i,n} \in [0, 1]$ ($\forall i \in K_n$) are depending on the iterate x_n. Strong convergence results regarding this algorithm are proved, but the hypotheses made on the variable blocks $(K_n)_{n\geqslant 0}$ and the operators $(T_i)_{i\in K}$ are restrictive, except when H is finite-dimensional.

The purpose of our work is to study the asymptotic convergence of the two viscosity-like methods (1.4) and (1.6). Under suitable conditions on the involved parameters we establish the convergence in norm of x_t (as $t \to 0$) defined by (1.4) and that of (x_n) (as $n \to \infty$) given by (1.6) to the unique fixed point of the map $P_S \circ C$. It turns out that our convergence results cover all the known ones as special cases of (1.4) and (1.6) for many finitely nonexpansive operators. Moreover, by (1.5) and (1.7) we provide iterative processes of practical interest from the computational point of view for solving (1.1). The proposed methods are also complementary to the one defined by iteration (1.8), since the techniques used are completely different.

To begin with, we make the following useful remark.

Remark 1.1. A self-mapping $T : D \to D$ satisfies the demiclosedness principle means that if (x_n) converges weakly to $q \in D$ and $(x_n - Tx_n)$ converges strongly to 0, then q is a fixed point of T. It is well known that any nonexpansive mapping $T : D \to D$ is demiclosed on D.

2. The fixed point equation

In this section, strong convergence results of the solution x_t of (1.4) (as $t \to 0$) are proved. To this end, we need some preliminaries.

Lemma 2.1. *The solution x_t of* (1.4) *is bounded* (*as $t \to 0$*), *besides it has at most one strong limit point in S.*

Proof. Let q in S. From (1.4), we have

$$\begin{aligned}|x_t - q|^2 &= t\langle Cx_t - q, x_t - q\rangle + \sum_{i\geqslant 0} w_{i,t}\langle T_i x_t - q, x_t - q\rangle \\ &\leqslant t\langle Cx_t - q, x_t - q\rangle + \Bigl(\sum_{i\geqslant 0} w_{i,t}\Bigr)|x_t - q|^2;\end{aligned}$$

since $\sum_{i\geqslant 0} w_{i,t} = 1 - t$, it follows that

$$|x_t - q|^2 \leqslant \langle Cx_t - q, x_t - q\rangle. \tag{2.1}$$

As C is a contraction with modulus $\varrho \in (0, 1)$, the previous inequality yields

$$|x_t - q|^2 \leqslant \langle Cx_t - Cq, x_t - q\rangle + \langle Cq - q, x_t - q\rangle \leqslant \varrho|x_t - q|^2 + \langle Cq - q, x_t - q\rangle,$$

hence

$$|x_t - q|^2 \leqslant \frac{1}{1-\varrho}\langle Cq - q, x_t - q\rangle, \tag{2.2}$$

so that

$$|x_t - q| \leqslant \frac{1}{1-\varrho}|Cq - q|,$$

which proves the boundedness of (x_t). Assume q_1, q_2 are two strong limit points of (x_t) in S. Thanks to (2.1), we have the following two inequalities:

$$|q_1 - q_2|^2 \leqslant \langle Cq_1 - q_2, q_1 - q_2\rangle, \qquad |q_2 - q_1|^2 \leqslant \langle Cq_2 - q_1, q_2 - q_1\rangle.$$

By adding these inequalities, we get

$$\begin{aligned}2|q_1 - q_2|^2 &\leqslant \bigl\langle (Cq_1 - Cq_2) + (q_1 - q_2), q_1 - q_2\bigr\rangle \\ &\leqslant (\varrho + 1)|q_1 - q_2|^2,\end{aligned}$$

so that $|q_1 - q_2| = 0$, which proves uniqueness of a strong limit point of (x_t). □

Lemma 2.2. *Let $t_n \in (0, 1)$ such that $t_n \to 0$* (*as $n \to +\infty$*) *and assume the following condition* (L) *holds*:

$$\text{(L)} \quad \forall i \in \mathcal{N}_I, \quad \lim_{n\to+\infty} \frac{t_n}{w_{i,t_n}} = 0.$$

Then the solution x_t of (1.4) *satisfies*

$$\lim_{n\to+\infty} |x_{t_n} - T_i x_{t_n}| = 0 \quad \textit{for each } i \in \mathcal{N}_I. \tag{2.3}$$

Proof. By definition of scheme (1.4), we have

$$x_t + \sum_{i\geqslant 0} w_{i,t}(x_t - T_i x_t) - (1-\alpha_t)x_t = tCx_t,$$

that is

$$\sum_{i\geqslant 0} w_{i,t}(x_t - T_i x_t) = t(Cx_t - x_t).$$

Given any $q \in S$, it is then immediate that

$$\sum_{i\geqslant 0} w_{i,t}\langle x_t - T_i x_t, x_t - q\rangle = t\langle Cx_t - x_t, x_t - q\rangle. \tag{2.4}$$

Moreover, for any nonexpansive (even quasi-nonexpansive) self-mapping $T : D \to D$ such that $\mathrm{Fix}(T) \neq \emptyset$ and for all $(p, x) \in \mathrm{Fix}(T) \times D$, we have

$$\begin{aligned}|x-p|^2 \geqslant |Tx-p|^2 &= \left|(Tx-x)+(x-p)\right|^2\\ &= |Tx-x|^2 + |x-p|^2 + 2\langle Tx-x, x-p\rangle,\end{aligned}$$

so that

$$|Tx-x|^2 \leqslant 2\langle x-Tx, x-p\rangle \quad \forall p\in \mathrm{Fix}(T),\ \forall x \in D. \tag{2.5}$$

Thanks to this last inequality, from (2.4) we obtain

$$\frac{1}{2}\sum_{i\geqslant 0} w_{i,t}|T_i x_t - q|^2 \leqslant t\langle Cx_t - x_t, x_t - q\rangle, \tag{2.6}$$

hence, for all $i \in \mathcal{N}_I$ and for t small enough, since $w_{i,t} \neq 0$ we have

$$|T_i x_t - q|^2 \leqslant \frac{t}{w_{i,t}}\langle Cx_t - x_t, x_t - q\rangle. \tag{2.7}$$

By Lemma 2.1, the solution x_t is bounded (as $t \to 0$), then so it is for the quantity $\langle Cx_t - x_t, x_t - q\rangle$; when condition (L) is also satisfied, it is easily deduced from (2.7) that $|T_i x_{t_n} - x_{t_n}| \to 0$, as $t_n \to 0$, for all $i \in \mathcal{N}_I$, that is the desired result. □

Theorem 2.3. *Under the hypotheses of Lemma* 2.2*, the solution x_t of Eq.* (1.4) *satisfies*

$$\lim_{n\to+\infty} |x_{t_n} - \bar{x}| = 0, \tag{2.8}$$

where $\bar{x}$ is the unique fixed point of the contraction $P_S \circ C$, P_S being the metric projection from H onto S.

Proof. Set $y_n := x_{t_n}$. According to Lemma 2.2, we have $\lim_{n\to+\infty} |y_n - T_i y_n| = 0$ (for all $i \in \mathcal{N}_I$) under condition (L). By Lemma 2.1 noticing that (y_n) is a bounded sequence, there exists a subsequence of (y_n) (labeled (y_{n_k})) which converges weakly to a point $\bar{x}$ in S, because of the demiclosedness of the mappings T_i (for all $i \in \mathcal{N}_I$). From (2.2), we then have

$$(1-\varrho)|y_{n_k} - \bar{x}|^2 \leqslant \langle C\bar{x} - \bar{x}, y_{n_k} - \bar{x}\rangle. \tag{2.9}$$

As $\langle C\bar{x} - \bar{x}, y_{n_k} - \bar{x}\rangle \to 0$ by weak convergence of (y_{n_k}) to $\bar{x}$, inequality (2.9) shows that y_{n_k} strongly converges to $\bar{x}$. Adding to the fact that any strong cluster-point of (y_n) is in S and since by Lemma 2.1 (y_n) has a unique strong cluster-point in S, we deduce the strong convergence of the whole sequence (y_n) to $\bar{x}$. It remains to characterize the limit $\bar{x}$ of (x_{t_n}). Let q be any element in S. By (2.7), it is easily seen that

$$\langle x_{t_n} - Cx_{t_n}, x_{t_n} - q\rangle \leqslant 0 \quad \text{since } t_n \in (0,1);$$

passing to the limit as $t_n \to 0$, we get

$$\langle \bar{x} - C\bar{x}, \bar{x} - q\rangle \leqslant 0 \quad \forall q \in S, \tag{2.10}$$

so that $\bar{x} = P_S(C\bar{x})$, which ends the proof. □

Corollary 2.4. *Assume the following condition* (L′) *is satisfied*:

$$(\mathrm{L}') \quad \forall i \in \mathcal{N}_I, \quad \lim_{t\to 0} \frac{t}{w_{i,t}} = 0.$$

Then, as $t \to 0$, the solution x_t of Eq. (1.4) *converges strongly to the unique fixed point, $\bar{x}$, of the contraction $P_S \circ C$, where P_S is the metric projection from H onto S.*

Proof. This result is a straightforward consequence of Theorem 2.3. □

Corollary 2.5. *If $\alpha_n \sum_{k=1}^n \gamma_k \to 0$ as $n \to \infty$, then the solution (x_n) of* (1.5) *converges strongly to $\bar{x}$, the unique fixed point of the contraction $P_S \circ C$.*

Proof. The solution (x_n) of (1.5) corresponds to that of (1.4) when $t = \alpha_n$ and $w_{i,(t=\alpha_n)} = \frac{1-\alpha_n}{\sum_{k=1}^n \gamma_k}\gamma_i$ if $0 \leqslant i \leqslant n$, $w_{i,(t=\alpha_n)} = 0$ otherwise. It is then clear that $\sum_{i\geqslant 0} w_{i,(t=\alpha_n)} = 1$. Therefore, the convergence result on (x_n) is obtained from Theorem 2.3 since

$$\lim_{n\to+\infty} \frac{\alpha_n}{w_{i,(t=\alpha_n)}} = \lim_{n\to+\infty} \frac{\alpha_n \sum_{k=1}^n \gamma_k}{(1-\alpha_n)\gamma_i} = 0 \quad (\forall i \geqslant 0),$$

provided that $\alpha_n \sum_{k=1}^n \gamma_k \to 0$ as $n \to \infty$; that is the desired result. □

3. The iterative method

In this section, we prove strong convergence results regarding the sequence (x_n) obtained with (1.6), by involving the following conditions:

$$(\mathrm{Q1}) \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

(Q2) $$\left[\begin{array}{l}\text{For all } i \in \mathcal{N}_I,\\[1ex] \bullet\ \ \dfrac{1}{w_{i,n}}\left|1-\dfrac{\alpha_{n-1}}{\alpha_n}\right| \to 0, \quad \text{or} \quad \displaystyle\sum_n \frac{1}{w_{i,n}}|\alpha_n-\alpha_{n-1}| < \infty,\\[1ex] \bullet\ \ \dfrac{1}{\alpha_n}\left|\dfrac{1}{w_{i,n}}-\dfrac{1}{w_{i,n-1}}\right| \to 0, \quad \text{or} \quad \displaystyle\sum_n \left|\frac{1}{w_{i,n}}-\frac{1}{w_{i,n-1}}\right| < \infty,\\[1ex] \bullet\ \ \dfrac{1}{w_{i,n}\alpha_n}\displaystyle\sum_{k\geqslant 0}|w_{k,n}-w_{k,n-1}| \to 0, \quad \text{or} \quad \sum_n \frac{1}{w_{i,n}}\sum_{k\geqslant 0}|w_{k,n}-w_{k,n-1}| < \infty.\end{array}\right.$$

(Q3) $$\frac{\alpha_n}{w_{i,n}} \to 0 \quad (\text{for all } i \in \mathcal{N}_I).$$

The next lemmas will be needed in the proof of the main result of this section.

Lemma 3.1. *Let* (s_n), $(c_n) \subset \mathbb{R}_+$, $(a_n) \subset (0,1)$ *and* $(b_n) \subset \mathbb{R}$ *be sequences such that*

$$s_{n+1} \leqslant (1-a_n)s_n + b_n + c_n \quad \textit{for all } n \geqslant 0. \tag{3.1}$$

Assume $\sum_{n\geqslant 0}|c_n| < \infty$. *Then the following results hold*:

(1) *If* $b_n \leqslant \beta a_n$ (*where* $\beta \geqslant 0$), *then* (s_n) *is a bounded sequence.*
(2) *If we have*

$$\sum_{n=0}^{\infty} a_n = \infty \quad \textit{and} \quad \limsup_{n\to\infty} \frac{b_n}{a_n} \leqslant 0,$$

then $\lim_{n\to\infty} s_n = 0$.

Proof. Let us prove (1). Set $\gamma_{n,k} := \prod_{j=k}^{n}(1-a_j)$ (for $n \geqslant k \geqslant 0$). If $b_n \leqslant \beta a_n$, then by a simple induction we have

$$\begin{aligned} s_{n+1} &\leqslant (\gamma_{n,0})s_0 + \sum_{j=0}^{n-1}(\gamma_{n,j+1})(a_j\beta + c_j) + (a_n\beta + c_n) \\ &= (\gamma_{n,0})s_0 + \beta\left(\sum_{j=0}^{n-1}(\gamma_{n,j+1} - \gamma_{n,j}) + a_n\right) + \sum_{j=0}^{n-1}\gamma_{n,j+1}c_j + c_n, \end{aligned}$$

hence

$$s_{n+1} \leqslant (\gamma_{n,0})s_0 + \beta(1-\gamma_{n,0}) + \sum_{j=0}^{n-1}\gamma_{n,j+1}c_j + c_n. \tag{3.2}$$

Since $\gamma_{n,j} \leqslant 1$ for $0 \leqslant j \leqslant n$, we deduce

$$s_{n+1} \leqslant s_0 + \beta + \sum_{j=0}^{n} c_j;$$

so that (s_n) is bounded since $\sum_j c_j < \infty$, which proves (1). By now we prove (2). Let ϵ be any positive real number. If $\limsup_{n\to\infty} \frac{b_n}{a_n} \leqslant 0$ then there exists $p = p(\epsilon)$ in $\mathbb{N}$ such that $b_n \leqslant \epsilon a_n$ for all $n \geqslant p$; hence by (3.2) we get

$$s_{n+1} \leqslant (\gamma_{n,p})s_p + \epsilon(1-\gamma_{n,p}) + \sum_{j=p}^{n-1} \gamma_{n,j+1}c_j + c_n. \tag{3.3}$$

Moreover, since $\sum_j c_j < \infty$, then there exists q_ϵ in $\mathbb{N}$ such that

$$q_\epsilon \geqslant p \quad \text{and} \quad \sum_{j\geqslant q_\epsilon+1} c_j < \epsilon,$$

hence

$$\forall n > q_\epsilon, \quad \sum_{j=p}^{n-1} \gamma_{n,j+1}c_j + c_n \leqslant \gamma_{n,q_\epsilon+1}\sum_{j=p}^{q_\epsilon} c_j + \sum_{j\geqslant q_\epsilon+1} c_j + \epsilon \leqslant \gamma_{n,q_\epsilon+1}\sum_{j\geqslant 0} c_j + 2\epsilon.$$

Combining this last inequality with (3.3), for $n > q_\epsilon$ we then obtain

$$s_{n+1} \leqslant (\gamma_{n,p})s_p + \epsilon(1-\gamma_{n,p}) + \gamma_{n,q_\epsilon+1}\sum_{j\geqslant 0} c_j + 2\epsilon. \tag{3.4}$$

It is also seen that $\lim_{n\to+\infty} \gamma_{n,p} = 0$ and $\lim_{n\to+\infty} \gamma_{n,q_\epsilon+1} = 0$ if $\sum_{n=0}^{\infty} a_n = \infty$; consequently, using (3.4) we deduce $\lim_{n\to+\infty} s_n = 0$, that is (2). □

Lemma 3.2. *The sequence* (x_n) *generated by scheme* (1.6) *is bounded.*

Proof. From (1.6) and given any $p \in S$, we have

$$x_{n+1} - p = \alpha_n(Cx_n - p) + \sum_{i\in J} w_{i,n}(T_i x_n - p), \tag{3.5}$$

since $\alpha_n + \sum_{i\geqslant 0} w_{i,n} = 1$; so that

$$\begin{aligned} |x_{n+1} - p| &\leqslant \alpha_n|Cx_n - Cp| + \alpha_n|Cp - p| + \sum_{i\in J} w_{i,n}|T_i x_n - p| \\ &\leqslant \alpha_n\varrho|x_n - p| + \alpha_n|Cp - p| + (1-\alpha_n)|x_n - p| \\ &= \big(1 - (1-\varrho)\alpha_n\big)|x_n - p| + \alpha_n|Cp - p|. \end{aligned}$$

Applying Lemma 3.1, we deduce the boundedness of the sequence (x_n). □

Lemma 3.3. *If conditions* (Q1)–(Q2) *hold, then the sequence* (x_n) *given by scheme* (1.6) *satisfies*

$$\frac{1}{w_{i,n}}|x_{n+1} - x_n| \to 0 \quad \textit{for all } i \in \mathcal{N}_I. \tag{3.6}$$

Proof. By definition of scheme (1.6), we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n Cx_n - \alpha_{n-1}Cx_{n-1} + \sum_{i\geqslant 0} w_{i,n}T_i x_n - \sum_{i\geqslant 0} w_{i,n-1}T_i x_{n-1} \\ &= \alpha_n(Cx_n - Cx_{n-1}) + (\alpha_n - \alpha_{n-1})Cx_{n-1} \\ &\quad + \sum_{i\geqslant 0} w_{i,n}(T_i x_n - T_i x_{n-1}) + \sum_{i\geqslant 0}(w_{i,n} - w_{i,n-1})T_i x_{n-1}. \end{aligned}$$

The operators T_i being nonexpansive, C being a contraction with modulus ϱ, and since $\sum_{i\geqslant 0} w_{i,n} = 1-\alpha_n$, we then obtain

$$\left|x_{n+1}-x_n\right| \leqslant \bigl(1-(1-\varrho)\alpha_n\bigr)|x_n-x_{n-1}| + |\alpha_n-\alpha_{n-1}| \times |Cx_{n-1}| + \sum_{i\geqslant 0} |w_{i,n}-w_{i,n-1}| \times |T_i x_{n-1}|.$$

Thus, for all $i \in \mathcal{N}_I$ and n large enough, we get

$$\begin{aligned}\frac{1}{w_{i,n}}|x_{n+1}-x_n| &\leqslant \bigl(1-(1-\varrho)\alpha_n\bigr)\frac{1}{w_{i,n-1}}|x_n-x_{n-1}| \\ &\quad + \bigl(1-(1-\varrho)\alpha_n\bigr)\left(\frac{1}{w_{i,n}}-\frac{1}{w_{i,n-1}}\right)|x_n-x_{n-1}| \\ &\quad + \frac{1}{w_{i,n}}|\alpha_n-\alpha_{n-1}||Cx_{n-1}| + \frac{1}{w_{i,n}}\sum_{i\geqslant 0}|w_{i,n}-w_{i,n-1}| \times |T_i x_{n-1}|.\end{aligned}$$

As (x_n) is bounded (see Lemma 3.2), then by nonexpansiveness of each mapping T_i and C, it is easily seen that the family $(T_i x_n)_{i,n\geqslant 0}$ and (Cx_n) are also bounded. Consequently, there exists a positive constant M such that

$$\begin{aligned}\frac{1}{w_{i,n}}|x_{n+1}-x_n| &\leqslant \bigl(1-(1-\varrho)\alpha_n\bigr)\left(\frac{1}{w_{i,n-1}}|x_n-x_{n-1}|\right) + M\left|\frac{1}{w_{i,n}}-\frac{1}{w_{i,n-1}}\right| \\ &\quad + M\left(\frac{1}{w_{i,n}}|\alpha_n-\alpha_{n-1}| + \frac{1}{w_{i,n}}\sum_{i\geqslant 0}|w_{i,n}-w_{i,n-1}|\right).\end{aligned}$$

Thanks to this last inequality and taking into account Lemma 3.1, the desired result follows. □

Lemma 3.4. *Assume conditions* (Q1)–(Q3) *hold. Then* (x_n) *given by scheme* (1.6) *satisfies*

$$\lim_{n\to\infty} |x_n - T_i x_n| = 0 \quad \forall i \in \mathcal{N}_I.$$

Proof. Using scheme (1.6), we have

$$x_{n+1} + \sum_{i\geqslant 0} w_{i,n}(x_n - T_i x_n) - (1-\alpha_n)x_n = \alpha_n C x_n,$$

that is

$$\sum_{i\geqslant 0} w_{i,n}(x_n - T_i x_n) = \alpha_n(Cx_n - x_n) + (x_n - x_{n+1});$$

hence for any q in S, we get

$$\sum_{i\geqslant 0} w_{i,n}\langle x_n - T_{i,n}x_n, x_n - q\rangle = \alpha_n\langle Cx_n - x_n, x_n - q\rangle + \langle x_n - x_{n+1}, x_n - q\rangle. \tag{3.7}$$

Moreover, as each T_i $(i \geqslant 0)$ is nonexpansive, by inequality (2.5) we have

$$|T_i x_n - x_n|^2 \leqslant 2\langle x_n - T_i x_n, x_n - q\rangle.$$

Combining this last inequality with (3.7), we get

$$\frac{1}{2}\sum_{i\geqslant 0} w_{i,n}|T_i x_n - x_n|^2 \leqslant \alpha_n\langle Cx_n - x_n, x_n - q\rangle + \langle x_n - x_{n+1}, x_n - q\rangle, \tag{3.8}$$

so that, for all $i \in \mathcal{N}_I$,

$$|T_i x_n - x_n|^2 \leqslant \frac{\alpha_n}{w_{i,n}}\langle Cx_n - x_n, x_n - q\rangle + \frac{1}{w_{i,n}}\langle x_n - x_{n+1}, x_n - q\rangle. \tag{3.9}$$

Hence, by Lemma 3.2 there exists a positive constant M_1 such that

$$|T_i x_n - x_n|^2 \leqslant M_1\left(\frac{\alpha_n}{w_{i,n}} + \frac{1}{w_{i,n}}|x_n - x_{n+1}|\right). \tag{3.10}$$

Using Lemma 3.3 and condition (Q3), we complete the proof. □

The main result of this section is given by the following theorem.

Theorem 3.5. *Under assumptions* (Q1)–(Q3)*, the sequence* (x_n) *given by scheme* (1.6) *converges strongly to* $\bar{x}$ *the unique fixed point of* $P_S \circ C$*, where* P_S *is the metric projection from* H *onto* S.

Proof. By scheme (1.6), we have

$$\begin{aligned} x_{n+1} - \bar{x} &= \alpha_n(Cx_n - \bar{x}) + (1-\alpha_n)\sum_{i\geqslant 0} w_{i,n}(T_i x_n - \bar{x}) \\ &= \left(\alpha_n(Cx_n - C\bar{x}) + (1-\alpha_n)\sum_{i\geqslant 0} w_{i,n}(T_i x_n - \bar{x})\right) + \alpha_n(C\bar{x} - \bar{x}). \end{aligned}$$

Recall that for any a, b in H, we have

$$|a+b|^2 - 2\langle b, a+b\rangle = |a|^2 - |b|^2, \tag{3.11}$$

so that

$$\begin{aligned} &|x_{n+1} - \bar{x}|^2 - 2\alpha_n\langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\quad\leqslant \left|\alpha_n(Cx_n - C\bar{x}) + \sum_{i\geqslant 0} w_{i,n}(T_i x_n - \bar{x})\right|^2 \leqslant \left(\alpha_n\varrho|x_n - \bar{x}| + \left(\sum_{i\geqslant 0} w_{i,n}\right)|x_n - \bar{x}|\right)^2 \\ &\quad\leqslant \big(1 - (1-\varrho)\alpha_n\big)^2|x_n - \bar{x}|^2; \end{aligned}$$

noting that $(1-(1-\varrho)\alpha_n)^2 \leqslant (1-(1-\varrho)\alpha_n)$, we deduce

$$|x_{n+1} - \bar{x}|^2 \leqslant \big(1 - (1-\varrho)\alpha_n\big)|x_n - \bar{x}|^2 + 2\alpha_n\langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle. \tag{3.12}$$

Otherwise, Lemma 3.4 shows that any weak limit point of (x_n) is in S because of the demiclosedness of each operator T_i; since $\bar{x} = P_S(C\bar{x})$, it is easily checked that

$$\limsup_{n\to\infty}\langle C\bar{x} - \bar{x}, x_n - \bar{x}\rangle \leqslant 0. \tag{3.13}$$

By (3.12), (3.13) and using Lemma 3.1, we conclude that (x_n) strongly converges to $\bar{x}$, which completes the proof. □

Corollary 3.6. *Assume the following conditions* (P1′) *is satisfied*:

$$(\mathrm{P1}')\quad \alpha_n \searrow 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then (x_n) *given by scheme* (1.7) *converges strongly to* $\bar{x}$ *the unique fixed point of* $P_S \circ C$, *where* P_S *is the metric projection from* H *onto* S.

Proof. Scheme (1.7) is a special case of (1.6) when $w_{i,n} = \alpha_{i-1} - \alpha_i$ for $0 \leqslant i \leqslant n$ and $w_{i,n} = 0$ for $i \geqslant n+1$. Thus we have $\sum_{i \geqslant 0} w_{i,n} = 1 - \alpha_n$, but also $w_{i,n} > 0$ for all $i \geqslant 0$ and for n large enough. In this setting, it is then immediate that conditions (Q1)–(Q3) in Theorem 3.5 are reduced to (P1′) and the following (P2):

$$(\mathrm{P2})\quad \frac{\alpha_{n+1}}{\alpha_n} \to 1 \quad \text{or} \quad \sum_{n \geqslant 0} |\alpha_{n+1} - \alpha_n| < \infty.$$

Note that $\sum_{n \geqslant 0} |\alpha_{n+1} - \alpha_n| < \infty$ if (α_n) is a positive decreasing sequence. As a consequence (P1′) yields (P2), which leads to the desired result. □

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