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Note

Two classes of hyperplanes of dual polar spaces without subquadrangular quads

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Abstract

Let Π be one of the following polar spaces: (i) a nondegenerate polar space of rank $n-1 \geqslant 2$ which is embedded as a hyperplane in $Q(2n, \mathbb{K})$; (ii) a nondegenerate polar space of rank $n \geqslant 2$ which contains $Q(2n, \mathbb{K})$ as a hyperplane. Let Δ and $DQ(2n, \mathbb{K})$ denote the dual polar spaces associated with Π and $Q(2n, \mathbb{K})$, respectively. We show that every locally singular hyperplane of $DQ(2n, \mathbb{K})$ gives rise to a hyperplane of Δ without subquadrangular quads. Suppose Π is associated with a nonsingular quadric $Q^-(2n+\epsilon, \mathbb{K})$ of $\mathrm{PG}(2n+\epsilon, \mathbb{K})$, $\epsilon \in \{-1, 1\}$, described by a quadratic form of Witt-index $n + \frac{\epsilon-1}{2}$, which becomes a quadratic form of Witt-index $n + \frac{\epsilon+1}{2}$ when regarded over a quadratic Galois extension of $\mathbb{K}$. Then we show that the constructed hyperplanes of Δ arise from embedding.

Keywords: Dual polar space; Hyperplane; Spin-embedding

1. Introduction

1.1. Basic definitions

Let Π be a nondegenerate polar space of rank $n \geqslant 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. The

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geometry Δ is called a *dual polar space* (Cameron [2]). There exists a bijective correspondence between the non-empty convex subspaces of Δ and the possibly empty singular subspaces of Π. If α is a possibly empty singular subspace of Π, then the set of all maximal singular subspaces containing α is a convex subspace of Δ. Conversely, every convex subspace of Δ is obtained in this way. The maximal distance between two points of a convex subspace A of Δ is called the *diameter* of A and is denoted as $\mathrm{diam}(A)$. (Distances $\mathrm{d}(\cdot,\cdot)$ are measured in the point or collinearity graph of Δ.) The convex subspaces of diameter 2, 3, respectively $n-1$, of Δ are called the *quads*, *hexes*, respectively *maxes*, of Δ. The points and lines contained in a quad define a generalized quadrangle (GQ). We refer to Payne and Thas [12] for the basic terminology regarding GQs to be used in this paper. If A is a convex subspace of Δ, then for every point x of Δ, there exists a unique point $\pi_A(x) \in A$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_A(x)) + \mathrm{d}(\pi_A(x),y)$ for every point y of A. The point $\pi_A(x)$ is called the *projection* of x onto A. If A_1 and A_2 are two disjoint maxes, then the restriction of π_{A_2} to A_1 defines an isomorphism between A_1 and A_2 (regarded as dual polar spaces of rank $n-1$). The set of convex subspaces through a point x of Δ defines a projective space of dimension $n-1$ which we will denote by $Res_\Delta(x)$. If x is a point of Δ and $i \in \mathbb{N}$, then $\Delta_i(x)$ denotes the set of points of Δ at distance i from x. We define $x^\perp := \Delta_0(x) \cup \Delta_1(x)$ for every point x of Δ.

A *hyperplane* of a point-line geometry is a proper subspace meeting each line (necessarily in a unique point or the whole line). Suppose H is a hyperplane of a thick dual polar space Δ of rank $n \geqslant 2$. By Shult [16, Lemma 6.1], we then know that H is a maximal subspace of Δ. A point x of H is called *deep* (*with respect to* H) if $x^\perp \subseteq H$. If H consists of all points of Δ at non-maximal distance from a given point x, then H is called the *singular hyperplane of* Δ *with deepest point* x.

If H is a hyperplane of a thick dual polar space Δ and if Q is a quad of Δ, then one of the following cases occurs: (i) $Q \subseteq H$; (ii) $Q \cap H = x^\perp \cap Q$ for a certain point $x \in Q$; (iii) $Q \cap H$ is an ovoid of Q; (iv) $Q \cap H$ is a subquadrangle of Q. If case (i), case (ii), case (iii) respectively case (iv) occurs, then Q is called *deep*, *singular*, *ovoidal*, respectively *subquadrangular* (*with respect to* H). If every quad is deep or singular with respect to H, then H is called *locally singular*.

A *full embedding* of a point-line geometry $\mathcal{S}$ into a projective space Σ is an injective mapping e from the point-set P of $\mathcal{S}$ to the point-set of Σ satisfying (i) $\langle e(P) \rangle = \Sigma$ and (ii) $e(L)$ is a line of Σ for every line L of $\mathcal{S}$. If e is a full embedding of $\mathcal{S}$, then for every hyperplane α of Σ, the set $e^{-1}(e(P) \cap \alpha)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $e^{-1}(e(P) \cap \alpha)$ *arises from the embedding* e.

If $e : \Delta \to \Sigma$ is a full embedding of a thick dual polar space Δ into a projective space Σ, then for every hyperplane H of Δ, $\langle e(H) \rangle$ is either Σ or a hyperplane of Σ. (Recall that every hyperplane of Δ is a maximal subspace of Δ.) If $\langle e(H) \rangle$ is a hyperplane α of Σ, then $e^{-1}(e(\Delta) \cap \alpha) = H$ (again since H is a maximal subspace of Δ). The full embedding e is called *polarized* if $\langle e(H) \rangle$ is a hyperplane of Σ for every singular hyperplane H of Δ.

Let $Q^+(2n+1,\mathbb{K})$, $n \geqslant 1$, denote a nonsingular hyperbolic quadric in $\mathrm{PG}(2n+1,\mathbb{K})$ (i.e., a quadric of Witt-index $n+1$). Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of generators (= maximal subspaces) of $Q^+(2n+1,\mathbb{K})$. Recall that two generators belong to the same family if they intersect in a subspace of even co-dimension. Let $\mathcal{S}^\epsilon$, $\epsilon \in \{+,-\}$, denote the point-line geometry with points the elements of $\mathcal{M}^\epsilon$ and with lines the $(n-2)$-dimensional subspaces contained in $Q^+(2n+1,\mathbb{K})$ (natural incidence). The geometries $\mathcal{S}^+$ and $\mathcal{S}^-$ are isomorphic and are called the *half-spin geometries* for $Q^+(2n+1,\mathbb{K})$. We will denote any of these geometries by $HS(2n+1,\mathbb{K})$. The half-spin geometry $HS(2n+1,\mathbb{K})$ admits a full embedding into

$\mathrm{PG}(2^n-1,\mathbb{K})$ which is called the *spin-embedding* of $HS(2n+1,\mathbb{K})$. We refer to Chevalley [4] or Buekenhout and Cameron [1] for the definition of this embedding.

In this paper, $Q(2n,\mathbb{K})$ ($n \geqslant 2$ and $\mathbb{K}$ a field) denotes a nonsingular quadric of Witt-index n in $\mathrm{PG}(2n,\mathbb{K})$ and $DQ(2n,\mathbb{K})$ denotes its associated dual polar space. Suppose $Q(2n,\mathbb{K})$ is obtained from a nonsingular hyperbolic quadric $Q^+(2n+1,\mathbb{K})$ of $\mathrm{PG}(2n+1,\mathbb{K})$ by intersecting it with a hyperplane of $\mathrm{PG}(2n+1,\mathbb{K})$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of generators of $Q^+(2n+1,\mathbb{K})$ and let $HS(2n+1,\mathbb{K})$ denote the half-spin geometry for $Q^+(2n+1,\mathbb{K})$ defined on the set $\mathcal{M}^+$. Let e denote the spin-embedding of $HS(2n+1,\mathbb{K})$ into $\mathrm{PG}(2^n-1,\mathbb{K})$. For every generator α of $Q(2n,\mathbb{K})$, let $\phi(\alpha)$ denote the unique element of $\mathcal{M}^+$ containing α. Then $e\circ\phi$ defines a full embedding of the dual polar space $DQ(2n,\mathbb{K})$ into $\mathrm{PG}(2^n-1,\mathbb{K})$. This embedding is called the *spin-embedding* of $DQ(2n,\mathbb{K})$.

By De Bruyn [7] (see also Shult and Thas [17]) the locally singular hyperplanes of $DQ(2n,\mathbb{K})$ are precisely the hyperplanes of $DQ(2n,\mathbb{K})$ which arise from its spin-embedding. By Pralle [13] (see also Shult [14] for the finite case) the dual polar space $DQ(6,\mathbb{K})$ has two types of locally singular hyperplanes, the singular hyperplanes and the so-called *hexagonal hyperplanes*. The points and lines contained in a hexagonal hyperplane define a split Cayley hexagon $H(\mathbb{K})$. We refer to Van Maldeghem [18] for the definition of this generalized polygon. Cardinali, De Bruyn and Pasini [3] showed that the dual polar space $DQ(8,\mathbb{K})$ has three types of locally singular hyperplanes.

1.2. Overview of the results

Suppose that one of the following holds for a certain field $\mathbb{K}$:

(I) Π is a nondegenerate polar space of rank $n-1 \geqslant 2$ which is embedded as a hyperplane in $Q(2n,\mathbb{K})$;
(II) Π is a nondegenerate polar space of rank $n \geqslant 2$ which contains $Q(2n,\mathbb{K})$ as a hyperplane.

Let Δ and $DQ(2n,\mathbb{K})$ denote the dual polar spaces associated with Π and $Q(2n,\mathbb{K})$, respectively. The following theorem summarizes Theorems 2.1 and 2.2 of Section 2.

Theorem 1.1. *(Section 2) Every locally singular hyperplane of $DQ(2n,\mathbb{K})$ gives rise to a hyperplane of Δ without subquadrangular quads.*

Suppose now that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$. Let θ denote the unique nontrivial element in the Galois group $Gal(\mathbb{K}'/\mathbb{K})$ and let $n \in \mathbb{N}\setminus\{0,1\}$. For all $i,j \in \{0,\dots,2n+1\}$ with $i \leqslant j$, let $a_{ij} \in \mathbb{K}$ such that $q(\overline{X}) = \sum_{0\leqslant i\leqslant j\leqslant 2n+1} a_{ij}X_iX_j$ is a quadratic form defining a quadric $Q^-(2n+1,\mathbb{K})$ of Witt-index n in $\mathrm{PG}(2n+1,\mathbb{K})$ and a quadric $Q^+(2n+1,\mathbb{K}')$ of Witt-index $n+1$ in $\mathrm{PG}(2n+1,\mathbb{K}')$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of maximal subspaces of $Q^+(2n+1,\mathbb{K}')$. Let $DQ^-(2n+1,\mathbb{K})$ denote the dual polar space associated with the quadric $Q^-(2n+1,\mathbb{K})$ and let $HS(2n+1,\mathbb{K}')$ denote the half-spin geometry for $Q^+(2n+1,\mathbb{K}')$ defined on the set $\mathcal{M}^+$. Let e denote the spin-embedding of $HS(2n+1,\mathbb{K}')$. For every generator α of $DQ^-(2n+1,\mathbb{K})$, let $\phi(\alpha)$ denote the unique element of $\mathcal{M}^+$ containing α. Then $e\circ\phi$ defines a full embedding of the dual polar space $DQ^-(2n+1,\mathbb{K})$ into $\mathrm{PG}(2^n-1,\mathbb{K}')$. This was shown by Cooperstein and Shult [6] for the finite case. De Bruyn [9] gave another proof (without invoking counting arguments) which also holds for infinite fields. The embedding $e\circ\phi$ is called the spin-embedding of $DQ^-(2n+1, \mathbb{K})$. In [9], it was also shown that the spin-embedding of $DQ^-(2n+1, \mathbb{K})$ is polarized.

In Section 3, we address the problem of whether the hyperplanes constructed in Section 2 arise from embedding.

Theorem 1.2. *(Section 3) If $\Pi = Q^-(2n-1, \mathbb{K})$ (Case I) or $\Pi = Q^-(2n+1, \mathbb{K})$ (Case II), then the constructed hyperplanes of Δ arise from its spin-embedding.*

2. Two constructions of hyperplanes

In this section, we will give two constructions of hyperplanes of dual polar spaces without subquadrangular quads. In both constructions, the hyperplanes arise from locally singular hyperplanes of $DQ(2n, \mathbb{K})$. The first construction generalizes a construction given in Pralle [13].

2.1. First construction

Suppose the polar space $Q(2n, \mathbb{K})$, $n \geqslant 2$, is embedded as a hyperplane in a nondegenerate thick polar space Π of rank n. Let Δ denote the dual polar space associated with Π. If $n = 3$, then every locally singular hyperplane of $DQ(2n, \mathbb{K}) = DQ(6, \mathbb{K})$ gives rise to a hyperplane of Δ by a construction given in Pralle [13, Theorem 3]. A similar construction can be given for arbitrary n. The aim of this section is to show that also for arbitrary n one can produce hyperplanes of Δ in this way. We also show that none of these hyperplanes admit subquadrangular quads.

For every singular subspace α of Π, let $\widetilde{\alpha}$ denote the corresponding convex subspace of Δ. If α is contained in $Q(2n, \mathbb{K})$, then we denote by $\bar{\alpha}$ the corresponding convex subspace of $DQ(2n, \mathbb{K})$. If X is a set of subspaces of Π, then we define $\widetilde{X} := \{\widetilde{\alpha} \mid \alpha \in X\}$. If every element of X is contained in $Q(2n, \mathbb{K})$, then we define $\overline{X} := \{\bar{\alpha} \mid \alpha \in X\}$.

Let $\mathcal{G}$ denote a set of generators of $Q(2n, \mathbb{K})$ such that $\overline{\mathcal{G}}$ is a locally singular hyperplane of $DQ(2n, \mathbb{K})$. From $\mathcal{G}$, a set $\mathcal{G}^+$ of maximal singular subspaces of Π can be derived in the following way. Let M denote a maximal singular subspace of Π.

- If $M \subseteq Q(2n, \mathbb{K})$, then $M \in \mathcal{G}^+$ if and only if $M \in \mathcal{G}$.
- If $M \nsubseteq Q(2n, \mathbb{K})$, then $M \in \mathcal{G}^+$ if and only if every generator of $Q(2n, \mathbb{K})$ through $Q(2n, \mathbb{K}) \cap M$ belongs to $\mathcal{G}$.

Theorem 2.1. *$\widetilde{\mathcal{G}^+}$ is a hyperplane of Δ without subquadrangular quads.*

Proof. Let L denote an arbitrary $(n-2)$-dimensional singular subspace of Π, then $\widetilde{L}$ is a line of Δ. We distinguish two cases.

(i) L is contained in $Q(2n, \mathbb{K})$.

If $\overline{L}$ is contained in the hyperplane $\overline{\mathcal{G}}$ of $DQ(2n, \mathbb{K})$, then every generator of $Q(2n, \mathbb{K})$ through L is contained in $\mathcal{G}$. Hence, also every maximal singular subspace of Π through L is contained in $\mathcal{G}^+$ and $\widetilde{L} \subseteq \widetilde{\mathcal{G}^+}$.

Suppose $\overline{L} \cap \overline{\mathcal{G}} = \{\overline{M}\}$, where M is a certain maximal subspace of $Q(2n, \mathbb{K})$ through L. Then M is the unique generator of $Q(2n, \mathbb{K})$ through L contained in $\mathcal{G}$. It follows that M is the unique maximal singular subspace of Π through L contained in $\mathcal{G}^+$. Hence, $|\widetilde{L} \cap \widetilde{\mathcal{G}^+}| = 1$.

(ii) L is not contained in $Q(2n,\mathbb{K})$.

Then $Q := L \cap Q(2n,\mathbb{K})$ is an $(n-3)$-dimensional subspace of $Q(2n,\mathbb{K})$ corresponding with a quad $\overline{Q}$ of $DQ(2n,\mathbb{K})$.

Suppose $\overline{Q}$ is a deep quad (with respect to $\overline{\mathcal{G}}$). Then every generator of $Q(2n,\mathbb{K})$ through Q belongs to $\mathcal{G}$. It follows that every maximal singular subspace of Π through L belongs to $\mathcal{G}^+$. Hence, $\widetilde{L} \subseteq \widetilde{\mathcal{G}^+}$.

Suppose $\overline{Q}$ is a singular quad (with respect to $\overline{\mathcal{G}}$) with deep point $\overline{M}$, where M is a generator of $Q(2n,\mathbb{K})$ through Q. If α is a maximal singular subspace of Π through L for which $\widetilde{\alpha} \in \mathcal{G}^+$, then every generator of $Q(2n,\mathbb{K})$ through $\alpha \cap Q(2n,\mathbb{K})$ belongs to $\mathcal{G}$. It follows that $\alpha \cap Q(2n,\mathbb{K})$ must be contained in M.

So, if β denotes the unique maximal singular subspace of Π through L meeting M in an $(n-2)$-dimensional subspace, then $\{\widetilde{\beta}\} = \widetilde{L} \cap \widetilde{\mathcal{G}^+}$.

This proves that $\widetilde{\mathcal{G}^+}$ is a hyperplane of Δ. We will now show that $\widetilde{\mathcal{G}^+}$ does not admit subquadrangular quads. Let Q denote an arbitrary $(n-3)$-dimensional singular subspace of Π, then $\widetilde{Q}$ is a quad of Δ. We distinguish two cases:

(i) $Q \subseteq Q(2n,\mathbb{K})$.

Then $\overline{Q}$ is a quad of $DQ(2n,\mathbb{K})$ which is either deep or singular with respect to $\overline{\mathcal{G}}$.

Suppose $\overline{Q}$ is deep with respect to $\overline{\mathcal{G}}$. Then every generator of $Q(2n,\mathbb{K})$ through Q belongs to $\overline{\mathcal{G}}$. It follows that every maximal singular subspace of Π through Q belongs to $\widetilde{\mathcal{G}^+}$. So, the quad $\widetilde{Q}$ of Δ is deep with respect to $\widetilde{\mathcal{G}^+}$.

Suppose $\overline{Q}$ is singular (with respect to $\overline{\mathcal{G}}$) with deep point $\overline{M}$, where M is a generator of $Q(2n,\mathbb{K})$ through Q. If α is a maximal singular subspace of Π through Q, then $\widetilde{\alpha}$ belongs to $\widetilde{\mathcal{G}^+}$ if and only if $\alpha \cap Q(2n,\mathbb{K})$ is contained in M. It follows that the quad $\widetilde{Q}$ of Δ is singular with deep point $\widetilde{M}$.

(ii) $Q \nsubseteq Q(2n,\mathbb{K})$.

Then $H := Q \cap Q(2n,\mathbb{K})$ is an $(n-4)$-dimensional subspace of $Q(2n,\mathbb{K})$ which determines a hex $\overline{H}$ of $DQ(2n,\mathbb{K})$.

If the quad $\widetilde{Q}$ of Δ does not contain two disjoint lines of $\widetilde{\mathcal{G}^+}$, then $\widetilde{Q}$ is either singular or ovoidal with respect to $\widetilde{\mathcal{G}^+}$.

Suppose now that the quad $\widetilde{Q}$ of Δ contains two disjoint lines $\widetilde{L_1}$ and $\widetilde{L_2}$ of $\widetilde{\mathcal{G}^+}$, where L_1 and L_2 are two $(n-2)$-dimensional singular subspaces of Π through Q. Put $Q_i := L_i \cap Q(2n,\mathbb{K})$, $i = 1,2$. Since $\widetilde{L_i} \subseteq \widetilde{\mathcal{G}^+}$, $\overline{Q_i}$ is a deep quad of $\overline{\mathcal{G}}$ (see above). So, $\overline{Q_1}$ and $\overline{Q_2}$ are two deep quads of the hex $\overline{H}$. If L is an $(n-2)$-dimensional subspace of $Q(2n,\mathbb{K})$ such that $\overline{L}$ is a common line of $\overline{Q_1}$ and $\overline{Q_2}$, then $\widetilde{\langle Q, L\rangle}$ is a common point of $\widetilde{L_1}$ and $\widetilde{L_2}$, a contradiction. Hence, $\overline{Q_1}$ and $\overline{Q_2}$ are two disjoint quads of $\overline{H}$ which are contained in $\overline{\mathcal{G}}$. Since $\overline{\mathcal{G}}$ is a locally singular hyperplane of $DQ(2n,\mathbb{K})$ containing two disjoint deep quads of $\overline{H}$, $\overline{H}$ must be contained in $\overline{\mathcal{G}}$. (Recall that every locally singular hyperplane of $DQ(6,\mathbb{K})$ is either singular or hexagonal and that none of these hyperplanes contains two disjoint quads.) It follows that the quad $\widetilde{Q}$ of Δ is deep with respect to $\widetilde{\mathcal{G}^+}$.

This proves the theorem. □

2.2. Second construction

Let Π be a nondegenerate thick polar space of rank $n-1$ which is embedded as a hyperplane in $Q(2n,\mathbb{K})$, $n \geqslant 3$. By Cohen and Shult [5, Theorem 5.12], Π arises as a section of $Q(2n,\mathbb{K})$ with a hyperplane of $\mathrm{PG}(2n,\mathbb{K})$ in which $Q(2n,\mathbb{K})$ is embedded. Let Δ and $DQ(2n,\mathbb{K})$ denote the dual polar spaces associated with Π and $Q(2n,\mathbb{K})$, respectively. For every subspace α of $Q(2n,\mathbb{K})$, let $\bar{\alpha}$ denote the corresponding convex subspace of $DQ(2n,\mathbb{K})$. If α is contained in Π, then we denote by $\widetilde{\alpha}$ the corresponding convex subspace of Δ. If X is a set of subspaces of $Q(2n,\mathbb{K})$, then we define $\overline{X} := \{\bar{\alpha} \mid \alpha \in X\}$. If every element of X is contained in Π, then we define $\widetilde{X} := \{\widetilde{\alpha} \mid \alpha \in X\}$.

Let $\mathcal{G}$ be a set of generators of $Q(2n,\mathbb{K})$ such that $\overline{\mathcal{G}}$ is a locally singular hyperplane of $DQ(2n,\mathbb{K})$. Let $\mathcal{G}'$ denote the set of all generators M of Π with the property that every generator of $Q(2n,\mathbb{K})$ through M belongs to $\mathcal{G}$.

Theorem 2.2. *$\widetilde{\mathcal{G}'}$ is a hyperplane of Δ without subquadrangular quads.*

Proof. Let α denote an $(n-3)$-dimensional subspace of Π, then $\widetilde{\alpha}$ is a line of Δ and $\bar{\alpha}$ is a quad of $DQ(2n,\mathbb{K})$. Let L_α denote the set of generators of Π through α, i.e., $\widetilde{L_\alpha}$ is the set of points of the line $\widetilde{\alpha}$. Then $\overline{L_\alpha}$ is a set of lines of the quad $\bar{\alpha}$ of $DQ(2n,\mathbb{K})$. Obviously, every point of $\bar{\alpha}$ is contained in a unique line of $\overline{L_\alpha}$, proving that $\overline{L_\alpha}$ is a spread of $\bar{\alpha}$. We distinguish now two cases:

- (i) The quad $\bar{\alpha}$ of $DQ(2n,\mathbb{K})$ is deep with respect to the hyperplane $\overline{\mathcal{G}}$. Then obviously, the line $\widetilde{\alpha}$ of Δ is contained in $\widetilde{\mathcal{G}'}$.
- (ii) The quad $\bar{\alpha}$ of $DQ(2n,\mathbb{K})$ is singular with respect to the hyperplane $\overline{\mathcal{G}}$. So, exactly one line of the spread $\overline{L_\alpha}$ of $\bar{\alpha}$ is contained in $\overline{\mathcal{G}}$. If β is the unique element of L_α such that $\bar{\beta} \subseteq \overline{\mathcal{G}}$, then $\widetilde{\beta}$ is the unique point of $\widetilde{\alpha}$ contained in $\widetilde{\mathcal{G}'}$.

This proves that $\widetilde{\mathcal{G}'}$ is a hyperplane of Δ. We will now prove that the hyperplane $\widetilde{\mathcal{G}'}$ does not admit subquadrangular quads.

Let α denote an $(n-4)$-dimensional subspace of Π, then $\widetilde{\alpha}$ is a quad of Δ and $\bar{\alpha}$ is a hex of $DQ(2n,\mathbb{K})$. There are three possibilities:

- (i) The hex $\bar{\alpha}$ of $DQ(2n,\mathbb{K})$ is contained in the hyperplane $\overline{\mathcal{G}}$ of $DQ(2n,\mathbb{K})$. Then the quad $\widetilde{\alpha}$ of Δ is deep with respect to $\widetilde{\mathcal{G}'}$.
- (ii) $\bar{\alpha} \cap \overline{\mathcal{G}}$ is a singular hyperplane of $\bar{\alpha}$. Let M be the generator of $Q(2n,\mathbb{K})$ such that $\overline{M}$ is the deep point of $\bar{\alpha}$ with respect to $\overline{\mathcal{G}}$. Put $N := M \cap \Pi$. If P is a generator of Π through α such that $\widetilde{P}$ and $\widetilde{N}$ have distance at most 1 in Δ, then $\dim(P \cap N) \geqslant n-3$ and hence also $\dim(P \cap M) \geqslant n-3$. So, every generator of $Q(2n,\mathbb{K})$ through P intersects M in a space of dimension at least $n-3$. It follows that $\widetilde{P} \in \widetilde{\mathcal{G}'}$. So, $\widetilde{\alpha}$ is either a deep quad or a singular quad with deep point $\widetilde{N}$. We will show that $\widetilde{\alpha}$ is a singular quad. Let P be a generator of Π through α such that $P \cap N = \alpha$. Then there exists precisely one generator P' of $Q(2n,\mathbb{K})$ through P meeting M in a subspace of dimension at least $n-3$. So, there exists precisely one generator P' of $Q(2n,\mathbb{K})$ through P for which $\overline{P'} \in \mathcal{G}$. This proves that $\widetilde{P} \notin \widetilde{\mathcal{G}'}$.
- (iii) $\bar{\alpha} \cap \overline{\mathcal{G}}$ is a hexagonal hyperplane of $\bar{\alpha}$. We will show that the quad $\widetilde{\alpha}$ of Δ is ovoidal with respect to $\widetilde{\mathcal{G}'}$. Let $\widetilde{L}$ denote an arbitrary line of $\widetilde{\alpha}$, where L is some $(n-3)$-dimensional subspace of Π through α. Since $\overline{L}$ is a quad of $DQ(2n,\mathbb{K})$ contained in $\bar{\alpha}$, $\overline{L}$ is singular with

respect to $\overline{\mathcal{G}}$. Let M be the generator of $Q(2n, \mathbb{K})$ such that $\overline{M}$ is the deepest point of $\overline{L}$ with respect to $\overline{\mathcal{G}}$. One readily verifies that $\widetilde{M \cap \Pi}$ is the unique point of $\widetilde{L}$ contained in $\widetilde{\mathcal{G}}'$. □

3. Proof of Theorem 1.2

In Section 2, we showed how a locally singular hyperplane of $DQ(2n, \mathbb{K})$ gives rise to a hyperplane of a dual polar space Δ whose associated polar space Π is either a hyperplane of $Q(2n, \mathbb{K})$ or contains $Q(2n, \mathbb{K})$ as a hyperplane. In this section, we will consider the case in which the polar space Π is associated with a nonsingular quadric in $\mathrm{PG}(2n + \epsilon, \mathbb{K})$, $\epsilon \in \{-1, 1\}$, described by a quadratic form of Witt-index $n' := n + \frac{\epsilon - 1}{2}$ which becomes a quadratic form of Witt-index $n' + 1$ when regarded over a quadratic Galois extension $\mathbb{K}'$ of $\mathbb{K}$. In this case, the dual polar space Δ admits a full embedding into $\mathrm{PG}(2^{n'} - 1, \mathbb{K}')$, the so-called spin-embedding of Δ.

3.1. First construction

Let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$, and let $n \in \mathbb{N} \setminus \{0, 1\}$. Since $\mathbb{K} \subseteq \mathbb{K}'$, every point of the projective space $\mathrm{PG}(2n + 1, \mathbb{K})$ can be regarded as a point of $\mathrm{PG}(2n + 1, \mathbb{K}')$. Every subspace α of $\mathrm{PG}(2n + 1, \mathbb{K})$ then generates a subspace α' of $\mathrm{PG}(2n+1, \mathbb{K}')$ with the same dimension as α. For all $i, j \in \{0, \ldots, 2n+1\}$ with $i \leqslant j$, let $a_{ij} \in \mathbb{K}$ such that $q(\overline{X}) = \sum_{0 \leqslant i \leqslant j \leqslant 2n+1} a_{ij} X_i X_j$ is a quadratic form defining a nonsingular quadric $Q^-(2n + 1, \mathbb{K})$ of Witt-index n in $\mathrm{PG}(2n + 1, \mathbb{K})$ which becomes a nonsingular quadric of Witt-index $n + 1$ in $\mathrm{PG}(2n + 1, \mathbb{K}')$ when regarded over the extension $\mathbb{K}'$ of $\mathbb{K}$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of generators of $Q^+(2n + 1, \mathbb{K}')$, and let $HS(2n + 1, \mathbb{K}')$ denote the half-spin geometry for $Q^+(2n + 1, \mathbb{K}')$ defined on the set $\mathcal{M}^+$. Let π be a hyperplane of $\mathrm{PG}(2n + 1, \mathbb{K})$ which intersects $Q^-(2n + 1, \mathbb{K})$ in a nonsingular quadric $Q(2n, \mathbb{K})$ of Witt-index n. Then π' intersects $Q^+(2n + 1, \mathbb{K}')$ in a nonsingular quadric $Q(2n, \mathbb{K}')$ of Witt-index n. Let $DQ^-(2n + 1, \mathbb{K})$, $DQ(2n, \mathbb{K})$ and $DQ(2n, \mathbb{K}')$ denote the dual polar spaces associated with the respective quadrics $Q^-(2n + 1, \mathbb{K})$, $Q(2n, \mathbb{K})$ and $Q(2n, \mathbb{K}')$. For every generator α of $Q(2n, \mathbb{K})$, we define $f(\alpha) := \alpha'$. Then f defines an isometric embedding of the dual polar space $DQ(2n, \mathbb{K})$ into the dual polar space $DQ(2n, \mathbb{K}')$.

Consider a set of generators of $Q(2n, \mathbb{K})$ which defines a locally singular hyperplane H_1 of $DQ(2n, \mathbb{K})$. By De Bruyn [10, Theorem 1.1], there exists a unique locally singular hyperplane H_2 of $DQ(2n, \mathbb{K}')$ such that $f(H_1) = f(P) \cap H_2$, where P denotes the point-set of $DQ(2n, \mathbb{K})$. Now, let H_3 denote the set of all elements α of $\mathcal{M}^+$ for which $\alpha \cap \pi' \in H_2$. By De Bruyn [8, Theorem 1.2], H_3 is a hyperplane of $HS(2n + 1, \mathbb{K}')$. Let e denote the spin-embedding of $HS(2n + 1, \mathbb{K}')$ into $\Sigma := \mathrm{PG}(2^n - 1, \mathbb{K}')$. By the main result of Shult [15] (see also Corollary 1.3 of [8] for an alternative proof) H_3 arises from a hyperplane η of Σ.

Now, by Theorem 2.1, the locally singular hyperplane H_1 of $DQ(2n, \mathbb{K})$ gives rise to a hyperplane H of $DQ^-(2n + 1, \mathbb{K})$. For every generator α of $Q^-(2n + 1, \mathbb{K})$, let $\phi(\alpha)$ denote the unique generator of $\mathcal{M}^+$ through α'. Then $e \circ \phi$ defines the spin-embedding of $DQ^-(2n + 1, \mathbb{K})$ into Σ.

Lemma 3.1. *If $\alpha \in H$, then $\phi(\alpha) \in H_3$.*

Proof. We distinguish two cases:

- $\alpha \subseteq Q(2n, \mathbb{K})$. Then $\phi(\alpha)$ is the unique generator of $\mathcal{M}^+$ through $\alpha' = f(\alpha)$. Since $\alpha \in H_1$, $f(\alpha) \in H_2$ and $\phi(\alpha) \in H_3$.
- α is not contained in $Q(2n, \mathbb{K})$. Put $\beta = \alpha \cap \pi$. Then β' defines a line of $DQ(2n, \mathbb{K}')$. Since every generator of $Q(2n, \mathbb{K})$ through β is contained in H_1, every generator of $Q(2n, \mathbb{K}')$ through β' is contained in H_2. Hence, every element of $\mathcal{M}^+$ through β is contained in H_3. In particular, $\phi(\alpha) \in H_3$. □

Corollary 3.2. *The hyperplane H of $DQ^-(2n+1, \mathbb{K})$ arises from the hyperplane η of Σ.*

Proof. By Lemma 3.1, $e \circ \phi(H) \subseteq \eta$. Since H is a maximal subspace of $DQ^-(2n+1, \mathbb{K})$, H must arise from the hyperplane η of Σ. □

3.2. Second construction

Let $n \geqslant 3$ and let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$. Let θ be the nontrivial element in $Gal(\mathbb{K}'/\mathbb{K})$. Let $Q(2n, \mathbb{K})$ be a nonsingular quadric of Witt-index n in $\mathrm{PG}(2n, \mathbb{K})$, let π be a hyperplane of $\mathrm{PG}(2n, \mathbb{K})$ intersecting $Q(2n, \mathbb{K})$ in a nonsingular quadric $Q^-(2n-1, \mathbb{K})$ of Witt-index $n-1$ which becomes a nonsingular hyperbolic quadric $Q^+(2n-1, \mathbb{K}')$ when regarded as quadric over $\mathbb{K}'$.

Lemma 3.3. *There exists a reference system in* $\mathrm{PG}(2n, \mathbb{K})$ *and a* $\delta \in \mathbb{K}' \setminus \mathbb{K}$ *such that* $Q(2n, \mathbb{K})$ *regarded as quadric has equation*

$$X_1X_3 + X_2^2 + \sum_{i=2}^{n} X_{2i}X_{2i+1} = 0$$

and $Q^-(2n-1, \mathbb{K})$ *has equation*

$$\begin{aligned}&\left(\delta + \delta^\theta\right)X_2 + \delta\delta^\theta X_3 - X_1 = 0,\\ &\left(X_2^2 + \left(\delta + \delta^\theta\right)X_2X_3 + \delta\delta^\theta X_3^2\right) + \sum_{i=2}^{n} X_{2i}X_{2i+1} = 0.\end{aligned}$$

Proof. Let α_1 and α_2 be two disjoint generators of $Q^-(2n-1, \mathbb{K})$. Let β_i, $i \in \{1, 2\}$, be a generator of $Q(2n, \mathbb{K})$ through α_i such that β_1 and β_2 are disjoint. The space $\langle \alpha_1, \alpha_2 \rangle$ intersects $Q(2n, \mathbb{K})$ in a nonsingular hyperbolic quadric $Q^+(2n-3, \mathbb{K})$ and $\langle \beta_1, \beta_2 \rangle$ intersects $Q(2n, \mathbb{K})$ in a nonsingular hyperbolic quadric $Q^+(2n-1, \mathbb{K})$. Let T_i, $i \in \{1, 2\}$, denote the tangent space of $Q(2n, \mathbb{K})$ at the subspace α_i. Then $\dim(T_1) = \dim(T_2) = n+1$ and hence $\dim(T_1 \cap T_2) \geqslant 2$. Since $(T_1 \cap T_2) \cap \alpha_1 = \emptyset$, $\dim(T_1 \cap T_2) = 2$, i.e., $T_1 \cap T_2$ is a plane. The unique generator of $Q^+(2n-1, \mathbb{K})$ through α_2 different from β_2 intersects β_1 in a point and this point belongs to $T_1 \cap T_2$. Similarly, the unique generator of $Q^+(2n-1, \mathbb{K})$ through α_1 different from β_1 intersects β_2 in a point of $T_1 \cap T_2$. Hence, $|T_1 \cap T_2 \cap Q(2n, \mathbb{K})| \geqslant 2$. Notice also that if $T_1 \cap T_2 \cap Q(2n, \mathbb{K})$ contains a line L, then $\langle L, \alpha_1 \rangle \subseteq Q(2n, \mathbb{K})$, in contradiction with $\dim(\langle L, \alpha_1 \rangle) = n$. Hence, $T_1 \cap T_2 \cap Q(2n, \mathbb{K})$ is a nonsingular quadric of $T_1 \cap T_2$ of Witt-index 1. Now, we can choose our reference system in such a way that: (i) $Q^+(2n-3, \mathbb{K})$ has equation $X_1' = X_2' = X_3' = 0$ and $\sum_{i=2}^{n} X_{2i}'X_{2i+1}' = 0$; (ii) $T_1 \cap T_2 \cap Q(2n, \mathbb{K})$ has equation $X_2'^2 + X_1'X_3' = 0$, $X_4' = \cdots = X_{2n+1}' = 0$. It readily follows that $Q(2n, \mathbb{K})$ has equation $X_2'^2 + X_1'X_3' + \sum_{i=2}^{n}(k \cdot X_{2i}'X_{2i+1}') = 0$

for some $k \in \mathbb{K} \setminus \{0\}$, i.e. equation $X_2^2 + X_1X_3 + \sum_{i=2}^n X_{2i}X_{2i+1} = 0$ after a rescaling of the variables. The hyperplane π has equation $\delta_1 X_1 + \delta_2 X_2 + \delta_3 X_3 = 0$ for some $\delta_1, \delta_2, \delta_3 \in \mathbb{K}$. Since $\pi \cap Q(2n, \mathbb{K}) = Q^-(2n-1, \mathbb{K})$, $\delta_1 \neq 0$. Without loss of generality, we may suppose that $\delta_1 = 1$. Then $Q^-(2n-1, \mathbb{K})$ has equation

$$\left(X_2^2 - \delta_2 X_2 X_3 - \delta_3 X_3^2\right) + \sum_{i=2}^n X_{2i}X_{2i+1} = 0,$$

i.e.

$$(X_2 + \delta X_3)\left(X_2 + \delta^\theta X_3\right) + \sum_{i=2}^n X_{2i}X_{2i+1} = 0$$

for a certain $\delta \in \mathbb{K}' \setminus \mathbb{K}$. □

Now, embed $\mathrm{PG}(2n, \mathbb{K})$ as a hyperplane in $\mathrm{PG}(2n+1, \mathbb{K})$. We denote a point of $\mathrm{PG}(2n+1, \mathbb{K})$ by $(X_0, X_1, \ldots, X_{2n}, X_{2n+1})$ and we suppose that $\mathrm{PG}(2n, \mathbb{K})$ has equation $X_0 = 0$. Now, consider the following quadric $Q^-(2n+1, \mathbb{K})$ in $\mathrm{PG}(2n+1, \mathbb{K})$:

$$(X_2 - \delta X_0)\left(X_2 - \delta^\theta X_0\right) + X_1X_3 + X_4X_5 + \cdots + X_{2n}X_{2n+1} = 0.$$

Notice that $Q(2n, \mathbb{K}) = Q^-(2n+1, \mathbb{K}) \cap \mathrm{PG}(2n, \mathbb{K})$. Let p be the point $(1, -\delta\delta^\theta, 0, 1, 0, \ldots, 0)$ of $Q^-(2n+1, \mathbb{K})$. One readily verifies that the tangent hyperplane at the point p intersects $\mathrm{PG}(2n, \mathbb{K})$ in the subspace π. Now, let $DQ^-(2n+1, \mathbb{K})$, $DQ(2n, \mathbb{K})$ and $DQ^-(2n-1, \mathbb{K})$ denote the dual polar spaces associated with $Q^-(2n+1, \mathbb{K})$, $Q(2n, \mathbb{K})$ and $Q^-(2n+1, \mathbb{K})$.

Let H be a locally singular hyperplane of $DQ(2n, \mathbb{K})$. Then H induces a hyperplane H_1 of $DQ^-(2n+1, \mathbb{K})$, see Theorem 2.1, and a hyperplane H_2 of $DQ^-(2n-1, \mathbb{K})$, see Theorem 2.2. Now, let M_p denote the max of $DQ^-(2n+1, \mathbb{K})$ consisting of all generators of $Q^-(2n+1, \mathbb{K})$ through p. If α is a generator through p, then α intersects π in a generator $\phi(\alpha)$ of $Q^-(2n-1, \mathbb{K})$. The map ϕ defines an isomorphism between M_p and $DQ^-(2n-1, \mathbb{K})$. Clearly, a generator α through p belongs to H_1 if and only if $\phi(\alpha)$ belongs to H_2.

Hence, the hyperplanes $H_1 \cap M_p$ of M_p and H_2 of $DQ^-(2n-1, \mathbb{K})$ are isomorphic. Now, the spin-embedding of $DQ^-(2n+1, \mathbb{K})$ induces an embedding of M_p which is isomorphic to the spin-embedding of $M_p \cong DQ^-(2n-1, \mathbb{K})$, see De Bruyn [11, Theorem 1.1]. Since the hyperplane H_1 arises from the spin-embedding of $DQ^-(2n+1, \mathbb{K})$, the hyperplane $H_1 \cap M_p$ of M_p arises from the spin-embedding of M_p. It follows that the hyperplane H_2 of $DQ^-(2n-1, \mathbb{K})$ arises from the spin-embedding of $DQ^-(2n-1, \mathbb{K})$.

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