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A duality between q-multiplicities in tensor products and q-multiplicities of weights for the root systems B, C or D

Cédric Lecouvey

Department of Mathematics, University of Caen, 14032 Caen Cedex, France

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Abstract

Starting from Jacobi–Trudi type determinantal expressions for the Schur functions of types B, C and D, we define a natural q-analogue of the multiplicity $[V(\lambda) : M(\mu)]$ when $M(\mu)$ is a tensor product of row or column shaped modules defined by μ. We prove that these q-multiplicities are equal to certain Kostka–Foulkes polynomials related to the root systems C or D. Finally we express the corresponding multiplicities in terms of Kostka numbers

Keywords: Representation theory; Characters; q-multiplicities; Lusztig q-analogues

1. Introduction

Given two partitions λ and μ of length n, the Kostka number $K_{\lambda,\mu}^{A_{n-1}}$ is equal to the dimension of the weight space μ in the finite dimensional irreducible sl_{n+1}-module $V(\lambda)$ of highest weight λ. The Schur duality is a classical result in representation theory establishing that $K_{\lambda,\mu}^{A_{n-1}}$ is also equal to the multiplicities of $V(\lambda)$ and $V(\lambda')$ respectively in the tensor products

$$V(\mu_1\Lambda_1)\otimes\cdots\otimes V(\mu_n\Lambda_1) \quad \text{and} \quad V(\Lambda_{\mu_1})\otimes\cdots\otimes V(\Lambda_{\mu_n}),$$

E-mail address: lecouvey@math.unicaen.fr.

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where λ' is the conjugate partition of λ and the Λ_i's, $i = 1, \dots, n-1$ are the fundamental weights of sl_{n+1}. Another way to define $K_{\lambda,\mu}^{A_{n-1}}$ is to use the Jacobi–Trudi identity which gives a determinantal expression of the Schur function $s_\mu = \mathrm{char}(V(\mu))$ in terms of the characters $h_k = \mathrm{char}(V(k\Lambda_1))$ of the kth symmetric power representation. This formula can be rewritten

$$s_\mu = \prod_{1 \leqslant i < j \leqslant n} (1 - R_{i,j}) h_\mu \tag{1}$$

where $h_\mu = h_{\mu_1} \cdots h_{\mu_n}$ and the $R_{i,j}$ are the commuting raising operators acting on $\mathbb{Z}^n$ (see 3.2). Then one can prove that it makes sense to write

$$h_\mu = \prod_{1 \leqslant i < j \leqslant n} (1 - R_{i,j})^{-1} s_\mu \tag{2}$$

which gives the decomposition of h_μ on the basis of Schur functions. From this decomposition we derive the following expression for $K_{\lambda,\mu}^{A_{n-1}}$:

$$K_{\lambda,\mu}^{A_{n-1}} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}^{A_{n-1}}(\sigma(\lambda + \rho) - (\mu + \rho)) \tag{3}$$

where $\mathcal{S}_n$ is the symmetric group of order n and $\mathcal{P}^{A_n}$ the ordinary Kostant partition function defined from the equality:

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1 - x^\alpha)} = \sum_\beta \mathcal{P}^{A_{n-1}}(\beta) x^\beta$$

with β running over the set of nonnegative integral combinations of positive roots of sl_n.

There exists a q-analogue $K_{\lambda,\mu}^{A_{n-1}}(q)$ of $K_{\lambda,\mu}^{A_{n-1}}$ obtained by replacing the ordinary Kostant partition function $\mathcal{P}^{A_{n-1}}$ by its q-analogue $\mathcal{P}_q^{A_{n-1}}$ satisfying

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1 - q x^\alpha)} = \sum_\beta \mathcal{P}_q^{A_{n-1}}(\beta) x^\beta.$$

So we have

$$K_{\lambda,\mu}^{A_{n-1}}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho) - (\mu + \rho)) \tag{4}$$

which is a polynomial in q with nonnegative integer coefficients [9,11]. In [13], Nakayashiki and Yamada have shown that $K_{\lambda,\mu}^{A_{n-1}}(q)$ can also be computed from the combinatorial R matrix corresponding to Kashiwara's crystals associated to some $U_q(\widehat{sl_n})$-modules.

For $g = so_{2n+1}, sp_{2n}$ or so_{2n} there also exist expressions similar to (3) for the multiplicities $K_{\lambda,\mu}^g$ of the weight μ in the finite dimensional irreducible module $V(\lambda)$ but such a simple duality as for sl_n does not exist although it is possible to obtain certain duality results between multiplicities of weights and tensor product multiplicities of representations by using duals pairs of algebraic groups [7]. This implies that the quantizations of weight multiplicities and tensor product multiplicities cannot coincide for the root systems B_n, C_n and D_n. The Kostka–Foulkes polynomials $K_{\lambda,\mu}^{g}(q)$ are the q-analogues of $K_{\lambda,\mu}^{g}$ defined as in (4) by quantizing the partition function corresponding to the root system associated to g (see 2.2). In [4], Hatayama et al. have introduced for type C_n a quantization $X_{\lambda,\mu}^{C_n}(q)$ of the multiplicity of $V(\lambda)$ in the tensor product

$$W(\mu_1\Lambda_1)\otimes\cdots\otimes W(\mu_n\Lambda_1)$$

where for any $i=1,\ldots,n$,

$$W(\mu_i\Lambda_1)=V(\mu_i\Lambda_1)\oplus V((\mu_i-2)\Lambda_1)\oplus\cdots\oplus V((\mu_i \bmod 2)\Lambda_1).$$

This quantization is based on the determination of the combinatorial R matrix of some $U_q'(\widehat{g})$-crystals in the spirit of [13]. Note that there also exist q-multiplicities for the sp_2-module $V(\lambda)$ in a tensor product

$$V(\Lambda_1)^{\otimes k}\otimes V(\Lambda_2)^{\otimes l}$$

where k, l are positive integers obtained by Yamada [17].

In this paper we first use Jacobi–Trudi type determinantal expressions for the Schur functions associated to g to introduce q-analogues of the multiplicity of $V(\lambda)$ in the tensor products

(i) $\mathfrak{h}(\mu)=V(\mu_1\Lambda_1)\otimes\cdots\otimes V(\mu_n\Lambda_1)$, $\mathfrak{H}(\mu)=W(\mu_1\Lambda_1)\otimes\cdots\otimes W(\mu_n\Lambda_1)$
(ii) $\mathfrak{e}(\mu)=V(\Lambda_{\mu_1'})\otimes\cdots\otimes V(\Lambda_{\mu_m'})$, $\mathfrak{E}(\mu)=W(\Lambda_{\mu_1'})\otimes\cdots\otimes W(\Lambda_{\mu_m'})$ with $n\geqslant|\mu|$

where

$$W(\Lambda_k)=V(\Lambda_k)\oplus V(\Lambda_{k-2})\oplus\cdots\oplus V(\Lambda_{k \bmod 2}).$$

With the condition $n\geqslant|\mu|$ for (ii), these multiplicities are independent of the type B_n, C_n or D_n of the Lie algebra considered. When $q=1$, we recover a remarkable property already used by Koike and Terada in [8]. Next we prove that these q-multiplicities are in fact equal to Kostka–Foulkes polynomials associated to the root systems of types C and D. It is possible to extend the definition (4) of the Kostka–Foulkes polynomials associated to the root system A_{n-1} by replacing μ by $\gamma\in\mathbb{N}^n$ where γ is not a partition. In this case $K_{\lambda,\gamma}^{A_{n-1}}(q)$ may have negative coefficients but $K_{\lambda,\gamma}^{A_{n-1}}(1)$ is equal to the dimension of the weight space γ in $V(\lambda)$ that is

$$K_{\lambda,\gamma}^{A_{n-1}}(1)=\begin{cases}K_{\lambda,\mu}^{A_{n-1}} & \text{if there exists a partition } \mu \text{ and } \sigma\in\mathcal{S}_n \text{ such that } \sigma(\mu)=\gamma,\\ 0 & \text{otherwise.}\end{cases}$$

Now if we extend (4) by replacing λ by $\xi\in\mathbb{N}^n$, the polynomial $K_{\xi,\mu}^{A_{n-1}}(q)$ is equal up to a sign to a Kostka–Foulkes polynomial $K_{\nu,\mu}^{A_{n-1}}(q)$ where ν is a partition. We obtained two expressions of the q-multiplicities defined above respectively in terms of the polynomials $K_{\lambda,\gamma}^{A_{n-1}}(q)$ and $K_{\xi,\mu}^{A_{n-1}}(q)$. By specializing at $q=1$, this yields expressions of the corresponding multiplicities in terms of Kostka numbers.

In Section 1 we recall the background on the root systems B_n, C_n and D_n and the corresponding Kostka–Foulkes polynomials. We review in Section 2 the determinantal identities for Schur functions that we need in the sequel and we introduce the formalism suggested in [1] to prove the expressions of Schur functions in terms of raising and lowering operators implicitly contained in [15]. Thanks to this formalism, we are able to obtain expressions for multiplicities similar to (3). We quantize these multiplicities to obtain the desired q-analogues in Section 3. We prove in Section 4 two duality theorems between our q-analogues and certain Kostka–Foulkes polynomials of types C and D. Finally we establish formulas expressing the associated multiplicities in terms of Kostka numbers.

Notation. In the sequel we frequently define similar objects for the root systems B_n, C_n and D_n. When they are related to type B_n (resp. C_n, D_n), we implicitly attach to them the label B (resp. the labels C, D). To avoid cumbersome repetitions, we sometimes omit the labels B, C and D when our definitions or statements are identical for the three root systems.

Note: While writing down this work, I have been informed that Shimozono and Zabrocki [16] have introduced independently and by using creating operators essentially the same tensor power multiplicities. Thanks to this formalism they recover in particular Jacobi–Trudi type determinantal expressions of the Schur functions associated to the root systems B, C and D which constitute the starting point of this article.

2. Background on the root systems B_n, C_n and D_n

2.1. Convention for the positive roots

Consider an integer $n \geqslant 1$. The weight lattice for the root system C_n (resp. B_n and D_n) can be identified with $P_{C_n} = \mathbb{Z}^n$ $\left(\text{resp. } P_{B_n} = P_{D_n} \left(\frac{\mathbb{Z}}{2}\right)^n\right)$ equipped with the orthonormal basis ε_i, $i = 1, \ldots, n$. We take for the simple roots

$$\begin{cases} \alpha_n^{B_n} = \varepsilon_n \text{ and } \alpha_i^{B_n} = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \ldots, n-1 \text{ for the root system } B_n, \\ \alpha_n^{C_n} = 2\varepsilon_n \text{ and } \alpha_i^{C_n} = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \ldots, n-1 \text{ for the root system } C_n, \\ \alpha_n^{D_n} = \varepsilon_n + \varepsilon_{n-1} \text{ and } \alpha_i^{D_n} = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \ldots, n-1 \text{ for the root system } D_n. \end{cases} \tag{5}$$

Then the sets of positive roots are

$$\begin{cases} R_{B_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leqslant i < j \leqslant n\} \cup \{\varepsilon_i \text{ with } 1 \leqslant i \leqslant n\} \quad \text{for the root system } B_n, \\ R_{C_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leqslant i < j \leqslant n\} \cup \{2\varepsilon_i \text{ with } 1 \leqslant i \leqslant n\} \quad \text{for the root system } C_n, \\ R_{D_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leqslant i < j \leqslant n\} \quad \text{for the root system } D_n. \end{cases}$$

Denote respectively by $P_{B_n}^+$, $P_{C_n}^+$ and $P_{D_n}^+$ the sets of dominant weights of so_{2n+1}, sp_{2n} and so_{2n}. Let θ be the involution in $\mathbb{Z}^n$ such that $\theta(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, -x_n)$. Then $R_{D_n}^+$ and $P_{D_n,}^+$ are stable under the action of θ.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with n parts. We will identify in the classical way λ with the dominant weight $\sum_{i=1}^n \lambda_i \varepsilon_i$. Note that there exists dominant weights associated to the orthogonal root systems whose coordinates on the basis ε_i, $i = 1, \ldots, n$ are not positive integers (hence which cannot be regarded as a partition). For each root system of type B_n, C_n or D_n, the set of weights having nonnegative integer coordinates on the basis $\varepsilon_1, \dots, \varepsilon_n$ can be identified with the set π_n^+ of partitions of length n. For any partition λ, the weights of the finite dimensional so_{2n+1}, sp_{2n} or so_{2n}-module of highest weight λ are all in $\pi_n = \mathbb{Z}^n$. For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \pi_n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$.

The conjugate partition of the partition λ is denoted λ' as usual. Consider λ, μ two partitions of length n and set $m = \max(\lambda_1, \mu_1)$. Then by adding to λ' and μ' the required numbers of parts 0 we will consider them as partitions of length m.

The Weyl group $W_{B_n} = W_{C_n}$ of so_{2n+1} and sp_{2n} is identified to the subgroup of the permutation group of the set $\{\overline{n}, \dots, \overline{2}, \overline{1}, 1, 2, \dots, n\}$ generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, \dots, n-1$ and $s_n = (n, \overline{n})$ where for $a \neq b$ (a, b) is the simple transposition which switches a and b. We denote by l_B the length function corresponding to the set of generators s_i, $i = 1, \dots, n$.

The Weyl group W_{D_n} of so_{2n} is identified to the subgroup of W_{B_n} generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, \dots, n-1$ and $s_n' = (n, \overline{n-1})(n-1, \overline{n})$. We denote by l_D the length function corresponding to the set of generators s_n' and s_i, $i = 1, \dots, n-1$.

Note that $W_{D_n} \subset W_{B_n}$ and any $w \in W_{B_n}$ verifies $w(\overline{i}) = \overline{w(i)}$ for $i \in \{1, \dots, n\}$. The action of w on $\beta = (\beta_1, \dots, \beta_n) \in P_n$ is given by

$$w \cdot (\beta_1, \dots, \beta_n) = (\beta_1^w, \dots, \beta_n^w),$$

where $\beta_i^w = \beta_{w(i)}$ if $\sigma(i) \in \{1, \dots, n\}$ and $\beta_i^w = -\beta_{w(\overline{i})}$ otherwise.

The half sums ρ_{B_n}, ρ_{C_n} and ρ_{D_n} of the positive roots associated to each root system B_n, C_n and D_n verify:

$$\rho_{B_n} = \left(n - \tfrac{1}{2}, n - \tfrac{3}{2}, \dots, \tfrac{1}{2}\right), \quad \rho_{C_n} = (n, n-1, \dots, 1) \quad \text{and} \quad \rho_{B_n} = (n-1, n-2, \dots, 0).$$

In the sequel we identify the symmetric group $\mathcal{S}_n$ with the subgroup of W_{B_n} or W_{D_n} generated by the s_i's, $i = 1, \dots, n-1$.

2.2. Schur functions and Kostka–Foulkes polynomials

We now briefly review the notions of Schur functions and Kostka–Foulkes polynomials associated to the roots systems B_n, C_n and D_n and refer the reader to [14] for more details. For any weight $\beta = (\beta_1, \dots, \beta_n) \in \pi_n$ we set $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ where $x_1, \dots, x_n$ are fixed indeterminates. We set

$$a_\beta^{B_n} = \sum_{w \in W_{B_n}} (-1)^{l(\sigma)} (w \cdot x^\beta)$$

where $w \cdot x^\mu = x^{w(\mu)}$. The Schur function $s_\beta^{B_n}$ is defined as in [14] by

$$s_\beta^{B_n} = \frac{a_{\beta+\rho_{B_n}}^{B_n}}{a_{\rho_{B_n}}^{B}}.$$

When $\nu \in \pi_n^+$, $s_\nu^{B_n}$ is the Weyl character of $V(\nu)$ the finite dimensional irreducible so_{2n+1}-module with highest weight ν. For any $w \in W_{B_n}$, the dot action of w on $\beta \in \pi_n$ is defined by

$$w \circ \beta = w \cdot (\beta + \rho_{B_n}) - \rho_{B_n}.$$

We have the following straightening law for the Schur functions. For any $\beta \in \pi_n$, $s_\beta^{B_n} = 0$ or there exists a unique $\nu \in \pi_n^+$ such that $s_\beta^{B_n} = (-1)^{l(w)} s_\nu^{B_n}$ with $w \in W_{B_n}$ and $\nu = w \circ \beta$. Set $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$ and write $\mathbb{K}[\pi_n]$ for the $\mathbb{K}$-module generated by the x^β, $\beta \in \pi_n$. Set $\mathcal{C}_{B_n} = \mathbb{K}[\pi_n]^{W_{B_n}} = \{f \in \mathbb{K}[\pi_n],\ w \cdot f = f \text{ for any } w \in W_{B_n}\}$. Then $\{s_\nu^{B_n}\}$, $\nu \in \pi_n^+$ is a basis of $\mathbb{K}[\pi_n]^{W_{B_n}}$.

We define $s_\beta^{C_n}$ and $s_\beta^{D_n}$ belonging to $\mathcal{C}_{C_n} = \mathcal{C}_{B_n}$ and $\mathcal{C}_{D_n}$ in the same way and we obtain similarly that $\{s_\nu^{C_n}, \nu \in \pi_n^+\}$ and $\{s_\nu^{D_n}, \nu \in \pi_n^+ \cup \theta(\pi_n^+)\}$ are respectively bases of $\mathcal{C}_{C_n}$ and $\mathcal{C}_{D_n}$.

The q-analogue $\mathcal{P}_q^{B_n}$ of Kostant partition function corresponding to the root system B_n is defined by the equality

$$\prod_{\alpha \in R_{B_n}^+} \frac{1}{1 - q x^\alpha} = \sum_{\beta \in \pi_n} \mathcal{P}_q^{B_n}(\beta) x^\beta.$$

Note that $\mathcal{P}_q^{B_n}(\beta) = 0$ if β is not a linear combination of positive roots of $R_{B_n}^+$ with nonnegative coefficients. We write similarly $\mathcal{P}_q^{C_n}$ and $\mathcal{P}_q^{D_n}$ for the q-partition functions associated respectively to the root systems C_n and D_n. Given λ and μ two partitions of length n, the Kostka–Foulkes polynomials of types B_n, C_n and D_n are then respectively defined by

$$K_{\lambda,\mu}(q) = \sum_{\sigma \in W} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda + \rho) - (\mu + \rho)).$$

Remarks.

(i) We have $K_{\lambda,\mu}(q) = 0$ when $|\lambda| < |\mu|$.

(ii) When $|\lambda| = |\mu|$, $K_{\lambda,\mu}^{B_n}(q) = K_{\lambda,\mu}^{C_n}(q) = K_{\lambda,\mu}^{D_n}(q) = K_{\lambda,\mu}^{A_{n-1}}(q)$ that is, the Kostka–Foulkes polynomials associated to the root systems B_n, C_n and D_n are Kostka–Foulkes polynomials associated to the root system A_{n-1} (see [15]).

3. Determinantal identities and multiplicities of representations

3.1. Determinantal identities for Schur functions

Consider $k \in \mathbb{Z}$. When k is a nonnegative integer, write $(k)_n = (k, 0, \dots, 0)$ for the partition of length n with a unique non-zero part equal to k. Then set

$$h_k^{B_n} = s_{(k)_n}^{B_n}, \quad h_k^{C_n} = s_{(k)_n}^{C_n}, \quad h_k^{D_n} = s_{(k)_n}^{D_n}$$

and

$$H_k^{B_n} = h_k^{B_n} + h_{k-2}^{B_n} + \cdots + h_{k \bmod 2}^{B_n}, \; H_k^{C_n} = h_k^{C_n} + h_{k-2}^{C_n} + \cdots + h_{k \bmod 2}^{B_n},$$

$$H_k^{D_n} = h_k^{D_n} + h_{k-2}^{D_n} + \cdots + h_{k \bmod 2}^{D_n}.$$

When k is a negative integer we set $h_k^{B_n} = h_k^{C_n} = h_k^{D_n} = 0$ and $H_k^{B_n} = H_k^{C_n} = H_k^{D_n} = 0$.

For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ define

$$u_\alpha^{B_n} = \det \begin{pmatrix} h_{\alpha_1}^{B_n} & h_{\alpha_1+1}^{B_n} + h_{\alpha_1-1}^{B_n} & \cdots & h_{\alpha_1+n-1}^{B_n} + h_{\alpha_1-n+1}^{B_n} \\ h_{\alpha_2-1}^{B_n} & h_{\alpha_2}^{B_n} + h_{\alpha_2-2}^{B_n} & \cdots & h_{\alpha_2+n-2}^{B_n} + h_{\alpha_2-n}^{B_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ h_{\alpha_n-n+1}^{B_n} & h_{\alpha_n-n+2}^{B_n} + h_{\alpha_n-n}^{B_n} & \cdots & h_{\alpha_n}^{B_n} + h_{\alpha_n-2n+2}^{B_n} \end{pmatrix}. \tag{6}$$

By using the equalities $h_k^{B_n} = H_k^{B_n} - H_{k-2}^{B_n}$ and simple computations on determinants we have also

$$u_\alpha^{B_n} = \det \begin{pmatrix} H_{\alpha_1}^{B_n} - H_{\alpha_1-2}^{B_n} & H_{\alpha_1+1}^{B_n} - H_{\alpha_1-3}^{B_n} & \cdots & H_{\alpha_1+n-1}^{B_n} - H_{\alpha_1-n-1}^{B_n} \\ H_{\alpha_2-1}^{B_n} - H_{\alpha_2-3}^{B_n} & H_{\alpha_2}^{B_n} - H_{\alpha_2-4}^{B_n} & \cdots & H_{\alpha_2+n-2}^{B_n} - H_{\alpha_2-n-2}^{B_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ H_{\alpha_n-n+1}^{B_n} - H_{\alpha_n-n-1}^{B_n} & H_{\alpha_n-n+2}^{B_n} - H_{\alpha_n-n-2}^{B_n} & \cdots & H_{\alpha_n}^{B_n} - H_{\alpha_n-2n-2}^{B_n} \end{pmatrix}. \tag{7}$$

We define $u_\alpha^{C_n}$ and $u_\alpha^{D_n}$ similarly by replacing $h_k^{B_n}$ respectively by $h_k^{C_n}$ and $h_k^{D_n}$.

Consider p and n two integers such that $n \geqslant 1$. When p is nonnegative and $n \geqslant p$, write $(1^p)_n = (1, \ldots, 1, 0, \ldots, 0)$ for the partition of length n having p non-zero parts equal to 1. Accordingly to Propositions 1.2.3, 1.2.4 and 1.2.5 of [8], we set

$$\begin{cases} e_p^{B_n} = s_{(1^p)_n}^{B_n}, e_p^{C_n} = s_{(1^p)_n}^{C_n} \quad \text{if } 0 \leqslant p \leqslant n, \; e_p^{D_n} = s_{(1^p)_n}^{D_n} \quad \text{if } 0 \leqslant p \leqslant n-1 \text{ and } e_n^{D_n} = s_{(1^n)_n}^{D_n} + s_{\theta(1^n)_n}^{D_n}, \\ e_p^{B_n} = e_{2n+1-p}^{B_n} \quad \text{if } n+1 \leqslant p \leqslant 2n+1, \; e_p^{C_n} = -e_{2n+2-p}^{C_n}, e_p^{D_n} = e_{2n-p}^{D_n} \quad \text{if } n+1 \leqslant p \leqslant 2n, \\ e_p^{B_n} = e_p^{C_n} = e_p^{D_n} = 0 \quad \text{otherwise} \end{cases}$$

and

$$E_k^{B_n} = e_k^{B_n} + e_{k-2}^{B_n} + \cdots + e_{k \bmod 2}^{B_n}, \quad E_k^{C_n} = e_k^{C_n} + e_{k-2}^{C_n} + \cdots + e_{k \bmod 2}^{B_n},$$

$$E_k^{D_n} = e_k^{D_n} + e_{k-2}^{D_n} + \cdots + e_{k \bmod 2}^{D_n}.$$

For any $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^n$ define

$$v_\beta^{B_n} = \det \begin{pmatrix} e_{\beta_1}^{B_n} & e_{\beta_1+1}^{B_n} + e_{\beta_1-1}^{B_n} & \cdots & e_{\beta_1+m-1}^{B_n} + e_{\beta_1-m+1}^{B_n} \\ e_{\beta_2-1}^{B_n} & e_{\beta_2}^{B_n} + e_{\beta_2-2}^{B_n} & \cdots & e_{\beta_2+m-2}^{B_n} + e_{\beta_2-m}^{B_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e_{\beta_m-m+1}^{B_n} & e_{\beta_m-m+2}^{B_n} + e_{\beta_n-m}^{B_n} & \cdots & e_{\beta_m}^{B_n} + e_{\beta_m-2m+2}^{B_n} \end{pmatrix}.$$

By using the equalities $e_k^{B_n} = E_k^{B_n} - E_{k-2}^{B_n}$ and simple computations on determinants we have also

$$v_\beta^{B_n} = \det\begin{pmatrix} E_{\beta_1}^{B_n} - E_{\beta_1-2}^{B_n} & E_{\beta_1+1}^{B_n} - E_{\beta_1-3}^{B_n} & \cdots & E_{\beta_1+m-1}^{B_n} - E_{\beta_1-m-1}^{B_n} \\ E_{\beta_2-1}^{B_n} - E_{\beta_2-3}^{B_n} & E_{\beta_2}^{B_n} - E_{\beta_2-4}^{B_n} & \cdots & E_{\beta_2+m-2}^{B_n} - E_{\beta_2-m-2}^{B_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ E_{\beta_m-m+1}^{B_n} - E_{\beta_m-m-1}^{B_n} & E_{\beta_m-m+2}^{B_n} - E_{\beta_m-m-2}^{B_n} & \cdots & E_{\beta_m}^{B_n} - E_{\beta_m-2m-2}^{B_n} \end{pmatrix}.$$

The determinants $v_\beta^{C_n}$, $v_\beta^{D_n}$ are defined similarly. Note that $v_\beta^{B_n}$, $v_\beta^{C_n}$, $v_\beta^{D_n}$ are polynomials in the indeterminates $x_1, \ldots, x_n, \frac{1}{x_n}, \ldots, \frac{1}{x_1}$.

Proposition 3.1.1 (*see Fulton and Harris [3, §24.2]*). *Consider λ a partition of length n and suppose that $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ is a partition of length m. Then for types B, C and D we have $u_\lambda = s_\lambda$ and $v_{\lambda'} = s_\lambda$.*

Lemma 3.1.2 (*straightening law for u_α and v_β*). *Consider $\alpha \in \pi_n$ then*

$$u_\alpha = \begin{cases} (-1)^{l(\sigma)} u_\lambda & \text{if there exists } \sigma \in \mathcal{S}_n \text{ and } \lambda \in \pi_n^+ \text{ such that } \sigma \circ \alpha = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\beta \in \pi_m$ then

$$v_\beta = \begin{cases} (-1)^{l(\sigma)} v_\nu & \text{if there exists } \sigma \in \mathcal{S}_m \text{ and } \nu \in \pi_m^+ \text{ such that } \sigma \circ \alpha = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By exchanging the rows i and $i+1$ in the determinant (7) we see that $u_{s_i \circ \alpha} = -u_\alpha$. This implies that $u_{\sigma \circ \alpha} = (-1)^{l(\sigma)} u_\alpha$ for any $\sigma \in \mathcal{S}_n$. Then it follows from the definition of the dot action that $u_\alpha = 0$ or there exists $\gamma \in \pi_n$ and $\sigma \in \mathcal{S}_n$ such that $\gamma_1 \geqslant \cdots \geqslant \gamma_n$ and $\gamma = \sigma \circ \alpha$. In this last case we have $u_\alpha = (-1)^{l(\sigma)} u_\gamma$. Now if $\gamma_i < 0$ for some i then $u_\gamma = 0$ since all the H_k which appear in the lowest row of (7) are equal to 0. Thus γ is a partition. The proof is similar for v_β. $\square$

3.2. Determinantal identities in terms of raising and lowering operators

Denote by $\mathcal{L}_n = \mathbb{K}[[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]]$ the vector space of formal Laurent series in the indeterminates $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. We identify the ring of polynomials $\mathcal{P}_n = \mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ with the sub-space of $\mathcal{L}_n$ containing the finite formal series. The vector space $\mathcal{L}_n$ is not a ring since the formal series are in the two directions. More precisely, the product $F_1 \cdots F_r$ of the formal series $F_i = \sum_{\beta_i \in E_i} x^{\beta_i}$ $i = 1, \ldots, r$ is defined if and only if for any $\gamma \in \mathbb{Z}^n$ the number N_γ of decompositions $\gamma = \beta_1 + \cdots + \beta_r$ such that $\beta_{i \in E_i}$ is finite and in this case we have

$$F_1 \cdots F_r = \sum_{\gamma \in \mathbb{Z}^n} N_\gamma x^\gamma.$$

In particular the product $P \cdot F$ with $P \in \mathcal{P}_n$ and $F \in \mathcal{L}_n$ is well defined.

Consider the following two determinants

$$\delta_n(\alpha)=\det\begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_1+1}+x_1^{\alpha_1-1} & \cdots & x_1^{\alpha_1+n-1}+x_1^{\alpha_1-n+1}\\ x_2^{\alpha_2+1} & x_2^{\alpha_2}+x_2^{\alpha_2-2} & \cdots & x_2^{\alpha_2+n-2}+x_2^{\alpha_2-n}\\ . & . & \cdots & .\\ . & . & \cdots & .\\ x_n^{\alpha_n-n+1} & x_n^{\alpha_n-n+2}+x_n^{\alpha_n-n} & \cdots & x_n^{\alpha_n}+x_n^{\alpha_n-2n+2}\end{pmatrix}$$

and

$$\Delta_n(\alpha)=\det\begin{pmatrix} x_1^{\alpha_1}-x_1^{\alpha_1-2} & x_1^{\alpha_1+1}-x_1^{\alpha_1-3} & \cdots & x_1^{\alpha_1+n-1}-x_1^{\alpha_1-n-1}\\ x_2^{\alpha_2-1}-x_2^{\alpha_2-3} & x_2^{\alpha_2}-x_2^{\alpha_2-4} & \cdots & x_2^{\alpha_2+n-2}-x_2^{\alpha_2-n-2}\\ . & . & \cdots & .\\ . & . & \cdots & .\\ x_n^{\alpha_n-n+1}+x_n^{\alpha_n-n-1} & x_n^{\alpha_{n-n+2}}-x_n^{\alpha_n-n} & \cdots & x_n^{\alpha_n}-x_n^{\alpha_n-2n-2}\end{pmatrix}$$

From a simple computation we derive the equalities:

$$\delta_n(\alpha)=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{x_i}{x_j}\right)\prod_{1\leqslant r<s\leqslant n}\left(1-\frac{1}{x_rx_s}\right)x^\alpha \quad\text{and}$$
$$\Delta_n(\alpha)=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{x_i}{x_j}\right)\prod_{1\leqslant r\leqslant s\leqslant n}\left(1-\frac{1}{x_rx_s}\right)x^\alpha. \tag{8}$$

We set $h_\alpha=h_{\alpha_1}\cdots h_{\alpha_n}$, $H_\alpha=H_{\alpha_1}\cdots H_{\alpha_n}$, $e_\alpha=e_{\alpha_1}\cdots e_{\alpha_n}$ and $E_\alpha=E_{\alpha_1}\cdots E_{\alpha_n}$.

Remarks.

(i) For any partition μ of length n, h_μ is the character of $\mathfrak{h}(\mu)=V(\mu_1\Lambda_1)\otimes\cdots\otimes V(\mu_n\Lambda_1)$ and H_μ is the character of $\mathfrak{H}(\mu)=W(\mu_1\Lambda_1)\otimes\cdots\otimes W(\mu_n\Lambda_1)$ where for any $k\in\mathbb{N}$, $W(k_1)=V(k\Lambda_1)\oplus V((k-2)\Lambda_1)\oplus\cdots\oplus V((k \bmod 2)\Lambda_1)$.

(ii) For any partition μ of length n such that μ' is of length m, $e_{\mu'}$ is the character of $\mathfrak{e}(\mu)=V(\Lambda_{\mu'_1})\otimes\cdots\otimes V(\Lambda_{\mu'_m})$ and $E_{\mu'}$ is the character of $\mathfrak{E}(\mu)=W(\Lambda_{\mu'_1})\otimes\cdots\otimes W(\Lambda_{\mu'_m})$ where for any $k\in\mathbb{N}$ with $k\leqslant n$, $W(\Lambda_k)=V(\Lambda_k)\oplus V(\Lambda_{k-2})\oplus\cdots\oplus V(\Lambda_{k \bmod 2})$.

For the root system B_n we introduce six linear maps $\mathrm{h}_{B_n},\mathrm{H}_{B_n},\mathrm{u}_{B_n}$ and $\mathrm{e}_{B_n},\mathrm{E}_{B_n},\mathrm{v}_{B_n}$ as follows:

$$\begin{cases}\mathrm{h}_{B_n}:\mathcal{L}_n\to\mathcal{C}_{B_n}\\ \quad x^\alpha\mapsto h_\alpha^{B_n}\end{cases},\quad \begin{cases}\mathrm{H}_{B_n}:\mathcal{L}_n\to\mathcal{C}_{B_n}\\ \quad x^\alpha\mapsto H_\alpha^{B_n}\end{cases},\quad \begin{cases}\mathrm{u}_{B_n}:\mathcal{L}_n\to\mathcal{C}_{B_n}\\ \quad x^\alpha\mapsto u_\alpha^{B_n}\end{cases}$$

and

$$\begin{cases}\mathrm{e}_{B_n}:\mathcal{L}_n\to\mathcal{C}_{B_n}\\ \quad x^\alpha\mapsto e_\alpha^{B_n}\end{cases},\quad \begin{cases}\mathrm{E}_{B_n}:\mathcal{L}_n\to\mathcal{C}_{B_n}\\ \quad x^\alpha\mapsto E_\alpha^{B_n}\end{cases},\quad \begin{cases}\mathrm{v}_{B_n}:\mathcal{L}_n\to\mathcal{C}_{B_n}\\ \quad x^\alpha\mapsto v_\alpha^{B_n}\end{cases}.$$

Note that the restriction of these maps on $\mathcal{P}_n$ are not ring homomorphisms. For the roots systems C_n and D_n we define respectively the maps $\mathrm{h}_{C_n},\mathrm{H}_{C_n},\mathrm{u}_{C_n},\mathrm{e}_{C_n},\mathrm{E}_{C_n},\mathrm{v}_{C_n}$ and $\mathrm{h}_{D_n},\mathrm{H}_{D_n},\mathrm{u}_{D_n},\mathrm{e}_{D_n},\mathrm{E}_{D_n},\mathrm{v}_{D_n}$ similarly.

Let ω_n and Ω_n be the endomorphisms of $\mathcal{L}_n$ corresponding respectively to the multiplication by

$$\phi_n = \prod_{1\leqslant i<j\leqslant n}\left(1-\frac{x_i}{x_j}\right)\prod_{1\leqslant r<s\leqslant n}\left(1-\frac{1}{x_r x_s}\right) \quad \text{and}$$
$$\Phi_n = \prod_{1\leqslant i<j\leqslant n}\left(1-\frac{x_i}{x_j}\right)\prod_{1\leqslant r\leqslant s\leqslant n}\left(1-\frac{1}{x_{ri} x_s}\right).$$

Proposition 3.2.1. *We have*

1. $\mathrm{u}_n = \mathrm{h}_n \cdot \omega_n$ *and* $\mathrm{u}_n = \mathrm{H}_n \cdot \Omega_n$,
2. $\mathrm{v}_n = \mathrm{e}_n \cdot \omega_n$ *and* $\mathrm{v}_n = \mathrm{E}_n \cdot \Omega_n$.

Proof. (1) We have seen that h_n is not a ring-homomorphism. Nevertheless we have by definition of the h_α

$$\mathrm{h}_n(x^\alpha) = \mathrm{h}_n(x_1^{\alpha_1})\cdots\mathrm{h}_n(x_n^{\alpha_n}) = h_{\alpha_1}\cdots h_{\alpha_n}.$$

More generally if $P_1,\dots,P_n$ are polynomials respectively in the indeterminates $x_1,\dots,x_n$, we have

$$\mathrm{h}_n(P_1(x_1)\cdots P_n(x_n)) = \mathrm{h}_n(P_1(x_1))\cdots\mathrm{h}_n(P_n(x_n))$$

by linearity of h_n. We can write

$$\delta_n(\alpha) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}x_{\sigma(1)}^{\alpha_1-\sigma(1)+1} \times\left(x_{\sigma(2)}^{\alpha_2-\sigma(2)+2}+x_{\sigma(2)}^{\alpha_2-\sigma(2)}\right)\cdots\left(x_{\sigma(n)}^{\alpha_n-\sigma(n)+n}+x_{\sigma(n)}^{\alpha_n-\sigma(n)-n+2}\right)$$

and by the previous argument

$$\mathrm{h}_n(\delta_n(\alpha)) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}h_{\alpha_1-\sigma(1)+1} \times(h_{\alpha_2-\sigma(2)+2}+h_{\alpha_2-\sigma(2)})\cdots(h_{\alpha_n-\sigma(n)+n}+h_{\alpha_n-\sigma(n)-n+2}) = u_\alpha$$

where the last equality follows from (6). By (8) we have $\delta_n(\alpha) = \omega_n(x^\alpha)$. Thus by applying h_n to this equality we obtain $\mathrm{h}_n(\omega_n(x^\alpha)) = u_\alpha = \mathrm{u}_n(x^\alpha)$. Hence $\mathrm{u}_n = \mathrm{h}_n\cdot\omega_n$. We derive the equality $\mathrm{u}_n = \mathrm{H}_n\cdot\Omega_n$ in a similar way starting from

$$\Delta_n(\alpha) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\left(x_{\sigma(1)}^{\alpha_1-\sigma(1)+1}+x_{\sigma(1)}^{\alpha_1-\sigma(1)-1}\right)\cdots\left(x_{\sigma(n)}^{\alpha_n-\sigma(n)+n}+x_{\sigma(n)}^{\alpha_n-\sigma(n)-n}\right).$$

(2) The arguments are the same as in 1 once replacing the characters h and H respectively by the characters e and E. □

Consider $\alpha = (\alpha_1,\dots,\alpha_n)\in\pi_n$ and two integers i, j such that $1\leqslant i\leqslant j\leqslant n$. The raising operator $R_{i,j}$ and the lowering operator $L_{i,j}$ are respectively defined on π_n by $R_{i,j}(\alpha) =$

$\alpha+\varepsilon_i-\varepsilon_j$ and $L_{i,j}(\alpha)=\alpha-\varepsilon_i-\varepsilon_j$. From the previous lemma we obtain:

Corollary 3.2.2. *For any partition $\mu=(\mu_1,\dots,\mu_n)$ we have*

$$s_\mu=\left(\prod_{1\leqslant i<j\leqslant n}(1-R_{i,j})\prod_{1\leqslant r<s\leqslant n}(1-L_{r,s})\right)h_\mu,$$

$$s_\mu=\left(\prod_{1\leqslant i<j\leqslant n}(1-R_{i,j})\prod_{1\leqslant r\leqslant s\leqslant n}(1-L_{r,s})\right)H_\mu,$$

$$s_\mu=\left(\prod_{1\leqslant i<j\leqslant m}(1-R_{i,j})\prod_{1\leqslant r<s\leqslant m}(1-L_{r,s})\right)e_{\mu'},$$

$$s_\mu=\left(\prod_{1\leqslant i<j\leqslant m}(1-R_{i,j})\prod_{1\leqslant r\leqslant s\leqslant m}(1-L_{r,s})\right)E_{\mu'}$$

where $\mu'=(\mu'_1,\dots,\mu'_m)$ is the conjugate partition of μ.

Proof. Let us write

$$\phi_n=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{x_i}{x_j}\right)\prod_{1\leqslant r<s\leqslant n}\left(1-\frac{1}{x_rx_s}\right)=\sum_{\alpha\in\pi_n}a(\alpha)x^\alpha.$$

Then by 1 of Proposition 3.2.1, we have for any $\mu\in\pi_n^+$,

$$\mathrm{u}_n(x^\mu)=\mathrm{h}_n\left(\sum_{\alpha\in\pi_n}a(\alpha)x^{\alpha+\mu}\right)=\sum_{\alpha\in\pi_n}a(\alpha)h_{\alpha+\mu}=u_\mu=s_\mu$$

where the last equality follows from Proposition 3.1.1. This is exactly equivalent to

$$s_\mu=\left(\prod_{1\leqslant i<j\leqslant n}(1-R_{i,j})\prod_{1\leqslant r<s\leqslant n}(1-L_{r,s})\right)h_\mu.$$

The arguments are essentially the same for the other equalities. □

3.3. Expressions for the multiplicities of representations

Lemma 3.3.1. *The products*

$$\phi_n^{-1}=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{x_i}{x_j}\right)^{-1}\prod_{1\leqslant r<s\leqslant n}\left(1-\frac{1}{x_rx_s}\right)^{-1}\quad\textit{and}$$

$$\Phi_n^{-1}=\prod_{1\leqslant i<j\leqslant n}\left(1-\frac{x_i}{x_j}\right)^{-1}\prod_{1\leqslant r\leqslant s\leqslant n}\left(1-\frac{1}{x_rx_s}\right)^{-1}$$

are well defined in $\mathcal{L}_n$.

Proof. Given any $\beta \in \mathbb{Z}^n$, the number of decompositions

$$\beta = \sum_{1 \leqslant i<j \leqslant n} a_{i,j}(\varepsilon_i - \varepsilon_j) - \sum_{1 \leqslant r<s \leqslant n} b_{r,s}(\varepsilon_r + \varepsilon_s)$$

with $a_{i,j}$ and $b_{r,s}$ some positive integers is finite. Thus ϕ_n^{-1} is well defined. The proof is similar for Φ_n^{-1}. $\square$

Write

$$\phi_n^{-1} = \sum_{\alpha \in \pi_n} f(\alpha) x^\alpha \quad \text{and} \quad \Phi_n^{-1} = \sum_{\alpha \in \pi_n} F(\alpha) x^\alpha.$$

Then ϕ_n^{-1} and Φ_n^{-1} belong to $\mathcal{L}_n$.

Lemma 3.3.2. *Consider μ a partition of length n with μ' of length m. We have*

$$\text{(i) } h_\mu = \sum_{\alpha \in \pi_n} f(\alpha) u_{\mu+\alpha}, \quad \text{(ii) } H_\mu = \sum_{\alpha \in \pi_n} F(\alpha) u_{\mu+\alpha},$$
$$\text{(iii) } e_{\mu'} = \sum_{\alpha \in \pi_m} f(\alpha) v_{\mu'+\alpha}, \quad \text{(iv) } E_{\mu'} = \sum_{\alpha \in \pi_m} F(\alpha) v_{\mu'+\alpha}.$$

Proof. Write $\overline{\omega}_n$ for the linear map

$$\overline{\omega}_n : \mathcal{P}_n \to \mathcal{L}_n, \quad P \mapsto \phi_n^{-1} P.$$

Then for any $\beta \in \pi_n$, we have $\omega_n(\overline{\omega}_n(x^\beta)) = x^\beta$. By Proposition 3.2.1 we know that $\mathrm{u}_n = \mathrm{h}_n \cdot \omega_n$. We derive

$$\mathrm{u}_n(\overline{\omega}_n(x^\beta)) = \mathrm{h}_n \cdot \omega_n(\overline{\omega}_n(x^\beta)) = \mathrm{h}_n(x^\beta) = h_\beta$$

for any $\beta \in \pi_n$. When $\beta = \mu$ this is equivalent to (i). We obtain (ii) similarly by using the linear map $\overline{\Omega}_n : P \mapsto \Phi_n^{-1} P$. The arguments are the same for the equalities (iii) and (iv). $\square$

The identities of the above lemma can be rewritten by using raising and lowering operators as in Corollary 3.2.2. Namely we have

$$h_\mu = \left(\prod_{1 \leqslant i<j \leqslant n} \frac{1}{1 - R_{i,j}} \prod_{1 \leqslant r<s \leqslant n} \frac{1}{1 - L_{r,s}} \right) s_\mu,$$
$$H_\mu = \left(\prod_{1 \leqslant i<j \leqslant n} \frac{1}{1 - R_{i,j}} \prod_{1 \leqslant r \leqslant s \leqslant n} \frac{1}{1 - L_{r,s}} \right) s_\mu,$$

$$e_{\mu'}=\left(\prod_{1\leqslant i<j\leqslant m}\frac{1}{1-R_{i,j}}\prod_{1\leqslant r<s\leqslant m}\frac{1}{1-L_{r,s}}\right)s_\mu \text{ and}$$
$$E_{\mu'}=\left(\prod_{1\leqslant i<j\leqslant m}\frac{1}{1-R_{i,j}}\prod_{1\leqslant r\leqslant s\leqslant m}\frac{1}{1-L_{r,s}}\right)s_\mu.$$

For any positive integer n write $\rho_l=(n,n-1,\dots,1)$.

Proposition 3.3.3. *Consider a partition μ of length n such that μ' has length m. Then for the three roots systems B_n, C_n and D_n we have*:

$$\text{(i)}\ \begin{cases} h_\mu=\sum_{\lambda\in\pi_n^+}\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}f(\sigma(\lambda+\rho_n)-\mu-\rho_n)u_\lambda\\ H_\mu=\sum_{\lambda\in\pi_n^+}\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}F(\sigma(\lambda+\rho_n)-\mu-\rho_n)u_\lambda\end{cases},$$
$$\text{(ii)}\ \begin{cases} e_{\mu'}=\sum_{\nu\in\pi_m^+}\sum_{\sigma\in\mathcal{S}_m}(-1)^{l(\sigma)}f(\sigma(\nu+\rho_m)-\mu'-\rho_m)v_\nu\\ E_{\mu'}=\sum_{\nu\in\pi_m^+}\sum_{\sigma\in\mathcal{S}_m}(-1)^{l(\sigma)}F(\sigma(\nu+\rho_m)-\mu'-\rho_m)v_\nu\end{cases}.$$

Proof. (i) Note first that the above relations do not depend on the root system considered. Indeed for any nonnegative integer n, we have $\rho_{B_m}=\rho_n-(\frac12,\dots,\frac12)$, $\rho_{C_n}=\rho_n$ and $\rho_{D_m}=\rho_n-(1,\dots,1)$. Thus $\sigma(\lambda+\rho_{B_n})-\mu-\rho_{B_n}=\sigma(\lambda+\rho_{C_n})-\mu-\rho_{C_n}=\sigma(\lambda+\rho_{D_n})-\mu-\rho_{D_n}=\sigma(\lambda+\rho_n)-\mu-\rho_n$. We have

$$h_\mu=\sum_{\alpha\in\pi_n}f(\alpha)u_{\mu+\alpha}.$$

From Lemma 3.1.2 we deduce that for any $\alpha\in\pi_n$ we have $u_{\mu+\alpha}=0$ or there exists a partition λ such that $\mu+\alpha=\sigma(\lambda+\rho_n)-\rho_n$ and $u_{\mu+\alpha}=(-1)^{l(\sigma)}u_\lambda$. By setting $\alpha=\sigma(\lambda+\rho_n)-\mu-\rho_n$ in the above sum we obtain $h_\mu=\sum_{\lambda\in\pi_n}\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}f(\sigma(\lambda+\rho_n)-\mu-\rho_n)u_\lambda$. The arguments are similar for the other assertions. $\square$

From relations (i) and by using the fact that $u_\lambda=s_\lambda$ for any partition λ of length n, we derive the equalities

$$h_\mu=\sum_{\lambda\in\pi_n}u_{\lambda,\mu}s_\lambda\quad\text{and}\quad H_\mu=\sum_{\lambda\in\pi_n}U_{\lambda,\mu}s_\lambda$$

where

$$u_{\lambda,\mu}=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}f(\sigma(\lambda+\rho_n)-\mu-\rho_n)\quad\text{and}$$
$$U_{\lambda,\mu}=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}F(\sigma(\lambda+\rho_n)-\mu-\rho_n)\tag{9}$$

are respectively the multiplicities of $V(\lambda)$ in $\mathfrak{h}(\mu)$ and $\mathfrak{H}(\mu)$. Note that $u_{\lambda,\mu}=0$ and $U_{\lambda,\mu}=0$ unless $|\mu|\geqslant|\lambda|$.

For the relations (ii) the situation is more complicated since the partitions ν obtained by applying straightening laws to the $v_{\mu'+\beta}$ yields polynomials v_ν where $\nu\in\pi_m^+$ is a partition of length m so cannot be necessarily regarded as the conjugate partition of a partition $\lambda \in \pi_n^+$. The straightening law of Lemma 3.1.2 implies that $|\nu| = |\mu'|$. Since $|\mu| = |\mu'|$, this problem disappears if we suppose $n \geqslant |\mu|$ for we will have $\nu_1 \leqslant |\nu| \leqslant n$ and thus $\nu' \in \pi_n^+$. We can then set $\nu = \lambda'$ with $\lambda \in \pi_n$ and obtain

$$e_{\mu'} = \sum_{\lambda \in \pi_n} v_{\lambda,\mu} s_\lambda \quad \text{and} \quad E_{\mu'} = \sum_{\lambda \in \pi_n} V_{\lambda,\mu} s_\lambda.$$

We deduce that

$$v_{\lambda,\mu} = u_{\lambda',\mu'} = \sum_{\sigma \in \mathcal{S}_m} (-1)^{l(\sigma)} f(\sigma(\lambda' + \rho_m) - \mu' - \rho_m), \tag{10}$$

$$V_{\lambda,\mu} = U_{\lambda',\mu'} = \sum_{\sigma \in \mathcal{S}_m} (-1)^{l(\sigma)} F(\sigma(\lambda' + \rho_m) - \mu' - \rho_m), \tag{11}$$

are respectively the multiplicities of $V(\lambda)$ in the tensor products $\mathfrak{e}(\mu)$ and $\mathfrak{E}(\mu)$ when $n \geqslant |\mu|$.

4. Quantization of the multiplicities

4.1. The functions f_q and F_q

Set

$$\phi_n(q) = \prod_{1 \leqslant i < j \leqslant n} \left(1 - q\frac{x_i}{x_j}\right) \prod_{1 \leqslant r < s \leqslant n} \left(1 - \frac{q}{x_i x_j}\right) \quad \text{and}$$

$$\Phi_n(q) = \prod_{1 \leqslant i < j \leqslant n} \left(1 - q\frac{x_i}{x_j}\right) \prod_{1 \leqslant r \leqslant s \leqslant n} \left(1 - \frac{q}{x_i x_j}\right).$$

The functions f_q and F_q are obtained by considering the formal series expansions of $\phi_n^{-1}(q)$ and $\Phi_n^{-1}(q)$. Namely we have

$$\phi_n^{-1}(q) = \sum_{\alpha \in \pi_n} f_q(\alpha) x^\alpha \quad \text{and} \quad \Phi_n^{-1}(q) = \sum_{\alpha \in \pi_n} F_q(\alpha) x^\alpha. \tag{12}$$

4.2. Some q-analogues of multiplicities of $V(\lambda)$ in $\mathfrak{h}(\mu)$, $\mathfrak{H}(\mu)$, $\mathfrak{e}(\mu)$ or $\mathfrak{E}(\mu)$

Given λ and μ two partitions of length n, let $c_{\lambda,\mu}(q)$ and $C_{\lambda,\mu}(q)$ be the two polynomials defined by

$$u_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} f_q(\sigma(\lambda + \rho_n) - \mu - \rho_n) \quad \text{and}$$

$$U_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} F_q(\sigma(\lambda + \rho_n) - \mu - \rho_n).$$

Then from the equalities (9), (10) and (11) we obtain:

Proposition 4.2.1. *Let λ and μ be two partitions of length n. Then*

1. *$u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ are q-analogues of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{h}(\mu)$ and $\mathfrak{H}(\mu)$,*
2. *$v_{\lambda,\mu}(q) = u_{\lambda,'\mu'}(q)$ and $V_{\lambda,\mu}(q) = U_{\lambda',\mu'}(q)$ are q-analogues of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{e}(\mu)$ and $\mathfrak{E}(\mu)$ when the condition $n \geqslant |\mu|$ is satisfied.*

The following example is obtained from the explicit computation of the function f_q when $n = 2$.

Example 4.2.2. Consider μ a partition of length 2 and set $\mathcal{E}_\mu = \{\lambda \in \pi_2^+, \lambda = (\mu_1 + r - s, \mu_2 - r - s), s \in \{0, \dots, \mu_2\}, r \in \{0, \dots, \mu_2 - s\}\}$. Then for any partition λ of length 2 we have:

$$u_{\lambda,\mu}(q) = \begin{cases} q^{\mu_1-\lambda_1} & \text{if } \lambda \in \mathcal{E}_\mu, \\ 0 & \text{otherwise.} \end{cases}$$

Remarks.

(i) It follows from the definition of the q-functions f_q and F_q that $u_{\lambda,\mu}(q) = U_{\lambda,\mu}(q) = 0$ if $|\lambda| > |\mu|$.

(ii) It is not trivial from the very definitions that $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ are polynomials in q with nonnegative integer coefficients. This property will be proved in Section 5 as a corollary of Theorem 5.1.5.

5. The duality theorems

5.1. A duality theorem for the q-multiplicities in $\mathfrak{h}(\mu)$ and $\mathfrak{H}(\mu)$

For any nonnegative integer n, set $\kappa_n = (1, \dots, 1) \in \pi_n$.

Lemma 5.1.1. *Consider λ, μ two partitions of length n such that $|\lambda| \geqslant |\mu|$. Let k be any integer such that $k \geqslant \frac{|\lambda|-|\mu|}{2}$. Then we have*

$$K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)). \tag{13}$$

Proof. Since $\mathcal{P}_q(\alpha) = 0$ if α is not a linear combination of positive roots with nonnegative coefficients, we have $\mathcal{P}_q(\alpha) = 0$ for any $\alpha \in \pi_n$ such that $|\alpha| < 0$. Consider $\delta = (\delta_1, \dots, \delta_n) \in \pi_n$ and $w \in W_n$. Write $w(\delta) = (\delta_1^w, \dots, \delta_n^w)$ and set $E_w = \{i, w(i) \notin \{1, \dots, n\}\}$. Define the sum $S_{w,\delta} = \sum_{i \in E_w} \delta_{i_k}$. Then $|w(\delta)| = |\delta| - 2S_{w,\delta}$. Now consider k a nonnegative integer and set $\delta = (\lambda + \rho_n + k\kappa_n)$. We have $\left|w(\lambda + \rho_n + k\kappa_n)\right| = \left|(\lambda + \rho_n + k\kappa_n)\right| - 2S_{w,\delta}$. But $S_{w,\delta} = S_{w,\lambda+\rho_n} + kp$ where $p = \mathrm{card}(E_w)$. Thus we obtain

$$\begin{aligned}&\left|w(\lambda+\rho_n+k\kappa_n)-(\mu+\rho_n+k\kappa_n)\right|\\&\quad=\left|(\lambda+\rho_n+k\kappa_n)\right|-2S_{w,\lambda+\rho_n}-\left|(\mu+\rho_n+k\kappa_n)\right|-2kp\\&\quad=|\lambda|-|\mu|-2S_{w,\lambda+\rho_n}-2kp.\end{aligned}$$

When $w\notin\mathcal{S}_n$, we have $p\geqslant 1$ and $S_{w,\lambda+\rho_n}\geqslant 1$ since the coordinates of $\lambda+\rho_n$ are all positive. Hence $\left|w(\lambda+\rho_n+k\kappa_n)-(\mu+\rho_n+k\kappa_n)\right|<|\lambda|-|\mu|-2k$ and is negative as soon as $k\geqslant\frac{|\lambda|-|\mu|}{2}$. For such an integer k the sum defining $K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q)$ normally running over W_n can be restricted to (13) and we obtain

$$K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q)=\sum_{\sigma\in S_n}(-1)^{l(\sigma)}\mathcal{P}_q(\sigma(\lambda+\rho_n+k\kappa_n)-(\mu+\rho_n+k\kappa_n)).$$

Since $\sigma\in\mathcal{S}_n$, we have $\sigma(k\kappa_n)=k\kappa_n$. Thus

$$K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q)=\sum_{\sigma\in S_n}(-1)^{l(\sigma)}\mathcal{P}_q(\sigma(\lambda+\rho_n)-(\mu+\rho_n)).\qquad\square$$

We define the involution I on π_n by $I(\alpha_1,\dots,\alpha_n)=(-\alpha_n,\dots,-\alpha_1)$ for any $\alpha=(\alpha_1,\dots,\alpha_n)\in\pi_n$.

Lemma 5.1.2. *For any $\alpha=(\alpha_1,\dots,\alpha_n)\in\pi_n$ we have*

$$f_q(\alpha)=\mathcal{P}_q^{D_n}(I(\alpha))\quad\textit{and}\quad F_q(\alpha)=\mathcal{P}_q^{C_n}(I(\alpha))$$

where $\mathcal{P}_q^{C_n}$ and $\mathcal{P}_q^{D_n}$ are the q-partition functions associated respectively to the root systems B_n and D_n.

Proof. By abuse of notation we also denote by I the ring automorphism of $\mathcal{L}_n$ defined by $I(x^\alpha)=x^{I(\alpha)}$. The images of the root systems C_n and D_n by I are respectively

$$\begin{cases}\{\varepsilon_i-\varepsilon_j,-\varepsilon_i-\varepsilon_j\text{ with }1\leqslant i<j\leqslant n\}\cup\{-2\varepsilon_i\text{ with }1\leqslant i\leqslant n\} & \text{for the root system }C_n,\\ \{\varepsilon_i-\varepsilon_j,-\varepsilon_i-\varepsilon_j\text{ with }1\leqslant i<j\leqslant n\} & \text{for the root system }D_n.\end{cases}\tag{14}$$

By applying I to the equality

$$\prod_{\alpha\in R^+_{C_n}}\frac{1}{1-qx^\alpha}=\sum_{\beta\in\pi_n}\mathcal{P}_q^{C_n}(\beta)x^\beta$$

we obtain

$$\prod_{1\leqslant i<j\leqslant n}\frac{1}{(1-q\frac{x_i}{x_j})}\prod_{1\leqslant r\leqslant s\leqslant n}\frac{1}{(1-\frac{q}{x_rx_s})}=\sum_{\beta\in\pi_n}\mathcal{P}_q^{C_n}(\beta)x^{I(\beta)}.$$

Set $\alpha=I(\beta)$. The equality becomes

$$\Phi_n^{-1}(q)=\sum_{\alpha\in\pi_n}\mathcal{P}_q^{C_n}(I(\alpha))x^\alpha$$

and from the definition (see 12) of the function F_q, we obtain $\mathcal{P}_q^{C_n}(I(\alpha)) = F_q(\alpha)$. The assertion with f_q is proved in the same way by considering the positive root system D_n. □

Given $\sigma \in \mathcal{S}_n$, denote by σ^* the permutation defined by

$$\sigma^*(k) = \sigma(n-k+1).$$

For any $i \in \{1, \ldots, n-1\}$, we have $s_i^* = s_{n-i}$. The following lemma is straightforward:

Lemma 5.1.3. *The map $\sigma \to \sigma^*$ is an involution of the group $\mathcal{S}_n$. Moreover we have $\sigma(I(\beta)) = I(\sigma^*(\beta))$ and $l(\sigma) = l(\sigma^*)$ for any $\beta \in \pi_n$, $\sigma \in \mathcal{S}_n$.*

Lemma 5.1.4. *Let λ, μ two partitions of length n and $\sigma \in \mathcal{S}_n$. Then*

$$(-1)^{l(\sigma)} f_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)) = (-1)^{l(\sigma^*)} \mathcal{P}_q^{D_n}(\sigma^*(I(\lambda)+\rho_n) - (I(\mu)+\rho_n))$$

and

$$(-1)^{l(\sigma)} F_q(\sigma(\lambda+\rho_n) - (\mu+\rho)) = (-1)^{l(\sigma^*)} \mathcal{P}_q^{C_n}(\sigma^*(I(\lambda)+\rho_n) - (I(\mu)+\rho_n)).$$

Proof. Since $l(\sigma) = l(\sigma^*)$, it suffices to prove the equalities

$$f_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)) = \mathcal{P}_q^{D_n}(\sigma^*(I(\lambda)+\rho_n) - (I(\mu)+\rho_n))$$

and

$$F_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)) = \mathcal{P}_q^{C_n}(\sigma^*(I(\lambda)+\rho_n) - (I(\mu)+\rho_n)).$$

Set $P = \mathcal{P}_q^{C_n}(\sigma^*(I(\lambda)+\rho_n) - (I(\mu)+\rho_n))$. From the above lemma we deduce

$$P = \mathcal{P}_q^{C_n}(I(\sigma(\lambda)) + \sigma^*(\rho_n) - I(\mu) - \rho_n).$$

Now an immediate computation shows that $\sigma^*(\rho_n) - \rho_n = I(\sigma(\rho_n) - \rho_n)$. Thus we derive

$$P = \mathcal{P}_q^{C_n}(I(\sigma(\lambda+\rho_n) - \mu - \rho_n)) = F_q(\sigma(\lambda+\rho_n) - \mu - \rho_n)$$

where the last equality follows from Lemma 5.1.2.

We obtain the equality $f_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)) = \mathcal{P}_q^{D_n}(\sigma(I(\lambda)+\rho_n) - (I(\mu)+\rho_n))$ in a similar way. □

Theorem 5.1.5. *Consider λ, μ two partitions of length n and set $m = \max(\lambda_1, \mu_1)$. Let k be any integer such that $k \geqslant \frac{|\mu|-|\lambda|}{2}$. Then $\widehat{\lambda} = (m-\lambda_n, \ldots, m-\lambda_1)$ and $\widehat{\mu} = (m-\mu_n, \ldots, m-\mu_1)$ are partitions of length n and*

$$\begin{cases} u_{\lambda,\mu}(q) = K^{D_n}_{\widehat{\lambda}+k\kappa_n, \widehat{\mu}+k\kappa_n}(q), \\ U_{\lambda,\mu}(q) = K^{C_n}_{\widehat{\lambda}+k\kappa_n, \widehat{\mu}+k\kappa_n}(q). \end{cases}$$

Proof. First $\widehat{\lambda}$ and $\widehat{\mu}$ are clearly partitions of length n since $m = \max(\lambda_1, \mu_1)$. It follows from the definition of $U_{\lambda,\mu}(q)$ and the above lemma that

$$\begin{aligned} U_{\lambda,\mu}(q) &= \sum_{\sigma\in\mathcal{S}_n} (-1)^{l(\sigma)} F_q(\sigma(\lambda+\rho_n) - \mu - \rho_n) \\ &= \sum_{\sigma^*\in\mathcal{S}_n} (-1)^{l(\sigma^*)} \mathcal{P}_q^{C_n}(\sigma^*(I(\lambda)+\rho_n)) - (I(\mu)+\rho_n)). \end{aligned}$$

Then by Lemma 5.1.3 we obtain

$$U_{\lambda,\mu}(q) = \sum_{\sigma\in\mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(I(\lambda)+\rho_n)) - (I(\mu)+\rho_n)).$$

We have $\sigma(I(\lambda)+\rho_n+m\kappa_n) = \sigma(I(\lambda)+\rho_n)+m\kappa_n$ since $\sigma\in\mathcal{S}_n$. So we can write

$$U_{\lambda,\mu}(q) = \sum_{\sigma\in\mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(I(\lambda)+m\kappa_n+\rho_n)) - (I(\mu)+m\kappa_n+\rho_n)).$$

Since $\widehat{\lambda} = I(\lambda)+m\kappa_n$ and $\widehat{\mu} = I(\mu)+m\kappa_n$ we derive

$$U_{\lambda,\mu}(q) = \sum_{\sigma\in\mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\widehat{\lambda}+\rho_n) - (\widehat{\mu}+\rho_n)) = K^{C_n}_{\widehat{\lambda}+k\kappa_n,\widehat{\mu}+k\kappa_n}(q)$$

by Lemma 5.1.1.

We obtain similarly the equality $u_{\lambda,\mu}(q) = K^{D_n}_{\widehat{\lambda}+k\kappa_n,\widehat{\mu}+k\kappa_n}(q)$ by replacing $\mathcal{P}_q^{C_n}$ by $\mathcal{P}_q^{D_n}$. $\square$

Example 5.1.6. Consider $\mu = (4,2,1)$ and $\lambda = (2,1,0)$. We have $m = 4$, $\widehat{\mu} = (3,2,0)$ and $\widehat{\lambda} = (4,3,2)$. We choose $k = 2$. Then we obtain the equalities

$$\begin{cases} u_{\lambda,\mu}(q) = K^{D_n}_{(6,5,4),(5,4,2)}(q) = q^3+q^2, \\ U_{\lambda,\mu}(q) = K^{C_n}_{(6,5,4),(5,4,2))}(q) = q^5+2q^4+3q^3+2q^2. \end{cases}$$

By using the fact that the Kostka–Foulkes polynomials have nonnegative integer coefficients [11] we obtain the following corollary.

Corollary 5.1.7. *The polynomials $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ have nonnegative integers coefficients.*

We also recover a property of the Kostka–Foulkes polynomials associated to the root system A_{n-1} proved in [9].

Corollary 5.1.8. *Consider λ,μ two partitions of length n such that $|\lambda| = |\mu|$ and set $m = \max(\lambda_1,\mu_1)$. Then the Kostka–Foulkes polynomials associated to the root system*

A_{n-1} *verifies*

$$K_{\lambda,\mu}^{A_{n-1}}(q) = K_{\widehat{\lambda},\widehat{\mu}}^{A_{n-1}}(q)$$

where $\widehat{\lambda} = (m - \lambda_n, \dots, m - \lambda_1)$ *and* $\widehat{\mu} = (m - \mu_n, \dots, m - \mu_1)$.

Proof. Suppose that β is a linear combination of $I(R_{C_n}^{+})$ with nonnegative coefficients such that $|\beta| = 0$. Then β is necessarily a linear combination of the roots $\varepsilon_i - \varepsilon_j$, $1 \leqslant i < j \leqslant n$ with nonnegative coefficients (see (14)) that is, a linear combination with nonnegative coefficients of the positive roots associated to the root system A_{n-1}. This implies that

$$f_q(\beta) = F_q(\beta) = \mathcal{P}_q^{A_{n-1}}(\beta)$$

where $\mathcal{P}_q^{A_{n-1}}$ is the q-partition function associated to the root system A_{n-1}. For any $\sigma \in \mathcal{S}_n$, we have $\left|\sigma(\lambda + \rho_n) - (\mu + \rho_n)\right| = 0$ since $|\lambda| = |\mu|$. Thus

$$\begin{aligned} f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) &= F_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) \\ &= \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) \end{aligned}$$

and the multiplicities $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ coincide with the Kostka–Foulkes polynomial $K_{\lambda,\mu}^{A_{n-1}}(q)$ when $|\lambda| = |\mu|$. Moreover by applying Theorem 5.1.5 with $|\lambda| = |\mu|$ and $k = 0$, we obtain $U_{\lambda,\mu}(q) = K_{\widehat{\lambda},\widehat{\mu}}^{C_n}(q) = K_{\widehat{\lambda},\widehat{\mu}}^{A_{n-1}}(q)$ where the last equality is due to the fact that the Kostka–Foulkes polynomials of types B_n, C_n or D_n are Kostka–Foulkes polynomials associated to the root system A_{n-1} when $|\lambda| = |\mu|$. So we derive the equality $K_{\lambda,\mu}^{A_{n-1}}(q) = K_{\widehat{\lambda},\widehat{\mu}}^{A_{n-1}}(q)$. $\square$

We have seen that $U_{\lambda,\mu}(q)$ can be regarded as a q-analogue of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{H}^{C_n}(\mu)$. In [4], Hatayama et al. have introduced another quantization $X_{\lambda,\mu}(q)$ of this multiplicity based on the determination of the combinatorial R matrix of the $U_q'(C_n^{(1)})$-crystals B_k. Considered as the crystal graph of the $U_q(C_n)$-module M_k, B_k can be identified with

$$B(k\Lambda_1) \oplus B((k-2)\Lambda_1) \oplus \cdots \oplus B(k \bmod 2\Lambda_1)$$

where for any $i \in \{k, k-2, \dots, k \bmod 2\}$, $B(\,k\Lambda_1)$ is the crystal graph of the irreducible finite dimensional $U_q(C_n)$-module of highest weight $k\Lambda_1$. Note that the character of M_k is equal to $H_k^{C_n}$.

Recall that the combinatorial R-matrix associated to crystals B_k is equivalent to the description of the crystal graph isomorphisms

$$\begin{cases} B_l \otimes B_k \overset{\simeq}{\to} B_k \otimes B_l, \\ b_1 \otimes b_2 \longmapsto b_2' \otimes b_1' \end{cases}$$

together with the energy function H on $B_l \otimes B_k$. The multiplicity of $V(\lambda)$ in $\mathfrak{H}^{C_n}(\mu)$ is then equal to the number of highest weight vertices of weight λ in the crystal $B_\mu = B_{\mu_1} \otimes \cdots \otimes B_{\mu_n}$. Then $X_{\lambda,\mu}(q)$ is defined by

$$X_{\lambda,\mu}(q) = \sum_{b \in E_\lambda} q^{\sum_{0 \leqslant i<j \leqslant n} H(b_i \otimes b_j^{(i+1)})}$$

where E_λ is the set of highest weight vertices $b = b_1 \otimes \cdots \otimes b_n$ in B_μ of highest weight λ, $b_j^{(i)}$ is determined by the crystal isomorphism

$$B_{\mu_i} \otimes B_{\mu_{i+1}} \otimes B_{\mu_{i+2}} \otimes \cdots \otimes B_{\mu_j} \to B_{\mu_i} \otimes B_{\mu_j} \otimes B_{\mu_{i+1}} \cdots \otimes B_{\mu_{j-1}},$$
$$b_i \otimes b_{i+1} \otimes \cdots \otimes b_j \to b_j^{(i)} \otimes b_i' \otimes \cdots \otimes b_{j-1}'$$

and for any $j = 1, \ldots, n$, $H(b_0 \otimes b_j^{(1)})$ depends only on $b_j^{(1)}$.

Many computations suggest the following conjecture

Conjecture 5.1.9. *For any partition λ and μ of length n with $|\mu| \geqslant |\lambda|$*

$$U_{\lambda,\mu}(q) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}(q).$$

Note that the conjecture is in particular true for all the examples given in the tables of [4].

5.2. *A duality theorem for the q-multiplicities in $\mathfrak{e}(\mu)$ and $\mathfrak{E}(\mu)$*

Consider λ, μ two partitions of length l such that $l \geqslant |\mu| \geqslant |\lambda|$. Write $n = \max(\lambda_1, \mu_1)$. Then by adding to λ' and μ' the required numbers of parts 0 we can consider them as partitions of length n. Set $m = \max(\lambda_1', \mu_1')$. We define the partitions $\widetilde{\lambda}$ and $\widetilde{\mu}$ belonging to π_n by $\widetilde{\lambda} = (m - \lambda_n', \ldots, m - \lambda_1')$ and $\widetilde{\mu} = (m - \mu_n', \ldots, m - \mu_1')$.

Theorem 5.2.1. *With the above notations, we have for any integer $k \geqslant \frac{|\mu|-|\lambda|}{2}$*

$$\begin{cases} \text{(i) } v_{\lambda,\mu}(q) = K^{D_n}_{\widetilde{\lambda}+k\kappa_n, \widetilde{\mu}+k\kappa_n}(q), \\ \text{(ii) } V_{\lambda,\mu}(q) = K^{C_n}_{\widetilde{\lambda}+k\kappa_n, \widetilde{\mu}+k\kappa_n}(q). \end{cases}$$

Proof. Since $l \geqslant |\mu|$, we have by Proposition 4.2.1 the equality $v_{\lambda,\mu}(q) = u_{\lambda',\mu'}(q)$. Moreover we have $m \geqslant \max(\lambda_1', \mu_1')$ and $k \geqslant \frac{|\mu'|-|\lambda'|}{2}$ for $|\lambda'| = |\lambda|$ and $|\mu'| = |\mu|$. Hence by applying Theorem 5.1.5 we obtain $v_{\lambda,\mu}(q) = K^{D_n}_{\widehat{\lambda'}+k\kappa_n, \widehat{\mu'}+k\kappa_n}(q)$ where $\widehat{\lambda'} = (m - \lambda_n', \ldots, m - \lambda_1') = \widetilde{\lambda}$ and $\widehat{\mu'} = (m - \mu_n', \ldots, m - \mu_1') = \widetilde{\mu}$. So (i) is proved. We obtain (ii) similarly. $\square$

Example 5.2.2. For $\lambda = (2, 1, 0, 0, 0)$ and $\mu = (2, 2, 1, 0, 0)$ we have $l = 5, n = 2$. Moreover $\lambda' = (2, 1)$, $\mu' = (3, 2)$ and $m = 3$. So $\widetilde{\lambda} = (2, 1)$ and $\widetilde{\mu} = (1, 0)$. Hence for

$k=1$

$$\begin{cases} \text{(i) } v_{\lambda,\mu}(q) = K^{D_n}_{(3,2),(2,1)}(q) = q, \\ \text{(ii) } V_{\lambda,\mu}(q) = K^{C_n}_{(3,2),(2,1)}(q) = q^2+q. \end{cases}$$

Remark. When λ, μ are considered as weights associated to the root system C_l, the above theorem is essentially the quantization of a duality result explicited by Foulle [2] from results of [7] for the dual pair $(Sp(2l), Sp(2n))$.

6. Identities for the q-multiplicities $U_{\lambda,\mu}(q)$ and $u_{\lambda,\mu}(q)$

6.1. A relations between q-partition functions

Consider a nonnegative integer k and define the finite sets

$$\begin{cases} \mathcal{C}^n_k = \{\beta \in \pi_n,\ \beta = \sum_{1\leqslant r\leqslant s\leqslant n} e_{r,s}(\varepsilon_r+\varepsilon_s) \text{ with } e_{r,s}\geqslant 0 \text{ and } |\beta| = 2k\}, \\ \mathcal{D}^n_k = \{\beta \in \pi_n,\ \beta = \sum_{1\leqslant r< s\leqslant n} e_{r,s}(\varepsilon_r+\varepsilon_s) \text{ with } e_{r,s}\geqslant 0 \text{ and } |\beta| = 2k\}. \end{cases}$$

Note that each $\beta\in\mathcal{C}^n_k$ (resp. $\beta\in\mathcal{D}^n_k$) verifies $|\beta| = 2\sum_{1\leqslant r\leqslant s\leqslant n} e_{r,s}$ (resp. $|\beta| = 2\sum_{1\leqslant r<s\leqslant n} e_{r,s}$). This implies that

$$\prod_{1\leqslant r\leqslant s\leqslant n} \frac{1}{\left(1-\frac{q}{x_r x_s}\right)} = \sum_{k\geqslant 0}\sum_{\beta\in\mathcal{C}^n_k} c^{C_n}_\beta q^k x^\beta \quad \text{and}$$

$$\prod_{1\leqslant r< s\leqslant n} \frac{1}{\left(1-\frac{q}{x_r x_s}\right)} = \sum_{k\geqslant 0}\sum_{\beta\in\mathcal{C}^n_k} c^{D_n}_\beta q^k x^\beta$$

where $c^{C_n}_\beta$ (resp. $c^{D_n}_\beta$) is the number of ways to decompose β as $\beta = \sum_{1\leqslant r\leqslant s\leqslant n} e_{r,s}(\varepsilon_r+\varepsilon_s)$ (resp. $\beta = \sum_{1\leqslant r<s\leqslant n} e_{r,s}(\varepsilon_r+\varepsilon_s)$) with $e_{r,s}\geqslant 0$.

Lemma 6.1.1. *For any $\beta\in\pi_n$ with $|\beta| = 2k\geqslant 0$, we have*

$$F_q(\beta) = \sum_{\delta\in\mathcal{C}^n_k} c^{C_n}_\delta q^k \mathcal{P}^{A_{n-1}}_q(\beta+\delta) \quad \textit{and} \quad f_q(\beta) = \sum_{\delta\in\mathcal{D}^n_k} c^{D_n}_\delta q^k \mathcal{P}^{A_{n-1}}_q(\beta+\delta).$$

Proof. We have:

$$\prod_{1\leqslant i<j\leqslant n} \frac{1}{\left(1-q\frac{x_i}{x_j}\right)} \prod_{1\leqslant r\leqslant s\leqslant n} \frac{1}{\left(1-\frac{q}{x_r x_s}\right)} = \sum_{\eta\in\pi_n}\sum_{\delta\in\pi_n} c^{C_n}_\delta q^{|\delta|/2}\mathcal{P}^{A_{n-1}}_q(\eta)x^{\eta-\delta}$$

which implies the equality $F_q(\beta) = \sum_{\eta-\delta=\beta} c^{C_n}_\delta q^{|\delta|/2}\mathcal{P}^{A_{n-1}}_q(\eta)$. Since $\mathcal{P}^{A_{n-1}}_q(\eta) = 0$ when $|\eta|\neq 0$, we can suppose $|\eta| = 0$ and $|\delta| = |\beta|$ in the previous sum. Then $\delta\in\mathcal{C}^n_k$ and the result follows immediately. The proof for $f_q(\beta)$ is similar. $\square$

6.2. Expressions of the multiplicities $u_{\lambda,\mu}$ and $U_{\lambda,\mu}$ in terms of Kostka numbers

Suppose that ξ and γ belong to π_n. Then we can define the polynomial

$$K_{\xi,\gamma}^{A_{n-1}}(q)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\mathcal{P}_q^{A_{n-1}}\big(\sigma(\xi+\rho_n)-(\gamma+\rho_n)\big).$$

Note that the coefficients of $K_{\xi,\gamma}^{A_{n-1}}(q)$ may be negative. When $\xi=\lambda$ is a partition, $K_{\lambda,\gamma}^{A_{n-1}}=K_{\lambda,\gamma}^{A_{n-1}}(1)$ is equal to the dimension of the weight space of weight γ in $V(\lambda)$. When $\gamma=\mu$ is a partition, we have

$$\begin{cases} K_{\xi,\mu}^{A_{n-1}}(q)=(-1)^{l(\tau)}K_{\nu,\mu}^{A_{n-1}}(q) & \text{if } \xi=\tau\circ(\nu) \text{ with } \tau\in\mathcal{S}_n \text{ and } \nu \text{ a partition},\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 6.2.1. *Consider λ, μ two partitions of length n such that $k=|\mu|-|\lambda|\geqslant 0$. Then*

$$u_{\lambda,\mu}(q)=\sum_{\delta\in\mathcal{D}_k^n}c_\delta^{D_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda,\mu-\delta}^{A_{n-1}}(q)=\sum_{\delta\in\mathcal{D}_k^n}c_\delta^{D_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda+\delta,\mu}^{A_{n-1}}(q)$$

and

$$U_{\lambda,\mu}(q)=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda,\mu-\delta}^{A_{n-1}}(q)=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda+\delta,\mu}^{A_{n-1}}(q).$$

Proof. By definition we have

$$U_{\lambda,\mu}(q)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}F_q\big(\sigma(\lambda+\rho_n)-(\mu+\rho_n)\big).$$

Hence from the above lemma we derive

$$U_{\lambda,\mu}(q)=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{|\delta|/2}\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\mathcal{P}_q^{A_{n-1}}\big(\sigma(\lambda+\rho_n)-(\mu-\delta+\rho_n)\big) \tag{15}$$

which yields the first desired equality since $K_{\lambda,\mu-\delta}^{A_{n-1}}(q)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\mathcal{P}_q^{A_{n-1}}(\sigma(\lambda+\rho_n)-(\mu-\delta+\rho_n))$. For any $\sigma\in\mathcal{S}_n$, we have $\sigma(\mathcal{C}_k^n)=\mathcal{C}_k^n$ and $c_{\sigma(\delta)}^{C_n}=c_\delta^{C_n}$. Thus (15) can also be rewritten

$$\begin{aligned}U_{\lambda,\mu}(q)&=q^{|\delta|/2}\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}\mathcal{P}_q^{A_{n-1}}\big(\sigma(\lambda+\rho_n+\delta)-(\mu+\rho_n)\big)\\ &=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda+\delta,\mu}^{A_{n-1}}(q).\end{aligned}$$

The proof is similar for $u_{\lambda,\mu}(q)$. $\square$

By setting $q=1$ in the above relations we obtain the following expressions of the multiplicities $U_{\lambda,\mu}$ and $u_{\lambda,\mu}$ in terms of Kostka numbers.

Corollary 6.2.2.

$$\begin{cases} U_{\lambda,\mu} = \sum_{\delta\in\mathcal{C}_k^n} c_\delta^{C_n} K_{\lambda,\mu-\delta}^{A_{n-1}} = \sum_{\delta\in\mathcal{C}_k^n} c_\delta^{C_n} K_{\lambda+\delta,\mu}^{A_{n-1}}, \\ v_{\lambda,\mu} = \sum_{\delta\in\mathcal{D}_k^n} c_\delta^{D_n} K_{\lambda,\mu-\delta}^{A_{n-1}} = \sum_{\delta\in\mathcal{D}_k^n} c_\delta^{D_n} K_{\lambda+\delta,\mu}^{A_{n-1}}. \end{cases}$$

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