Annals of Pure and Applied Logic 51 (1991) 215–240
North-Holland

The strength of the failure of the Singular Cardinal Hypothesis

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Communicated by T. Jech
Received 28 March 1990

Abstract

Gitik, M., The strength of the failure of the Singular Cardinal Hypothesis, Annals of Pure and Applied Logic 51 (1991) 215–240.

We show that $o(\kappa)=\kappa^{++}$ is necessary for $\neg$SCH. Together with previous results it provides the exact strength of $\neg$SCH.

0. Introduction

The singular cardinal hypothesis (SCH) is a descendant of the generalized continuum hypothesis. It states that $\kappa^{\mathrm{cf}\,\kappa}=\kappa^{+}+2^{\mathrm{cf}\,\kappa}$ for a singular cardinal κ. In particular, a power of a singular strong limit cardinal κ is always κ^{+}. We refer to [2, 5, 7, 8, 9, 10, 11, 14, 16] for the motivation and previous results.

Our aim will be to show the following:

Main Theorem. *Assume that* $\neg(\exists\alpha\, o(\alpha)=\alpha^{++})$. *Let* $\kappa>2^{\aleph_0}$ *be a singular cardinal. Then* $\mathrm{pp}(\kappa)=\kappa^{+}$.

The strength of $\neg$SCH can be deduced from the Main Theorem and the following theorem of Shelah.

Theorem (Shelah). *Suppose that κ is the least singular cardinal satisfying* $\kappa^{\mathrm{cf}\,\kappa}>\kappa^{+}+2^{\mathrm{cf}\,\kappa}$. *Then* $\mathrm{pp}(\kappa)\geqslant\kappa^{++}$; $\mathrm{cf}\,\kappa=\aleph_0$ *and for every* $\mu<\kappa$, $\mu^{\aleph_0}\leqslant\mu^{+}+2^{\aleph_0}$.

The case $\kappa<\aleph_\kappa$ appears in [16] and the general one in [17].

Corollary (to the Shelah Theorem and Main Theorem). *The strength of* $\neg$SCH *is at least* "$\exists\alpha\, o(\alpha)=\alpha^{++}$".

[5] provides the opposite direction.

Previously, Mitchell [14] showed that strength of $\neg$SCH is at least "$\exists\kappa\ \{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in κ" and the strength of existence of a cardinal of uncountable cofinality violating SCH is at least "$\exists\alpha\, o(\alpha) = \alpha^{++}$".

Our proof relies heavily on the Covering Lemma of Mitchell and uses some of the ideas of Shelah developed for studying the cardinal arithmetic. We are grateful to both of them for sharing with us (directly or indirectly) their deep insights. We would also like to thank Mitchell and the referee of the paper for various corrections and suggestions they made reading an earlier version of the paper.

The paper is organized as follows. In Section 1 the main technical definitions will be given, the Mitchell Covering Lemma and some of the facts following from it will be stated. Section 2 is devoted to the proof of the Main Theorem. Some generalizations and further directions will be discussed at the end of this section. In Section 3 we present some forcing constructions related to cases which appear in Section 2 and to a question of Mitchell about existence of accumulation points.

1. Preliminaries

Let κ be a singular cardinal of cofinality ω. Let $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ be a sequence of cardinals below κ and let D be a filter over ω. Define an order on $\prod_{n<\omega} \kappa_n$ as follows:

$$g <_D f \quad \text{iff} \quad \{n \mid g(n) < f(n)\} \in D.$$

If D is the filter of cobounded sets, then let us denote $<_D$ by $<$.

A set $A \subseteq \prod_{n<\omega} \kappa_n$ is called unbounded for D if for every $g \in \prod_{n<\omega} \kappa_n$ there is $f \in A, f_D > g$. The true cofinality $\operatorname{tcf}(\langle \prod_{n<\omega} \kappa_n, <_D \rangle)$ is the least λ such that there exists an unbounded linear ordered subset A of $\prod_{n<\omega} \kappa_n$ of cardinality λ.

Note that, if D is an ultrafilter, then the true cofinality is always defined. The following notions were introduced by Shelah [18] in order to refine the usual power set:

$$\operatorname{pp}^*_D(\kappa) = \sup\Big\{ \operatorname{tcf}\Big(\Big\langle \prod_{n<\omega} \kappa_n, <_D \Big\rangle\Big) \;\Big|\; \kappa_i = \operatorname{cf} \kappa_i < \kappa, \text{ for every } \mu < \kappa$$
$$\{i \mid \kappa_i \geq \mu\} \in D \text{ and } \Big\langle \prod_{i<\omega} \kappa_i, <_D \Big\rangle \text{ has a true cofinality} \Big\}$$

$$\operatorname{pp}(\kappa) = \sup\{\operatorname{pp}^*_D(\kappa) \mid D \supseteq \text{cobounded subset of } \omega\}.$$

The proof will heavily rely on the Covering Lemma of Mitchell for the core model with the maximal sequences of measures $\mathcal{K}(\mathcal{F})$ and the properties of models appearing in it. Let us only state some basic definitions and the facts that we are going to use. We refer to the Mitchell papers' [11–15] for a detailed presentation.

The Mitchell Covering Lemma. *Let $N < H_\lambda$ (for some $\lambda \geq \kappa^+$) be such that*

(a) ${}^\omega N \subseteq N$,

(b) $|N| < \kappa$,

(c) $N \cap \kappa$ *is cofinal in* κ.

Then there are a function $h^N \in \mathcal{K}(\mathcal{F})$, an ordinal $\delta^N < |N|^+$ and a system of indiscernibles $\mathbb{C}^N$ such that $N \cap H_\kappa \cap \mathcal{K}(\mathcal{F})$, $N \cap \mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F}) \subseteq h^{N\prime\prime}(\delta^N, \mathbb{C}^N)$.

Fact 1(a). *$\mathbb{C}^N$ is a function with domain a subset of the domain of $\mathcal{F} \upharpoonright \kappa + 1$. For every $(\alpha, \beta) \in \operatorname{dom} \mathbb{C}^N$, $\mathbb{C}^N(\alpha, \beta)$ is a subset of α so that the following holds*

$$\forall f \in \mathcal{K}(\mathcal{F})\, \exists \delta < \alpha\, \forall \nu \in \mathbb{C}^N(\alpha, \beta) \setminus \delta\, \forall x \in f''\nu \quad \nu \in x \leftrightarrow x \cap \alpha \in \mathcal{F}(\alpha, \beta).$$

Further we shall confuse $\mathbb{C}^N$ and $\bigcup \{\mathbb{C}^N(\alpha, \beta) \mid (\alpha, \beta) \in \operatorname{dom} \mathbb{C}^N\}$. Elements of $\mathbb{C}^N$ (i.e. of $\mathbb{C}^N(\alpha, \beta)$ for some (α, β)) are called indiscernibles.

Note that $\mathbb{C}^N \upharpoonright \kappa$ is a system of indiscernibles of the mouse related to N, but over κ itself $\mathbb{C}^N(\alpha, \beta)$ is connected with the measure $\mathcal{F}(\kappa, \beta)$ of $\mathcal{K}(\mathcal{F})$ rather than those of the mouse. Thus $\mathbb{C}^N \upharpoonright \{\kappa\}$ is what is called in [15] the maximal system of indiscernibles of N. Also instead of dealing with the Skolem function of mice deal only with its restriction to $H_\kappa \times H_\kappa$ and replace the ordinal values of it above κ by assignments introduced in [15]. Let us not give the definition of these notions but instead state the properties of such 'combined' functions h^N that we are going to use further.

Fact 1(b). (i) *The ordinal values of $h^N \upharpoonright \kappa$ are in* $\max(\kappa, o^{\mathcal{F}}(\kappa))$.

(ii) *If $c \in \mathbb{C}^N(\alpha, \beta)$ and $x \in h^{N\prime\prime}(c \cap N)$, then $c \in x$ iff $c \in \mathcal{F}(\alpha, \beta)$.*

(iii) *If $c \in \mathbb{C}^N(\alpha, \beta)$, then $h^{N\prime\prime}(c) \cap (\alpha \setminus c) = \emptyset$.*

(iv) *If $\delta \in \kappa \cap N \setminus \mathbb{C}^N$, then for some $\delta_1, \ldots, \delta_n \in \mathbb{C}^N \cap \delta$, $\delta = h^N(\delta_1, \ldots, \delta_n)$.*

(v) *For every $c \in \mathbb{C}^N$ there exists a unique pair $(\alpha, \beta) \in h^{N\prime\prime}c$ such that $c \in \mathbb{C}^N(\alpha, \beta)$. Actually $\alpha \in h^{N\prime\prime}(c \cap N)$ and if $\alpha < \kappa$ then also $\beta \in h^{N\prime\prime}(c \cap N)$. Note that in this presentation for $\alpha = \kappa$ this β may be not in N. Let us denote α by $\alpha^N(c)$ and β by $\beta^N(c)$.*

(vi) *If $c \in \mathbb{C}^N(\alpha', \beta')$ for some $\alpha' \neq \alpha^N(c)$, then $\alpha' \in \mathbb{C}^N(\alpha^N(c), \beta'')$ for some $\beta'' > \beta^N(c)$ such that $\beta' = c(\alpha^N(c), \beta^N(c), \beta'')(\alpha')$, where $c(\text{-}, \text{-}, \text{-})$ is the coherence function for $\mathcal{F}$, i.e., $c(\alpha, \beta, \gamma)$ is the least function f in $\mathcal{K}(\mathcal{F})$ so that $[f]_{\mathcal{F}(\alpha, \gamma)} = \beta$.*

(vii) *If $f : \kappa \to \sigma^{\mathcal{F}}(\kappa)$ belongs to $N \cap \mathcal{K}(\mathcal{F})$, then there is some $\tau < \kappa$ so that $f(\xi) = h^N(\langle \tau, \xi \rangle)$ for every $\xi < \kappa$. Let us call further such a τ a support of f in N. Notice that τ need not be in N.*

This property is not stated explicitly in [14, 15] but it follows easily using the techniques of this paper. Proceed as follows. Let F denote the set $N \cap \{f \mid f : \kappa \to \sigma^{\mathcal{F}}(\kappa)\}$. Using the Covering Lemma, find $F^* \in \mathcal{K}(\mathcal{F}) \cap \{f \mid f : \kappa \to \sigma^{\mathcal{F}}(\kappa)\}$ containing F and of cardinality κ. Let $d : \kappa \to F$ be some function in $\mathcal{K}(\mathcal{F})$. Plug it into h^N.

Fact 2. *Let N, N' be models as in the Covering Lemma* (further, we shall deal only with such models, so N, N', M, etc., will always denote such models). *Then*

(1) $\{c \in N \cap N' \mid c \in \mathbb{C}^N \setminus \mathbb{C}^{N'}\}$ *is finite.*

(2) $\{\alpha \mid \exists c \in \mathbb{C}^N \cap \mathbb{C}^{N'}\ \alpha = \alpha^N(c) \neq \alpha^{N'}(c)\}$ *is finite.*

(3) $\{\alpha \mid \exists c \in \mathbb{C}^N \cap \mathbb{C}^{N'}\ \alpha = \alpha^N(c) = \alpha^{N'}(c)$ *and* $\beta^N(c) \neq \beta^{N'}(c)\}$ *is finite.*

(4) *If D is an unbounded subset of α, $c \in \mathbb{C}^N(\alpha, \beta_c)$ for each $c \in D$, and D and $\langle \beta_c \mid c \in D\rangle$ are in both N and N', then $\{c \in D \mid c \notin \mathbb{C}^{N'}(\alpha, \beta_c)\}$ is bounded in α.*

(5) *If $\langle c_\nu \mid \nu < \delta\rangle \in N$ is an increasing sequence of indiscernibles, $c = \bigcup_{\nu<\delta} c_\nu$ and $c_\nu \in \mathbb{C}^N(\alpha, \beta_\nu)$ for each ν where $\{\beta_\nu \mid \nu < \delta\} \in N$ is a nondecreasing sequence, then either $c = \alpha$ or $c \in \mathbb{C}^N(\alpha, \beta)$ for some β such that $\beta > \beta_\nu$ for all $\nu < \delta$.*

Set $s^N(\alpha, \beta, \gamma) = \min(\mathbb{C}^N(\alpha, \beta) \setminus (\gamma + 1))$. It is called the least indiscernible function. An indiscernible $c \in N$ is called an accumulation point for (α, β) if for every $\nu \in N \cap c$ for every $\gamma \in \beta \cap N \cap h^{N''}c$ there are an indiscernible $c' \in N$ and an ordinal β' such that $\nu < c' < c$, $\gamma \leq \beta' < \beta$ and $c' \in \mathbb{C}^N(\alpha, \beta')$. Further, by (α, β)-accumulation point we shall mean an accumulation point for (α, β) which is not an accumulation point for $(\alpha, \beta + 1)$. $a^N(\alpha, \beta, \gamma)$ denotes the least (α, β)-accumulation point above γ.

Fact 3. *Let N, N' be two models as in the Covering Lemma. Then there is $\xi < \kappa$ so that for every $(\alpha, \beta, \gamma) \in N \cap N'$ with $\gamma > \xi$*

$$s^N(\alpha, \beta, \gamma) = s^{N'}(\alpha, \beta, \gamma) \quad \text{and} \quad a^N(\alpha, \beta, \gamma) = a^{N'}(\alpha, \beta, \gamma).$$

2. The proof of the Main Theorem

We are going to prove a slightly more general statement. By the theorem of Shelah, stated in the beginning, it will easily imply the Main Theorem.

Theorem A. *Let $\kappa > 2^{\aleph_0}$ be a singular cardinal of cofinality $\aleph_0$. Suppose that* $\mathrm{pp}(\kappa) > \kappa^+$. *Then one of the following two conditions holds.*

(1) $o(\kappa) = \kappa^{++}$ *in an inner model.*

(2) *There are unboundedly many cardinals $\mu < \kappa$ so that for a regular cardinal δ which is a limit of measurable in $\mathcal{K}(\mathcal{F})$ cardinals $\mu^+ < \delta < \mu^\omega$.*

Remark. The condition (2) seems to be much stronger than just $o(\kappa) = \kappa^{++}$. With an appropriate generalization of the Mitchell Covering Lemma to models with extenders, (2) should imply the existence of an extender of measurable length in an inner model.

Proof. Suppose otherwise. Then by [18], there are an increasing sequence of regular cardinals $\langle \kappa_n \mid n < \omega\rangle$ with limit κ and an ultrafilter D containing all cobounded subsets of ω so that $\mathrm{tcf}(\prod_{n<\omega} \kappa_n, <_D) = \kappa^{++}$. Let $\langle f_\alpha \mid \alpha < \kappa^{++}\rangle$ be a sequence witnessing this. It is possible to replace D by the filter of cobounded sets. It follows from [20] or [1], or just force an ω-sequence almost contained in every set in D. Since it can be done by ccc forcing, nothing above would be effected.

So, let us assume that D is just the filter of cobounded subsets of ω.

In the following, by a model we shall always mean a model as in the Covering Lemma which contains $\langle \kappa_n \mid n < \omega \rangle$. For a model N let us denote by ch^N the characteristic function of N, i.e.,

$$\mathrm{ch}^N(n) = \sup(N \cap \kappa_n) \quad \text{for every } n < \omega.$$

Up to Claim 22 we can restrict ourselves to models of cardinality $2^{\aleph_0}$.

Before proceeding further, let us describe the scheme of the proof. We shall start with any scale $\langle f_\alpha \mid \alpha < \kappa^{++} \rangle$, i.e., a sequence witnessing $\mathrm{tcf}(\prod_{n<\omega} \kappa_n, <) = \kappa^{++}$. Then it will be replaced from time to time by better and better scales. That is, first, by sequences of characteristic functions of models like in the Covering Lemma. Such a function ch^N will consist of limit indiscernibles $\langle \mathrm{ch}^N(n) \mid n < \omega \rangle$ with $\alpha^{N'}(\mathrm{ch}^N(n)) \geq \kappa_n$ where $N' \supseteq N \cup \{\mathrm{ch}^N, h^N\}$. The fact that the number of h^N's is small is crucial for this.

Then the proof splits into two cases. The first deals with so-called independent sequences of indiscernibles. Intuitively this means that indiscernibles (or at least many of them) in the intervals (κ_n, κ_{n+1}) are not connected with indiscernibles below κ_n. A little bit more precisely, a sequence $\langle c_n \mid n < \omega \rangle$ of indiscernibles is independent if $\kappa_{n-1} < c_n < \kappa_n$ and the index of the measure for which c_n is indiscernible, i.e. $\beta(c_n)$, does not depend on $\langle c_0, \ldots, c_{n-1} \rangle$. The typical example for this is the situation when $o^{\mathcal{F}}(\kappa_n) \geq \kappa_n$ and the indiscernibles for all the measures appear. Thus a sequence $\langle c_n \mid n < \omega \rangle$, $\kappa_{n-1} < c_n < \kappa_n$ of indiscernibles such that $\beta(c_n) = 0$, is independent. Using Fact 4, it will be shown that the number of independent sequences is κ^+. And then the contradiction will be derived. Here is actually the place where we are using the fact that the number of functions in $\mathcal{K}(\mathcal{F})$ from κ to $o(\kappa)$ is small. Note that the model of [5] constructed using $o(\kappa) = \kappa^{++}$, has κ^{++} independent sequences. The same is true about the models of [21] and [6] with wider gaps between κ and 2^κ. The number of independent sequences in this model is 2^κ.

The second and actually the main case is the case when the indiscernibles in (κ_n, κ_{n+1}) are connected with indiscernibles below κ_n. The typical situation here is as follows: $o^{\mathcal{F}}(\kappa_{n+1}) = \kappa_n$ and indiscernibles for all the measures appear. The scale of characteristic functions here will be replaced by a better one. It will consist of so-called diagonal sequences. Intuitively, this means a function $f \in \prod_{n<\omega} \kappa_n$, so that for some model N with $f \in N$, $\langle f(n) \mid n < \omega \rangle$ is a sequence of indiscernibles of N so that $\alpha^N(f(n)) = \kappa_n$, $\beta^N(f(n)) = f(n-1)$ and $f(n) = s^N(\kappa_n, f(n-1), \kappa_{n-1})$, i.e., $f(n)$ is the least indiscernible above κ_{n-1} for measure $\mathcal{F}(\kappa_n, f(n-1))$. Note that such functions are actually 'the trouble makers', since a disagreement between two such functions in the beginning cannot be fixed later.

A diagonal function f is called faithful if $\mathrm{cf}(f(n)) = \mathrm{cf}(f(n-1))$ for all but finitely many n's. It will be shown that it is possible to construct a scale $\langle f_\alpha \mid \alpha < \kappa^{++} \rangle$ consisting of faithful diagonal functions. Then for δ's below κ^{++} of cofinality κ^+, a least upper bounds f^*_δ of $\langle f_\alpha \mid \alpha < \delta \rangle$ will be considered. There will be only three possibilities for such f^*_δ. Namely:

(a) f^*_δ is a faithful diagonal sequence;

(b) f^*_δ is a diagonal function but it is not faithful;

(c) $\langle f^*_\delta(n) \mid n < \omega \rangle$ is a sequence of accumulation points.

Possibility (a) can be ruled out immediately, since $\gamma^\omega < \kappa$ for every $\gamma < \kappa$. The possibilities (b) and (c) are treated similarly in Claims 22 and 23. We consider $(2^{\aleph_0})^+$ such δ's and produce an increasing sequence $\langle f^*_{\delta_i}(n) \mid i < \omega_1 \rangle$ for infinitely many n's. Then $\gamma_n = \sup\{f^*_{\delta_i}(n) \mid i < \omega_1\}$ will be a regular cardinal in $\mathscr{K}(\mathscr{F})$ with $o^{\mathscr{F}}(\gamma_n) < \kappa_{n-1}$, for unboundedly many n's. This will lead to the contradiction.

Claim 1. *For every set A of cardinality less than κ there exists a sequence of models $\langle N_\alpha \mid \alpha < \kappa^{++} \rangle$ so that*

(1) *$A \in N_\alpha$ for every $\alpha < \kappa^{++}$;*

(2) *for every $\alpha, \beta < \kappa^{++}$, $\delta^{N_\alpha} = \delta^{N_\beta}$ and $h^{N_\alpha} \cap H_\kappa = h^{N_\beta} \cap H_\kappa$;*

(3) *$\langle \mathrm{ch}^{N_\alpha} \mid \alpha < \kappa^{++} \rangle$ witnesses $\mathrm{tcf}(\langle \prod_{n<\omega} \kappa_n, < \rangle) = \kappa^{++}$.*

Proof. Let $\langle f_\alpha \mid \alpha < \kappa^{++} \rangle$ be any sequence witnessing that the true cofinality of $\langle \prod_{n<\omega} \kappa_n, < \rangle$ is κ^{++}. Define by induction an increasing sequence of ordinals below κ^{++}, $\langle \beta_\alpha \mid \alpha < \kappa^{++} \rangle$ and a sequence of models as follows:

(a) $A \in N_\alpha$,

(b) $\beta_\alpha = \min\{\gamma < \kappa^{++} \mid f_\gamma > ch^{N_\nu} \text{ for every } \nu < \alpha\}$,

(c) $f_{\beta_\alpha} \in N_\alpha$.

Clearly, $\mathrm{ch}^{N_\alpha} > f_{\beta_\alpha}$ and since $\langle f_{\beta_\alpha} \mid \alpha < \kappa^{++} \rangle$ witnesses the true cofinality of $\langle \prod_{n<\omega} \kappa_n, < \rangle$, $\langle \mathrm{ch}^{N_\alpha} \mid \alpha < \kappa^{++} \rangle$ will be such as well. Now, the number of δ^{N_α}'s and h^{N_α}'s is small. So for some $S \subseteq \kappa^{++}$ of cardinality κ^{++} they all are the same. Then $\langle N_\alpha \mid \alpha \in S \rangle$ is as desired. □ Claim 1

Claim 2. *For every $g \in \prod_{n<\omega} \kappa_n$ there exists N and a sequence of indiscernibles $\langle c_n \mid n < \omega \rangle$ in N so that*

(1) *$\langle c_n \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_n$;*

(2) *$\alpha^N(c_n) \geqslant \kappa_n$;*

(3) *$c_n \geqslant g(n)$ for all but finitely many n's.*

Proof. Let $g \in \prod_{n<\omega} \kappa_n$. Let $\langle N_\alpha \mid \alpha < \kappa^{++} \rangle$ be as in Claim 1, with $g \in N_\alpha$ for every $\alpha < \kappa^{++}$. Let $h = h^{N_\alpha}$ for every $\alpha < \kappa^{++}$. Assume for simplicity that $\delta^{N_\alpha} = 0$ $(\alpha < \kappa^{++})$.

Now, define a sequence $\langle \xi_n \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_n$. Let $n < \omega$. ξ_n would be the limit of the sequence $\langle \xi^i_n \mid i < \omega \rangle$ which is defined as follows: $\xi^0_n = g(n)$,

$\xi_n^{i+1} = \sup(h''(\xi_n^i) \cap \kappa_n)$ for every $i < \omega$. Since κ_n is a regular cardinal and $g(n) < \kappa_n$, all ξ_n^i's and ξ_n will be below κ_n. Clearly, $\sup(h''(\xi_n) \cap \kappa_n) = \xi_n$.

Let $n < \omega$. Set $c_n = \min(N_\gamma \setminus \xi_n)$. Then $c_n < \kappa_n$. Also c_n is an indiscernible in N_γ with $\alpha^{N_\gamma}(c_n) \geq \kappa_n$. Since, otherwise $\sup(h''(N_\gamma \cap c_n) \cap \kappa_n) > c_n$, but $N_\gamma \cap c_n \subseteq \xi_n$, $\xi_n \leq c_n$ and $\sup(h''(\xi_n) \cap \kappa_n) = \xi_n$. So the sequence $\langle c_n \mid n < \omega \rangle$ satisfies the conclusion of the claim. □ Claim 2

The following is an easy consequence of Claim 2.

Claim 3. *All but finitely many of κ_n's are inaccessible in $\mathcal{K}(\mathcal{F})$.*

Proof. Suppose otherwise. Let $A \subseteq \omega$ be an infinite set so that for every $n \in A$, $\kappa_n = (\lambda_n^+)^{\mathcal{K}(\mathcal{F})}$. Define a function $g \in \prod_{n<\omega} \kappa_n$ as follows:

$$g(n) = \begin{cases} \lambda_n & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let N be a model for g as in the previous claim. Then there would be indiscernibles in N between λ_n and $\kappa_n = (\lambda_n^+)^{\mathcal{K}(\mathcal{F})}$ for infinitely many n's in A, which is impossible. Contradiction. □ Claim 3

Claim 4. *There exists a function $f \in \prod_{n<\omega} \kappa_n$ and a sequence $\{\alpha_n \mid n < \omega\rangle$ so that for every N for every sequence of indiscernibles $\langle c_n \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_n$ if $c_n > f(n)$ and $\alpha^N(c_n) \geq \kappa_n$ for all but finitely many n's, then*

$$\alpha^N(c_n) = \alpha_n \quad \textit{for all but finitely many } n\textit{'s}.$$

Proof. Let N be a model. Note that if $c < c'$ are two indiscernibles in N and $\alpha^N(c) > c'$, then $\alpha^N(c) \geq \alpha^N(c')$ since $\alpha^N(c) \in h^{N''}(c)$, $\alpha^N(c') \in h^{N''}(c')$ and $h^{N''}(c') \cap [c', \alpha^N(c')] = \emptyset$.

Define now $f \in \prod_{n<\omega} \kappa_n$ and the sequence $\langle \alpha_n \mid n < \omega \rangle$ as follows:

Let $\alpha_n = 0 = f(n)$ unless there is an indiscernible $c \in N$, $\kappa_{n-1} \leq c < \kappa_n$ with $\alpha^N(c) \geq \kappa_n$. In the last case set α_n to be the minimal value $\alpha^N(c)$ for $c \in N$, $\kappa_{n-1} \leq c < \kappa_n$ and $\alpha^N(c) \geq \kappa_n$. Let $f(n) = c$ for c, $\kappa_{n-1} \leq c < \kappa_n$, with $\alpha^N(c) = \alpha$.

Let us show that the sequence $\langle \alpha_n \mid n < \omega \rangle$ is as desired. Suppose otherwise. Let N', $\langle c_n \mid n < \omega \rangle$ be witnessing this. Denote $\alpha^{N'}(c_n)$ by α_n' and $\beta^N(c_n)$ by β_n'. Let A be an infinite set so that $n \in A$ implies $\alpha_n \neq \alpha_n'$. Then, by Fact 1, for every $g \in \mathcal{K}(\mathcal{F})$ there exists $n < \omega$ so that for every $m \geq n$ for every $X \in g''(c_m)$, $c_m \in X$ iff $X \cap \alpha_m' \in \mathcal{F}(\alpha_m', \beta_m')$.

So H_{κ^+} satisfies the following statement:

$$\exists A \subset \omega \text{ infinite } \exists \langle c_n \mid n < \omega \rangle \; \exists \langle \alpha_n' \mid n < \omega \rangle \; \exists \langle \beta_n' \mid n < \omega \rangle \; \exists h' \in \mathcal{K}(\mathcal{F})$$
$$\forall n < \omega \; \alpha_n', \beta_n' \in h'''(c_n') \wedge \forall n \in A \; (c_n > f(n) \wedge \alpha_n' \geq \kappa_n \wedge \alpha_n' \neq \alpha_n)$$
$$\wedge \; (\forall g \in \mathcal{K}(\mathcal{F}) \; \exists n < \omega \; \forall m \geq n \; \forall X \in g''(c_m)(c_m \in X \leftrightarrow X \cap \alpha_m' \in \mathcal{F}(\alpha_m', \beta_m'))).$$

Since N is an elementary submodel of H_{κ^+} and the parameters are in N, this statement is true in N. Let A^*, $\langle c_n^* \mid n<\omega\rangle$, $\langle \alpha_n^* \mid n<\omega\rangle$, $\langle \beta_n^* \mid n<\omega\rangle$, $h^* \in N$ be so that N satisfies the following:

(a) for every $n \in A^*$, $c_n^* > f(n)$, $\alpha_n^* \geq \kappa_n$ and $\alpha_n^* \neq \alpha_n$;

(b) for every $g \in \mathcal{H}(\mathcal{F})$ for some $n<\omega$ for every $m \geq n$ for every $X \in g''(c_m^*)$, $c_m^* \in X$ iff $X \cap \alpha_m^* \in \mathcal{F}(\alpha_m^*, \beta_m^*)$;

(c) $h^* \in \mathcal{H}(\mathcal{F})$ and for every $n<\omega$, $\langle \alpha_m^*, \beta_m^*\rangle \in h^{*''}(c_m^*)$.

Then (a), (b) and (c) are true in H_{κ^+}. In particular applying (b) for h^N there exists $n<\omega$ so that for every $m \geq n$ for every $X \in h^{N''}(c_m^*)$, $c_m^* \in X$ iff $X \cap \alpha_m^* \in \mathcal{F}(\alpha_m^*, \beta_m^*)$. It implies that every c_m^* with $m \geq n$ is an indiscernible in N. Since otherwise $c_m^* = h^N(\vec{c}_m)$ for some $\vec{c}_m$ strictly below c_m^*. Hence $c_m^* \in h^{N''}(c_m^*)$ and so $\kappa \setminus c_m^* \in h^{N''}(c_m^*)$. But, clearly, $\alpha_m^* \setminus c_m^* \in \mathcal{F}(\alpha_m^*, \beta_m^*)$ which provides a contradiction.

Now, let $m \in A \setminus n$ be above a support of h^*. Then $\alpha_m^* \in h^{*''}(c_m^*)$ and hence $\alpha_m^* \in h^{N''}(c_m^*)$. This together with (b) implies that $\alpha_m^* = \alpha^N(c_m^*)$. Recall now the definition of $f(m)$ and α_m. Since $\alpha^N(c_m^*) \geq \kappa_m$, $f(m)$ is chosen to be an indiscernible c with $\alpha^N(c)$ least possible $\geq \kappa_m$ and $\alpha_m = \alpha^N(c)$. So, $f(m) = c < c_m^*$ then $\alpha_m \leq \alpha_m^*$. But then, necessarily $\alpha_m = \alpha_m^*$, which is impossible. Contradiction. □ Claim 4

Let us assume for simplication of notation that $f(n) = 0$ for all n's, since we can restrict ourselves to the functions above f.

Claim 5. *For every N there exists $N^* \supseteq N$ so that for every $N' \supseteq N^* \cup \{h^{N^*}, \mathrm{ch}^{N^*}\}$, $\mathrm{ch}^{N^*}(n)$ is an indiscernible in N' for α_n for all but finitely many n's.*

Proof. Suppose otherwise. Let N be a model witnessing this. Define by induction a sequence of models $\langle N_i \mid i<\omega_1\rangle$ so that

(0) $N_0 = N$;

(1) for every $i<j<\omega_1$, $N_j \supseteq N_i \cup \{h^{N_i}, \mathrm{ch}^{N_i}\}$;

(2) for every $i<\omega_1$, N_{i+1} contains a sequence of indiscernibles $\langle c_n^i \mid n<\omega\rangle$ so that $\alpha^{N_{i+1}}(c_n^i) = \alpha_n$ and $c_n^i > \mathrm{ch}^{N_i}(n)$ for all but finitely many n's.

There is no problem in the induction. Use Claims 2, 4 in order to satisfy (2). Set $N^* = \bigcup_{i<\omega_1} N_i$. Let $N' \supseteq N^* \cup \{h^{N^*}, \mathrm{ch}^{N^*}\}$. By Fact 2, for every $i<\omega_1$ there is $n(i)$ so that for every $n \geq n(i)$, c_n^i is an indiscernible in N' and $\alpha^{N^*}(c_n^i) = \alpha^{N_{i+1}}(c_n^i) = \alpha_n$. Let $S \subseteq \omega_1$, $|S| = \aleph_1$, $n^* < \omega$ be so that $n(i) = n^*$ for every $i \in S$. Then $\mathrm{ch}^{N^*}(n) = \bigcup_{i\in S} c_n^i$, so $\langle \mathrm{ch}^{N^*}(n) \mid n \geq n^*\rangle$ are indiscernibles in N'. By Claim 4, $\alpha^{N'}(\mathrm{ch}^{N^*}(n)) = \alpha_n$, for every $n \geq n^*$. Contradiction. □ Claim 5

Further let us restrict ourselves only to models N^* like in Claim 5, i.e., $\mathrm{ch}^{N^*}(n)$ is a limit indiscernible for α_n.

Claim 6. *There exists a function $f \in \prod_{n<\omega} \kappa_n$ satisfying the following:*
$(*)$ *for every N there is $N^* \supset N$ so that for every $N' \supset N^* \cup \{h^{N^*}, \mathrm{ch}^{N^*}\}$, $\beta^{N'}(\mathrm{ch}^{N^*}(n)) \in h^{N''}(f(n))$ for all but finitely many n's.*

Proof. Suppose otherwise, let us define by induction sequences $\langle f_i \mid i < \omega_1 \rangle$ and $\langle N_i \mid i < \omega \rangle$ as follows:

(1) for every $i < j$, $N_j \supseteq N_i \cup \{h^{N_i}, \mathrm{ch}^{N_i}\}$;
(2) for every $i < j$ for every $n < \omega$, $\max(\mathrm{ch}^{N_i}(n), f_i(n)) < f_j(n)$;
(3) for every $i < \omega_1$, $f_i \in \prod_{n<\omega} \kappa_n$;
(4) for every $i < \omega_1$, N_i witnesses the failure of $(*)$ for f_i.

Let $N^* = \bigcup_{i<\omega_1} N_i$ and $N' \supseteq N^* \cup \{h^{N^*}, \mathrm{ch}^{N^*}\}$. Then $\mathrm{ch}^{N^*}(n) = \bigcup_{i<\omega_1} f_i(n)$ for every n. Let $i_n < \omega_1$ be such that $\beta^{N'}(\mathrm{ch}^{N^*}(n)) \in h^{N''}(f_{i_n}(n))$, where $n < \omega$. Set $i^* = \bigcup_{n<\omega} i_n$. Then for every $n < \omega$, $\beta(\mathrm{ch}^{N^*}(n)) \in h^{N''}(f_{i^*}(n))$. But $N^* \supseteq N_{i^*}$ and N_{i^*} was picked to be a counterexample for f_{i^*}. Contradiction. □ Claim 6

Further let us deal only with models like N^* of the claim.

Claim 7. *For every N, there exists $g \in \prod_{n<\omega} \kappa_n$ so that for every $N' \supseteq N \cup \{h^N, \mathrm{ch}^N\}$, if $\langle c_n \mid n < \omega \rangle$ is a sequence of indiscernibles in N' so that for all but finitely many n's*

(a) $\alpha^N(c_n) = \alpha_n$,
(b) $c_n < \mathrm{ch}^N(n)$,
(c) $\beta^{N'}(c_n) \geq \beta^{N'}(\mathrm{ch}^N(n))$,

then $g > \langle c_n \mid n < \omega \rangle$.

Proof. Let N be a model. Pick some $N^* \supseteq N \cup \{h^N, \mathrm{ch}^N\}$ containing a cofinal subset of $N \cap \mathrm{ch}^N(n)$ for every $n < \omega$. Since $\mathrm{ch}^N(n)$ is a limit indiscernible of cofinality $> \aleph_0$, for all but finitely many n's, by Fact 2(5), the indiscernibles c for α_n with $\beta^{N^*}(c) \geq \beta^{N^*}(\mathrm{ch}^N(n))$ are bounded in $\mathrm{ch}^N(n)$. Define $g(n) \in N$ to be such a bound. Let us show that this g is desired. Suppose otherwise. Let M witness this. Let $M' \supseteq M \cup N^* \cup \{h^M, \mathrm{ch}^M, h^{N^*}, h^N, \mathrm{ch}^N\}$ be so that there exists a sequence of indiscernibles $\langle c_n \mid n < \omega \rangle$ in M' and an infinite set $A \subseteq \omega$ satisfying the following for every $n \in A$:

(a) $\alpha^{M'}(c_n) = \alpha_n$,
(b) $c_n < \mathrm{ch}^N(n)$,
(c) $\beta^{M'}(c_n) \geq \beta^{M'}(\mathrm{ch}^N(n))$,
(d) $c_n \geq g(n)$.

Using Fact 2, we can assume that for every $n \in A$, $\beta^{M'}(\mathrm{ch}^N(n)) = \beta^{N^*}(\mathrm{ch}^N(n))$.
Let us proceed as in Claim 5.
The following statement is true in H_{κ^+}:

$$\exists h \in \mathscr{H}(\mathscr{F})\, \exists \langle c_n \mid n < \omega \rangle\, \exists \langle \beta_n \mid n < \omega \rangle\, \forall n \in A$$
$$(\langle \alpha_n, \beta_n \rangle \in h''(c_n) \wedge c_n < \mathrm{ch}^N(n) \wedge \beta_n \geq \beta^{N^*}(\mathrm{ch}^N(n)) \wedge c_n \geq g(n))$$
$$\wedge\, \forall t \in \mathscr{H}(\mathscr{F})\, \exists n_0\, \forall n \geq n_0\, \forall X \in t''(c_n)\, (c_n \in X \leftrightarrow X \in \mathscr{F}(\alpha_n, \beta_n)).$$

Then the same statement is true in N^*. Let h, $\langle c_n^* \mid n<\omega\rangle$, $\langle \beta_n \mid n<\omega\rangle \in N^*$ be witnessing this. Going back to H_{κ^+} with h^{N^*} instead of t, we obtain, as in Claim 4, that c_n^* is an indiscernible with $\alpha^{N^*}(c_n)=\alpha_n$ and $\beta^{N^*}(c_n^*)=\beta_n$ for every $n\in A$ big enough. But $c_n^*<\mathrm{ch}^N(n)$, $\beta_n \geqslant \beta^{N^*}(\mathrm{ch}^N(c))$ and still $c_n \geqslant g(n)$. This contradicts the choice of g. □ Claim 7

Claim 8. *There exists $g\in\prod_{n<\omega}\kappa_n$ so that for every N there exists $N'\supseteq N$ with $g^{N'}\leqslant g$, where $g^{N'}$ is a function given N' by Claim 7.*

Proof. Suppose otherwise. Define by induction sequences $\langle g_i \mid i<\omega_1\rangle$ and $\langle N_i \mid i<\omega_1\rangle$ so that

(1) for every $i<j$ for every $n<\omega$, $g_i(n)<g_j(n)$;

(2) for every $i<j$, $N_j\supseteq N_i\cup\{h^{N_i}, \mathrm{ch}^{N_i}, g^{N_i}\}$;

(3) for every $i<\omega_1$, $g_i(n)>\mathrm{ch}^{N_i}(n)$, $g^{N_i}(n)$ for every $n<\omega$;

(4) N_{i+1} witnesses the failure of g_i to satisfy the requirements of the claim.

Let $N_{\omega_1}=\bigcup_{i<\omega_1} N_i$. Since $\mathrm{ch}^{N_{\omega_1}}(n)=\bigcup_{i<\omega_1} g_i(n)$ and $g^{N_{\omega_1}}(n)<\mathrm{ch}^{N_{\omega_1}}(n)$, there exists $i^*<\omega_1$, so that $g^{N_{\omega_1}}\leqslant g_{i^*}$. But $N_{\omega_1}\supseteq N_{i^*+1}$. It contradicts condition (4). □ Claim 8

So we obtain a function $g^*\in\prod_{n<\omega}\kappa_n$ so that for every N there is $N'\supseteq N$ satisfying the following:

(*) For every $N''\supseteq N'\cup\{h^{N'}, \mathrm{ch}^{N'}\}$, if $\langle c_n \mid n<\omega\rangle$ is a sequence of indiscernibles in N'' so that for all but finitely many n's

(a) $\alpha^{N''}(c_n)=\alpha_n$,

(b) $c_n<\mathrm{ch}^{N'}(n)$,

(c) $\beta^{N''}(c_n)\geqslant\beta^{N'}(\mathrm{ch}^N(n))$,

then $g^*>\langle c_n \mid n<\omega\rangle$.

Assume that g^* is above the function of Claim 6. Further let us consider only models containing g* and satisfying (*) for this g^*. Then the following holds:

Claim 9. *Let $N'\supseteq N\cup\{h^N, \mathrm{ch}^N\}$. Then $\mathrm{ch}^N(n)=s^{N'}(\alpha_n, \beta^{N'}(\mathrm{ch}^N(n)), g^*(n))$ and $\beta^{N'}(\mathrm{ch}^N(n))\in h^{N'''}(g^*(n))$, for all but finitely many n's, where s^N is the least indiscernible function for N'.*

Let us split now the proof into two cases.

Case 1. For every $h\in\mathscr{H}(\mathscr{F})$ there is N containing h so that, for infinitely many n's, N has an indiscernible c_n such that $g^*(n)<c_n<\kappa_n$ for α_n with $\beta^N(c_n)>\sup(o^{\mathscr{F}}(\alpha_n)\cap h''(g^*(n)))$.

For example, if $\alpha_n=\kappa_n$, $\mathrm{cf}\, o^{\mathscr{F}}(\kappa_n)\geqslant\kappa_n$ and indiscernibles for all measures appear then the above holds.

Definition 10. Let $A \subseteq \omega$. A sequence $\langle \xi_n \mid n \in A \rangle$ is called independent if for some N, $\langle \xi_n \mid n < \omega \rangle$ is a sequence of indiscernibles in N satisfying the following conditions:

(1) $g^*(n) < \xi_n < \kappa_n$;

(2) $\alpha^N(\xi_n) = \alpha_n$;

(3) $\xi_n = s^N(\alpha_n, \beta^N(\xi_n), g^*(n))$;

(4) for every $n \in A$ there is $\gamma_n < \kappa_{\min A}$ so that $\beta^N(\xi_n)$ is the least $\beta \geqslant h^N(\gamma_n, \alpha_n, g^*(n))$ for which there are indiscernibles for α_n in N above $g^*(n)$.

Intuitively, this means that each ξ_n $(n \in A)$ is independent of indiscernibles below $g^*(n)$.

Claim 11. *For every $f \in \prod_{n<\omega} \kappa_n$ there are $A \subseteq \omega$, $|A| = \aleph_0$ and an independent sequence of indiscernibles $\langle \xi_n \mid n \in A \rangle$ so that for every $n \in A$, $f(n) < \xi_n$.*

Proof. Let $f \in \prod_{n<\omega} \kappa_n$. Pick N to be a model containing f. Let $N^* \supseteq N \cup \{h^N, \mathrm{ch}^N\}$. Pick $N^{**} \supseteq N^* \cup \{h^{N^*}\}$ to be as in Case 1 for $h = h^{N^*}$. Actually, any $N^{**} \supseteq N^* \cup \{h^{N^*}\}$ will be O.K. Let $A \subseteq \omega$, $|A| = \aleph_0$ be the set consisting of n's so that there exists an indiscernible c_n for α_n in N^{**} satisfying $g^*(n) < c_n < \kappa_n$ and $\beta^{N^{**}}(c_n) > \sup(o^{\mathscr{F}}(\alpha_n) \cap h^{N^*}{}''(g^*(n)))$.

For $n \in A$ denote $\sup(o^{\mathscr{F}}(\alpha_n) \cap h^{N^*}{}''(g^*(n)))$ by β_n. Then, since $h^{N^*} \in \mathscr{H}(\mathscr{F}) \cap N^{**}$, also the function $t(\gamma, \delta) = \sup(o^{\mathscr{F}}(\gamma) \cap h^{N^*}{}''\delta)$ is in $\mathscr{H}(\mathscr{F}) \cap N^{**}$. Then, by Fact 1(bvii), there is $\xi < \kappa$ so that $t(\gamma, \delta) = h^{N^*}(\xi, \gamma, \delta)$. Hence, $\beta_n = h^{N^{**}}(\xi, \alpha_n, g^*(n))$. Removing a finite subset of A if necessary, we can assume that $\kappa_{\min A}$ is above ξ and points of disagreement between N, N^*, N^{**}. Let $n \in A$. Since $\beta^{N^{**}}(\mathrm{ch}^N(n)) = \beta^{N^*}(\mathrm{ch}^N(n)) \in h^{N^*}{}''(g^*(n))$, by the choice of n and Claim 9, $\beta^{N^{**}}(\mathrm{ch}^N(n)) < \beta_n$. By the choice of g^*, then for every $\beta_n' \geqslant \beta_n$, indiscernibles for (α_n, β_n') which are above $g^*(n)$ are also above $\mathrm{ch}^N(n)$. Let $\beta_n^* \geqslant \beta_n$ be the least ordinal so that there exists an indiscernible c_n, $g^*(n) < c_n < \kappa_n$ for $\langle \alpha_n, \beta_n^* \rangle$. Set $\xi_n = s^{N^{**}}(\alpha_n, \beta_n^*, g^*(n))$. Hence $\langle \xi_n \mid n \in A \rangle$ is an independent sequence.

Then for all but finitely many n's in A, $\xi > f(n)$. □ Claim 11

Claim 12. *Let A, $\langle \xi_n \mid n \in A \rangle$, N, $\langle \gamma_n \mid n \in A \rangle$ be as in Definition* 10. *Then for every M, s.t. $\langle h^N(\gamma_n, g^*(n)) \mid n \in A \rangle \in M$, for all but finitely many n's in A, ξ_n, $\beta^N(\xi_n) \in M$ and $\xi_n = s^M(\alpha_n, \beta_n^N(\xi_n), g^*(n))$.*

Proof. Define β_n^M to be the least $\beta \geqslant h^N(\gamma_n, g^*(n))$, $\beta \in M$ for which there are indiscernibles in M above $g^*(n)$. As in Claim 4, then $\beta_n^M = \beta_n^N(\xi_n)$ for all but finitely many n's in A. Now the claim follows from Fact 3. □ Claim 12

Claim 13. *The number of independent sequences is κ^+.*

Proof. It follows from Fact 4, and Claim 12 and the cardinal arithmetic since the number of $h^N \cap \kappa \times \mathrm{On}$'s is κ^+. Recall that $h^N \cap \kappa \times \mathrm{On}$ maps a subset of κ into $\max(\kappa, o^{\mathscr{F}}(\kappa))$. □ Claim 13

Now it easy to derive the contradiction in Case 1. Let $\langle f_\alpha \mid \alpha < \kappa^{++} \rangle$ be a sequence witnessing that the true cofinality of $\langle \prod \kappa_n, < \rangle$ is κ^{++}. Using Claims 11, 13 find $S \subseteq \kappa^{++}$, $|S| = \kappa^{++}$, $A \subseteq \omega$, $|A| = \aleph_0$ and an independent sequence $\langle \xi_n \mid n \in A \rangle$ so that for every $\alpha \in S$, $n \in A$, $f_\alpha(n) < \xi_n$. It is clearly impossible since there should be $\alpha \in S$ with $f_\alpha(n) > \xi_n$ for all but finitely many n's in A.

Case 2. For some $h \in \mathscr{H}(\mathscr{F})$ for every N containing h there are only finitely many n's such that N has indiscernible c_n, $g^*(n) < c_n < \kappa_n$ for α_n with $\beta^N(c_n) \geqslant \sup(o^{\mathscr{F}}(\alpha_n) \cap h''(g^*(n)))$.

Let us fix some such function h. Further, we are not going to use the fact that h is in $\mathscr{H}(\mathscr{F})$. Even more, we shall replace h by some other functions which should not be in $\mathscr{H}(\mathscr{F})$.

Claim 14. *There exists a function $g^{**} \in \prod_{n<\omega} (g^*(n) + 1)$ so that*

(a) *for every N containing h and g^{**} there are only finitely many n's such that N has indiscernible c_n, $g^*(n) < c_n < \kappa_n$ for α_n with $\beta^N(c_n) \geqslant \sup(o^{\mathscr{F}}(\alpha_n) \cap h''(g^{**}(n)))$;*

(b) *for every $t \in \prod_{n<\omega} g^{**}(n)$ there exists N which has for infinitely many n's indiscernible c_n, $g^*(n) < c_n < \kappa_n$ for α_n so that $\beta^N(c_n) \geqslant \sup(o^{\mathscr{F}}(\alpha_n) \cap h''(t(n))))$.*

Proof. Let N be a model containing h. Define $g^{**}(n)$ to be the least element of $N \leqslant g^*(n)$ so that N has no indiscernibles c_n with $\beta^N(c_n) \geqslant \sup(o^{\mathscr{F}}(\alpha_n) \cap h''(g^{**}(n)))$. Using the argument similar to the argument of Claim 4, it is not hard to see that g^{**} satisfies (a) and (b). □ Claim 14

Replacing $g^{**}(n)$ by its cofinality, we can assume that every $g^{**}(n)$ is a regular cardinal. Notice that this may replace h by a function which is not in $\mathscr{H}(\mathscr{F})$.

Let $\langle \kappa_n^{(1)} \mid n < \omega \rangle$ be the sequence obtained from $\langle g^{**}(n) \mid n < \omega \rangle$ by removing all its members which appear in $\{\kappa_n \mid n < \omega\}$. We can use the previous argument and define $g^{(1)**}$ for $\langle \kappa_n^{(1)} \mid n < \omega \rangle$ in the same fashion as g^{**} was defined for $\langle \kappa_n \mid n < \omega \rangle$. Continue, removing from $\langle g^{(1)**}(n) \mid n < \omega \rangle$ all the members that appear in $\{\kappa_n \mid n < \omega\} \cup \{\kappa_n^{(1)} \mid n < \omega\}$. Define $\langle \kappa_n^{(2)} \mid n < \omega \rangle$, and so on. It is possible to show that the process terminates after countably many stages. Let us use a simpler argument, suggested by Mitchell which does all this at once.

Let N be a model containing h. Define $D \subseteq N$ to be the smallest set containing $\{\kappa_n \mid n < \omega\}$ such that for each $\gamma > \kappa_0$ in D if there is an ordinal $\nu \in N$ with $\kappa_0 < \nu < \gamma$ such that for some $f \in N$ the set $\{s^N(\alpha^N(\gamma), f(\xi), g^*(\gamma)) \mid \xi \in N \cap \nu\}$ is unbounded in $N \cap \gamma$, where $g^*(\gamma)$ is defined in N like g^* was defined above, then the least such ν is in D. Denote such ν by $\sigma(\gamma)$.

Clearly, D is a countable set consisting of regular in $\mathcal{H}(\mathcal{F})$ cardinals. Since ${}^{\omega}N \subseteq N$, D belongs to N. Fix for each γ in D a function $f_\gamma : \sigma(\gamma) \to o(\alpha^N(\gamma))$ as in the definition of D. Combine all f_γ's together and still denote the result by h. By Fact 3 and elementarity final segments of D do not depend on the particular N. Let $\langle \kappa_i \mid i < \nu \rangle$ be an increasing enumeration of D. (To prevent the confusion let us denote the original κ_n's by κ_n^*'s.) Then $\nu < \omega_1$ and $\kappa_{j_i} = \sigma(\kappa_i) < \kappa_i$ for some $j_i < i$. Let us view σ as a partial function on ν, i.e., $\sigma(i) = j_i$.

The next observation, which shows that the order type of D, i.e. ν, should be ω, is also due to Mitchell.

Claim 14′. $\nu = \omega$.

Proof. First note that it is impossible to have δ and an infinite set of κ_i's above it with $\kappa_{\sigma(i)} < \delta$, since then picking an N containing δ we would get a contradiction to regularity of all but finitely many of these κ_i's.

Suppose now that $\nu > \omega$. Consider $\{\kappa_i \mid i < \omega\}$. By the above there are only finitely many κ_i's with $i > \omega$ so that $\sigma(i) < \omega$. Let $\kappa_{i_1} < \cdots < \kappa_{i_n}$ be all such κ_i's. Pick $m > \max\{\sigma(i_1), \ldots, \sigma(i_n)\}$. Also assume that m is above the indexes of finitely many κ_n^*'s which are below κ_ω. But then the set $\{\kappa_i \mid m < i < \omega\}$ can be removed from D. Which contradicts its minimality. Contradiction. □ Claim 14′

By the proof of Claim 14′, for every $n < \omega$, $\sigma^{-1}(\{n\})$ is finite. Hence, by König's theorem, there is a subsequence $\langle \kappa_{i_n} \mid n < \omega \rangle$ of $\langle \kappa_i \mid i < \omega \rangle$ so that $\sigma(i_{n+1}) = i_n$ for every $n < \omega$.

Claim 15. *For every ultrafilter D over ω, containing all cobounded subsets of ω*

$$\mathrm{cf}\Big(\prod_{n<\omega} \kappa_{i_n}, <_D\Big) = \kappa^{++}.$$

Proof. Let D be an ultrafilter over ω and $\langle f_\alpha \mid \alpha < \lambda \rangle$ be a D-increasing sequence unbounded in $\langle \prod_{n<\omega} \kappa_{i_n}, <_D \rangle$. For every $\alpha < \lambda$ pick a model N_α so that $N_\alpha \supseteq \{f_\alpha\}$. Notice that if on a set in D, $\{n \mid \sigma^{-1}(\{i_n\}) = \emptyset\} \in D$, then an infinite set of original κ_n^*'s is in D. Hence $\lambda = \kappa^{++}$.

Set $A_0 = \{i_n \mid n < \omega\}$ and by induction define $A_{n+1} = \sigma^{-1\prime\prime}(A_n)$. Clearly, $\bigcup_{n<\omega} A_n$ contains infinitely many κ_n^*'s. For every $\alpha < \lambda$ let us define simultaneously $\langle t_{\alpha n} \mid n < \omega \rangle$ so that $t_{\alpha n} \in \prod_{i \in A_n} \kappa_i$. Set $t_{\alpha_0} = f_\alpha$. Suppose that $t_{\alpha m}$ is defined for every $m < n$. Define $t_{\alpha n}$ as follows. Set $t_{\alpha n}(i)$ $(i \in A_n)$ to be the least indiscernible c_i of N_α so that

(i) $g^*(i) < c_i < \kappa_i$,

(ii) $\beta^{N_\alpha}(c_i) \geq \sup(o^{\mathcal{F}}(\alpha^{N_\alpha}(\kappa_i)) \cap h''(t_{\alpha n-1}(\sigma(i))))$, if such c_i exists and 0 otherwise. Note that for all but finitely many i's, c_i does exist.

Let $\langle j_n \mid n<\omega\rangle$ be the indexes of all κ_n^*'s appearing in $\bigcup_{n<\omega} A_n$. For $m<\omega$ define $t_\alpha(j_m)=t_{\alpha n}(j_m)$ where n is the minimal such that $j_m \in A_n$. So for every $\alpha<\lambda$, $t_\alpha \in \prod_{n<\omega} \kappa_{j_n}$.

Let us show that $\langle t_\alpha \mid \alpha<\lambda\rangle$ is unbounded in $\prod_{n<\omega} \kappa_{j_n}$ which will give the contradiction if $\lambda=\kappa^+$, since the cofinality of every infinite subsequence of $\langle \kappa_n^* \mid n<\omega)$ is κ^{++}.

Let $f \in \prod_{n<\omega} \kappa_{j_n}$. Pick a model N containing $\{f, \langle t_{\alpha m} \mid \alpha<\lambda, n<\omega\rangle, \langle N_\alpha \mid \alpha<\lambda\rangle\}$. Let $B_0=\{\kappa_{j_n} \mid n<\omega\}$. For $n>0$ define $B_{n+1}=\sigma''(B_n)$. Define functions $f^{(n)} \in \prod B_n$ by induction as follows. $f^{(0)}=f$, for $i \in B_{n+1}$ set $f^{(n+1)}(i)=$ the least $\gamma_i<\kappa_{\sigma(i)}$ so that for some indiscernible c_i, $g^*(i)<c_i<\kappa_i$

$$\beta^N(f^{(n)}(i))<\beta^N(c_i)<\sup(o^{\mathscr{F}}(\alpha^N(\kappa_i)\cap h''(\gamma_i)),$$

if such γ_i exists and 0 otherwise.

Define now $\bar{f} \in \prod A_0$ as follows. $\bar{f}(i)=f^{(n)}(i)$, where n is the minimal number such that $i \in B_n$.

Then $\bar{f} \in N$, since ${}^\omega N \subseteq N$. Hence there is $\alpha \in N$, $\alpha<\lambda$ so that $t_{\alpha_0\, D}>\bar{f}$. Let $C=\{i \in A_0 \mid t_{\alpha_0}(i)>\bar{f}(i)\}$. N and N_α cannot disagree on an unbounded sequence of common indiscernibles. So, tracking back, it is not hard to see that t_α will be bigger than f on an infinite subset of B_0.

If $\lambda=\kappa^+$, then this contradicts the fact that the cofinality of every infinite subsequence of $\langle \kappa_n^* \mid n<\omega\rangle$ is κ^{++}. In order to derive a contradiction also in case $\lambda>\kappa^{++}$ let us point out that for $\alpha<\beta<\lambda$, $t_\alpha(n)\leqslant t_\beta(n)$ on a set of n's having σ-image in D. It follows from the definition of $\langle t_{\alpha_n} \mid \alpha<\lambda, n<\omega\rangle$. Now pick $f \in \prod_{n<\omega} \kappa_{j_n}$ which is bigger than λ many of t_α's. Then for every $\alpha<\lambda$, $f(n)>t_\alpha(n)$ on a set of n's having σ-image in D. But this is impossible since there is $\alpha \in N \cap \lambda$ so that $t_{\alpha_0\, D}>\bar{f}$. Hence λ should be κ^{++}. □ Claim 15

So for every infinite $b \subseteq \{\kappa_{i_n} \mid n<\omega\}$ and for every ultrafilter D on $\{\kappa_{i_n} \mid n<\omega\}$ with $b \in D$, $\mathrm{tcf}(\prod_{n<\omega} \kappa_{i_n}, <_D) \geqslant \kappa^{++}$. Hence the ideal $J^0_{<\kappa^{++}}[\{\kappa_{i_n} \mid n<\omega\}]=\{b \subseteq \{\kappa_{i_n} \mid n<\omega\} \mid$ there is no ultrafilter to which b belongs with $\mathrm{tcf}(\prod \{\kappa_{i_n} \mid n<\omega\}, <_D) \geqslant \kappa^{++}\}$ is the idea of bounded subsets of $\{\kappa_{i_n} \mid n<\omega\}$.

By [18–20], then the true cofinality of $\langle \prod_{n<\omega} \kappa_{i_n}, <\rangle$ is κ^{++}.

Let us assume that $i_n=n$ for every n, $\sigma(n)=n-1$ for every $n \geqslant 1$, $\alpha_n=\kappa_n$ for every n and h is the identity. In the general case, mainly the notation is more complicated.

Definition 16. A sequence $\langle \delta_n \mid n<\omega\rangle \in \prod_{n<\omega} \kappa_n$ is called a diagonal sequence if for some or equivalently for every N containing $\langle \delta_n \mid n<\omega\rangle$ there exists $n_0<\omega$ so that for every $n \geqslant n_0$, $\delta_{n+1}=s^N(\kappa_{n+1}, \delta_n^*, g^*(n+1))$ where δ_n^* is the least β, $o^{\mathscr{F}}(\kappa_{n+1})>\beta \geqslant \delta_n$ for which there exists an indiscernible $c_n>g^*(n+1)$ with $\beta^N(c_n)=\delta_n^*$ and g^* is as in Claim 8.

Definition 17. A diagonal sequence of indiscernibles $\langle \delta_n \mid n < \omega \rangle$ is called a faithful diagonal sequence, if there exists a model N so that for some $n_0 < \omega$ for every $n \geq n_0$, $\delta_n = \mathrm{ch}^N(n)$.

Notice, that $\mathrm{cf}\, \delta_n \leq |N| < \kappa$ for such a sequence.

Claim 18. *For every $f \in \prod_{n<\omega} \kappa_n$ there exists a faithful diagonal sequence $\langle \delta_n \mid n < \omega \rangle > f$.*

Proof. Let $f \in \prod_{n<\omega} \kappa_n$. Pick N containing f. Define an increasing sequence of models $\langle N_i \mid i < \omega_1 \rangle$ so that

(a) $N_0 = N$;

(b) $N_{i+1} \supseteq N_i \cup \{N_i\}$ for every i;

(c) $|N_i| = |N|$ for every i.

Let $N^* = \bigcup_{i<\omega_1} N_i$ and $N^{**} \supseteq N^* \cup \{\langle N_i \mid i < \omega_1 \rangle, N^*\}$. Find $S \subset \omega_1$, $|S| = \aleph_1$ and $n_0 < \omega$ so that for every $i \in S$, $n \geq n_0$, N^{**}, N^*, N_{i+1} and N_i agree about common indiscernibles above κ_{n_0} and $\mathrm{ch}^N(n) > f(n)$. Assume for simplicity that $n_0 = 0$ and $S = \omega_1$.

Set $\delta_0 = \mathrm{ch}^{N^*}(0)$. For every $n < \omega$ define then $\delta_{n+1} = s^{N^{**}}(\kappa_{n+1}, \delta_n^*, g^*(n+1))$. Clearly, such a defined sequence $\langle \delta_n \mid n < \omega \rangle$ is a diagonal sequence. Let us show that it is above f and that it is a good sequence.

Subclaim 18.1. $\langle \delta_n \mid n < \omega \rangle > f$.

Proof. Let us show that for every n, $\delta_n \geq \mathrm{ch}^{N^*}(n)$. Which is clearly enough since $\mathrm{ch}^{N^*}(n) = \bigcup_{i<\omega_1} \mathrm{ch}^{N_i}(n)$ and $\mathrm{ch}^{N_0} > f$. Let us show that $\delta_1 \geq \mathrm{ch}^{N^*}(1)$. Note that since $\beta^{N_{i+1}}(\mathrm{ch}^{N_i}(1)) \in N_{i+1}$ and it is less than κ_0, it should be less than $\mathrm{ch}^{N_{i+1}}(0)$, for every $i < \omega_1$. But $\delta_0 > \mathrm{ch}^{N_{i+1}}(0)$. Then $\delta_1 \geq \mathrm{ch}^{N_i}(1)$ by its definition and the choice of g^*, since $\beta^{N^*}(\delta_1) = \delta_0^* \geq \delta_0 > \beta^{N_{i+1}}(\mathrm{ch}^{N_i}(1)) = \beta^{N^{**}}(\mathrm{ch}^{N_i}(1))$. So for every $i < \omega_1$, $\delta_1 \geq \mathrm{ch}^{N_i}(1)$. Hence $\delta_1 \geq \mathrm{ch}^{N^*}(1)$. Continue by induction for every $n > 1$. □ Subclaim 18.1

Notice that for the diagonal sequence starting with $\delta_0^i = \mathrm{ch}^{\bigcup_{j<i} N_j}(0)$, for limit $i < \omega_1$, the proof of Subclaim 18.1 shows that $\delta_n^i \geq \mathrm{ch}^{\bigcup_{j<i} N_j}(n)$ for every n. Also $\langle \delta_n^i \mid n < \omega \rangle \in N_{i+1}$, and, hence $\delta_n^i < \mathrm{ch}^{N_{i+1}}(n)$ for every $n < \omega$.

Subclaim 18.2. *The sequence $\langle \delta_n \mid n < \omega \rangle$ is a faithful sequence.*

Proof. It is enough to show that $\delta_n = \bigcup \{\mathrm{ch}^{N_i}(n) \mid i < \omega_1,\ i \text{ is a limit ordinal}\} = \mathrm{ch}^{N^*}(n)$ for every n, since $\mathrm{ch}^{N^*}(n) = \bigcup_{i<\omega_1} \mathrm{ch}^{N_i}(n)$. It is clear for $n = 0$. Let us show this for $n = 1$. The general case is similar. Consider $\mathrm{ch}^{N^*}(1)$. Since $\delta_1^i < \mathrm{ch}^{N_{i+1}}(1) < \delta_1^j < \mathrm{ch}^{N_{j+1}}(1)$ for every limit ordinals $i < j < \omega_1$, $\mathrm{ch}^{N^*}(1) = \bigcup \{\delta_1^i \mid i < \omega_1, \text{limit}\}$.

The sequence $\langle \delta_1^i \mid i < \omega_1, \text{limit}\rangle \in N^{**}$. So

$$\beta^{N^{**}}(\mathrm{ch}^{N^*}(1)) \geqslant \bigcup_{\substack{i<\omega_1 \\ i \text{ limit}}} \beta^{N^{**}}(\delta_1^i),$$

by Fact 2. But $\beta^{N^{**}}(\delta_1^i) = (\delta_0^i)^* \geqslant \delta_0^i = \mathrm{ch}^{\bigcup_{j<i} N_j}(0)$. Hence $\beta^{N^{**}}(\mathrm{ch}^{N^*}(1)) \geqslant \delta_0$ and so $\geqslant \delta_0^*$. Remember that $\delta_1 = s^{N^{**}}(\kappa_1, \delta_0^*, g(1))$ and $\delta_1 \geqslant \mathrm{ch}^{N^*}(1)$. It means that $\delta_1 = \mathrm{ch}^{N^*}(1)$. □ Subclaim 18.2

The proof of the previous claim actually gives the following.

Claim 19. *For every N' there exists N containing $N' \cup \{N'\}$ so that ch^N is a faithful diagonal sequence.*

Further let us restrict ourselves only to the models of Claim 19. Let $\{f_\alpha \mid \alpha < \kappa^{++}\rangle$ be an increasing sequence of faithful diagonal sequences so that for every $f \in \prod_{n<\omega} \kappa_n$ there is $\alpha < \kappa^{++}$ with $f < f_\alpha$.

Let us consider least upper bounds of $\{f_\alpha \mid \alpha < \delta\rangle$ of δ's of cofinality κ^+. They exist by [16]. Recall that f_δ^+ is called a least upper bound of $\langle f_\alpha \mid \alpha < \delta\rangle$ if $f_\delta^+ > f_\alpha$ for every $\alpha < \delta$ and if $f < f_\delta^+$ then there is $\alpha < \delta$ so that $f < f_\alpha$.

Claim 20. *A faithful diagonal sequence cannot be a least upper bound of $\langle f_\alpha \mid \alpha < \delta\rangle$, where $\delta < \kappa^{++}$, $\mathrm{cf}\,\delta = \kappa^+$.*

Proof. Since otherwise, the true cofinality of $\prod_{n<\omega} \delta_n = \kappa^+$, for such a sequence $\langle \delta_n \mid n < \omega\rangle$. But it is impossible, since then also $\mathrm{tcf}(\prod_{n<\omega} (\mathrm{cf}\,\delta_n)) = \kappa^+$ and $\{\mathrm{cf}\,\delta_n \mid n < \omega\}$ is bounded in κ. □ Claim 20

Let $l^N(\tau, \nu, \xi)$ be the least indiscernible c in N above ξ with $\alpha^N(c) = \tau$ and $\beta^N(c) \geqslant \nu$. The following is similar to Fact 3.

Fact 4. *Let N, N' be two models. Then there are only boundedly many α's below κ in $N \cap N'$ such that for some $(\beta, \gamma) \in N \cap N'$*

$$l^N(\alpha, \beta, \gamma) \neq l^{N'}(\alpha, \beta, \gamma).$$

Claim 21. *Let $\delta < \kappa^{++}$ be an ordinal of cofinality κ^+, f be a least upper bound of $\langle f_\alpha \mid \alpha < \delta\rangle$, N be a model containing $\{\langle f_\alpha \mid \alpha < \delta\rangle, f, \delta\}$. Then one of the following two possibilities holds:*

(1) *$f(n) = l^N(\kappa_n, f(n-1), g^*(n))$ for all but finitely many n's;*

(2) *for infinitely many n's $f(n) = a^N(\kappa_n, \beta_n, \gamma_n)$ for some β_n, γ_n.*

Proof. Suppose that (2) does not hold. Let us prove (1).

It is not hard to see that for all but finitely many n's, $f(n)$ should be an indiscernible in N with $\alpha^N(f(n)) = \kappa_n$. Suppose for simplicity that this holds for every $n < \omega$. Also assume that $f(n)$ is not an accumulation point for every $n < \omega$. Denote $\beta^N(f(n))$ by β_n. Then $\beta_n < \kappa_{n-1}$. Since $f(n)$ is not a (κ_n, β) accumulation point for $\beta > \beta_n$ there is a minimal $d_n \in N$ an indiscernible below $f(n)$ so that for every $c \in N$, s.t. $d_n \leq c < f(n)$ and $\alpha^N(c) = \kappa_n$, $\beta^N(c) < \beta_n$. Then $f(n) = s^N(\kappa_n, \beta_n, d_n)$. Since $\langle d_n \mid n < \omega \rangle < f$, there is $\alpha < \delta$ so that $\langle d_n \mid n < \omega \rangle < f_\alpha$. Let n be big enough. Then $f(n) > f_\alpha(n) > d_n$. Since f_α is a diagonal sequence $f_\alpha(n) = s^N(\kappa_n, \beta', g^*(n))$, where $\beta' = (f_\alpha(n-1))^*$. Let us compare β_n and β'. If $\beta' \geq \beta_n$ then $f(n) \leq f_\alpha(n)$ since $f_\alpha(n) > d_n$ and $\beta^N(f_\alpha(n)) = \beta' \geq \beta_n$. So $\beta' < \beta_n$. But then $f(n) = s^N(\kappa_n, \beta_n, g^*(n))$ since $s^N(\kappa_n, \beta_n, g^*(n)) > s^N(\kappa_n, \beta', g^*(n)) = f_\alpha(n)$ (by choice of g^* and since f_α is a characteristic function) and $f_\alpha(n) > d_n$.

Subclaim 21.1. $\langle d_n \mid n < \omega \rangle \leq g^*$.

Proof. Suppose otherwise. Let $A = \{n < \omega \mid d_n > g^*(n)\}$ be infinite. Find some $\alpha < \delta$, $\alpha \in N$, so that $f(n) > f_\alpha(n) > d_n$ for all but finitely many n's. Let $n \in A$ be big enough. Then, there is an indiscernible c_n, for κ_n in N so that $g^*(n) < c_n \leq d_n$ and $\beta^N(c_n) \geq \beta^N(f(n))$. Since $c_n > g^*(n)$ and f_α is a characteristic function of some model, we can assume that n is big eoungh to satisfy $\beta^N(f_\alpha(n)) > \beta^N(c_n)$. But this contradicts the choice of d_n, since $f_\alpha(n) > d_n$ and $\beta^N(f_\alpha(n)) > \beta^N(c_n) \geq \beta^N(f(\alpha))$. Contradiction. □ Subclaim 21.1

Further let us assume that $d_n = g^*(n)$ for all n's.

Subclaim 21.2. *For all but finitely many n's, $\beta_{n+1} \geq f(n)$.*

Proof. Suppose otherwise. Let $A = \{n \mid \beta_{n+1} < f(n)\}$. Define a function $g \in \prod_{n<\omega} \kappa_n$ as follows

$$g(n) = \begin{cases} \max\{\beta_{n+1}, g^*(n)\} & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g < f$ and hence there is $\alpha < \delta$ such that $g < f_\alpha$. Let $n \in A$ be big enough. Then $\max(g^*(n), \beta_{n+1}) = g(n) < f_\alpha(n) < f(n)$ and also $f_\alpha(n+1) < f(n+1)$. But this is impossible, since

$$f_\alpha(n+1) = s^N(\kappa_{n+1}, (f_\alpha(n))^*, g^*(n+1)) < f(n+1) = s^N(\kappa_{n+1}, \beta_{n+1}, g^*(n+1)),$$
$$\beta^N((f_\alpha(n+1))^*) \geq f_\alpha(n) > \beta_{n+1} = \beta^N(f(n+1))$$

and $f_\alpha(n+1) \geq d_{n+1}$. Contradiction. □ Subclaim 21.2

We would like to have $\beta_n = f(n-1)$ for all but finitely many n's. This should not always be true. But still the following holds:

Subclaim 21.3. *For all but finitely many n's, $f(n) = s^N(\kappa_n, \beta_n, g^*(n)) = l^N(\kappa_n, f(n-1), g^*(n))$.*

Proof. For every $m < \omega$ define a function $t_m \in \prod_{n<\omega} \kappa_n$ as follows

$$t_m(n) = \begin{cases} 0 & \text{if } n \leqslant m, \\ l^N(\kappa_n, t_m(n-1), g^*(n)) & \text{if } n > m+1, \\ l^N(\kappa_n, f(n-1), g^*(n)) & \text{if } n = m+1. \end{cases}$$

Since ${}^{\omega}N \subseteq N$, $t_m \in N$ and, also, $\langle t_m \mid m < \omega \rangle \in N$.

Then $t_m(n) \leqslant f(n)$ for each n. This is immediate for $n \leqslant m$; for $n = m+1$ we have

$$\begin{aligned} f(m+1) &= s^N(\kappa_{m+1}, \beta_{m+1}, g^*(m+1)) = l^N(\kappa_{m+1}, \beta_{m+1}, g^*(m+1)) \\ &\geqslant l^N(\kappa_{m+1}, f(m), g^*(m+1)) = t_m(m+1); \end{aligned}$$

and for $n > m+1$,

$$\begin{aligned} f(n) &= s^N(\kappa_n, \beta_n, g^*(n)) = l^N(\kappa_n, \beta_n, g^*(n)) \\ &\geqslant l^N(\kappa_n, f(n-1), g^*(n)) \geqslant l^N(\kappa_n, t_m(n-1), g^*(n)) = t_m(n). \end{aligned}$$

Suppose that for every $m < \omega$ there is an infinite set $A_m \subseteq \omega$ so that for every $n \in A_m$, $t_m(n) < f(n)$. Define then a function $t'_m \in \prod_{n<\omega} \kappa_n$ as follows:

$$t'_m(n) = \begin{cases} t_m(n) & \text{if } n \in A_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $t'_m < f$. So there is $f_{\alpha_m} > t'_m$ for $\alpha_m < \delta$. Since $\langle f_\alpha \mid \alpha < \delta \rangle$, f, $\langle t'_m \mid m < \omega \rangle$ are in N we can find such a sequence $\langle f_{\alpha_m} \mid m < \omega \rangle$ inside N. Pick now $\alpha \in N \cap \delta$ above $\bigcup_{m<\omega} \alpha_m$. Let $m \in A$ be big enough so that for every $n \geqslant m$, $f(n) > f_\alpha(n)$. By the definition of t_m,

$$\begin{aligned} t_m(m+1) &= l^N(\kappa_{m+1}, f(m), g^*(m+1)) \\ &\geqslant l^N(\kappa_{m+1}, f_\alpha(m), g^*(m+1)) \\ &= s^N(\kappa_{m+1}, (f_\alpha(m))^*, g^*(m+1)) = f_\alpha(m+1). \end{aligned}$$

In the same fashion for every $n \geqslant m+1$, $t_m(n) \geqslant f_\alpha(n)$. But $f_\alpha > t'_m$, so for infinitely many n's, $t_m(n) < f_\alpha(n)$. Contradiction.

So, there are m_0, $n_0 < \omega$ such that for every $n \geqslant n_0$, $t_{m_0}(n) = f(n)$. It means that for every $n \geqslant n_0 + 1$,

$$f(n) = t_{m_0}(n) = l^N(\kappa_n, t_{m_0}(n-1), g^*(n)) = l^N(\kappa_n, f(n-1), g^*(n)).$$

□ Subclaim 21.3 □ Claim 21

Claim 21′. *If f satisfies* (1) *of Claim* 21, *then the set* $\{n \mid \mathrm{cf}(f(n)) > \mathrm{cf}(f(n-1))\}$ *is infinite.*

Proof. Otherwise there would be a bounded in κ sequence $\langle \delta_n \mid n < \omega \rangle$ with $\mathrm{tcf}(\prod_{n<\omega} \delta_n, <) = \kappa^+$, which is impossible. □ Claim 21′

Now we would like to show that (1), (2) of Claim 21 are impossible. It is not quite true in general. There are forcing providing f as in (2) or f as in (1) with $f(n)$'s regular.

The point which will lead to a contradiction is that there should be unboundedly many f's in $\prod_{n<\omega} \kappa_n$ satisfying (1) or (2). Actually it would be enough to have an increasing sequence of length $(2^{\aleph_0})^+$.

Claim 22. *There is no increasing sequence $\langle \delta_i \mid i < (2^{\aleph_0})^+ \rangle$ of ordinals below κ^{++} of cofinality κ^+ so that for every $i < (2^{\aleph_0})^+$ there is a least upper bound g_i of $\langle f_\alpha \mid \alpha < \delta_i \rangle$ satisfying* (1) *of Claim* 21.

Proof. Suppose otherwise. Let $\langle \delta_i \mid i < (2^{\aleph_0})^+ \rangle$ and $\langle g_i \mid i < (2^{\aleph_0})^+ \rangle$ be witnessing this. Let N be a model containing $\{\langle f_\alpha \mid \alpha < \kappa^{++} \rangle, \langle \delta_i \mid i < (2^{\aleph_0})^+ \rangle, \langle g_i(n) \mid i < (2^{\aleph_0})^+, n < \omega \rangle\}$. For $i < (2^{\aleph_0})^+$, let A_i be the set $\{n < \omega \mid g_i(n) = l^N(\kappa_n, g_i(n-1), g^*(n))$ and $\operatorname{cf} g_i(n) > \operatorname{cf} g_i(n-1)\}$. Shrinking the sequence $\langle g_i \mid i < (2^{\aleph_0})^+ \rangle$, if necessary, we can assume that all A_i's are the same infinite set A. Assume for simplicity that $A = \omega$. Using the Erdös–Rado theorem, find $S \subseteq (2^{\aleph_0})^+$, $|S| = \aleph_1$ and $n_0 < \omega$ so that for every $i < j$, $i, j \in S$, for every $n \geq n_0$, $g_i(n) < g_j(n)$. Notice that w.l.o.g. $S \in N$ since otherwise we can just replace N by some $N' \supseteq N \cup \{S\}$. Models N, N' will agree about these common indiscernibles above some κ_n, and we can deal with N' instead of N. Let us assume for simplification of the notations that $S = \omega_1$ and $n_0 = 0$.

Set $\tau_n = \bigcup_{i<\omega_1} g_i(n)$. Then $\tau_n \in N$, $\alpha^N(\tau_n) = \kappa_n$ and $\kappa_{n-1} > \beta^N(\tau_n) \geq \bigcup_{i<\omega_1} g_i(n-1)$, for every $n < \omega$. For $n < \omega$ big enough, pick a model M_n containing $\langle g_i \mid i < \omega_i \rangle$, $\langle \beta^N(g_i(n)) \mid i < \omega_1 \rangle$, $\beta^N(\tau_n)$ of cardinality less than κ_n and satisfying ${}^\omega M_n \subseteq M_n$. Here is the first time that we are going to use a model whose cardinality may be bigger than $2^{\aleph_0}$. In order to satisfy ${}^\omega M_n \subseteq M_n$ and $|M_n| < \kappa_n$, we use the condition (2) of Theorem A. Thus M_n can be taken of cardinality μ^ω, where $\mu = \beta^N(\tau_n) < \kappa_{n-1}$. If $\mu^\omega \geq \kappa_n$, then the condition (2) of the theorem holds, Since κ_n is a regular cardinal which is a limit of measurable in $\mathscr{K}(\mathscr{F})$ cardinals and also $\mu^\omega \geq \kappa_n \geq \kappa_{n-1}^+ > \kappa_{n-1} > \mu$.

Fact 2 and $\beta^N(g_i(n)) < \kappa_{n-1}$ for every i imply that there is $i_n < \omega_1$ so that for every $i \geq i_n$, $g_i(n)$ is an indiscernible for τ_n with $\beta^{M_n}(g_i(n)) = \beta^N(g_i(n))$ and also $g_i(n) = l^{M_n}(\tau_n, g_i(n-1), g^*(n))$. For every $i \geq i_n$ and $\gamma < g_i(n-1)$ let $d(i, \gamma) = l^{M_n}(\tau_n, \gamma, g^*(n))$. Clearly the sequence $\langle d(i, \gamma) \mid \gamma < g_i(n-1) \rangle$ is nondecreasing.

Pick now a model $M_n^* \supseteq M_n$, $|M_n^*| = |M_n|$, ${}^\omega M_n^* \subseteq M_n$ containing

$$\langle \langle d(i, \gamma) \mid \gamma < g_i(n-1) \rangle \mid i \geq i_n, i < \omega_1 \rangle, \qquad \langle \beta^{M_n}(d(i, \gamma)) \mid \gamma < g_i(n-1) \rangle.$$

Using Fact 2, find i_n^*, $\omega_1 > i_n^* \geq i_n$ so that M_n^* and M_n agree about all here mentioned common indiscernibles. Let $i \geq i_n^*$. Consider $d_n(i) = \bigcup \{d(i, \gamma) \mid \gamma < g_i(n-1)\}$. If the sequence $\{d(i, \gamma) \mid \gamma < g_i(n-1)\}$ does not have a last element, then $\operatorname{cf} d_n(i) \leq g_i(n-1)$ and $\beta^{M_n^*}(d_n(i)) \geq g_i(n-1)$, by Fact 2(5). Then $d_n(i) =$

$l^{M_n^*}(\tau_n, g_i(n-1), g^*(n))$ and hence $d_n(i) = g_i(n)$. But $\operatorname{cf} g_i(n) > g_i(n-1)$. Contradiction.

So for every $i \geq i_n^*$ there is $\gamma_{i,n} < g_i(n-1)$ so that for every γ, $\gamma_{i,n} \leq \gamma < g_i(n-1)$

$$l^{M_n^*}(\tau_n, \gamma, g^*(n)) = l^{M_n^*}(\tau_n, \gamma_{i,n}, g^*(n)) = d_n(i).$$

Let $i^* = \bigcup \{i_n^* \mid n < \omega\}$.

Pick now $N^* \supseteq N$, ${}^{\omega}N^* \supseteq N^*$ so that $\langle d(i, \gamma_{in}) \mid n < \omega, i < \omega_1 \rangle$, $\langle M_n^* \mid n < \omega \rangle$, $\langle \tau_n \mid n < \omega \rangle$, $\langle \gamma_{in} \mid i < \omega_1, n < \omega \rangle$ and $\langle \beta^{M_n^*}(d(i, \gamma_{in})) \mid i < \omega_1, n < \omega \rangle$ are in N^*. Suppose for simplicity that N^* and N agree about all common indiscernibles. For every $i \geq i^*$ pick $\alpha_i < \delta_i$ so that $f_{\alpha_i}(n-1) > \gamma_{i,n}$ for all but finitely many n's. Let $n_i < \omega$ be so that for every $n \geq n_i$

$$\gamma_{i,n} < f_{\alpha_i}(n-1) \leq (f_{\alpha_i}(n-1))^* < g_i(n-1).$$

Then

$$f_{\alpha_i}(n) = s^{N^*}(\kappa_n, (f_{\alpha_i}(n-1))^*, g^*(n)) < g_i(n) = l^{N^*}(\kappa_n, g_i(n-1), g^*(n)).$$

Obviously

$$d_n(i) = l^{N^*}(\kappa_n, \gamma_{in}, g^*(n)) \leq s^{N^*}(\kappa_n, (f_{\alpha_i}(n-1))^*, g^*(n)) = f_{\alpha_i}(n).$$

Replacing f_{α_i} by f_{α_i+1}, if necessary, we can assume that $d_n(i) < f_{\alpha_i}(n)$.

Find $S \subseteq \omega_1$, $|S| = \aleph_1$, $\min S \geq i^*$ and $n^* < \omega$ so that $n_i = n^*$ for every $i \in S$. Fix $n \geq n^*$. Pick a model M_n^{**} which contains $M_n^* \cup \{\langle f_{\alpha_i}(n) \mid i < \omega_1 \rangle, \langle (f_{\alpha_i}(n))^* \mid i < \omega_1, n < \omega \rangle, S\}$ and ${}^{\omega}M_n^{**} \subseteq M_n^{**}$. By Facts 2, 3 and 4, M_n^* and M_n^{**}, M_n^{**} and N^* will agree about all here mentioned common indiscernibles for τ_n above some $g_{i_0}(n)$. But it is impossible, since then for every $i \in S \setminus g(i_0)$ the following holds:

(a) for every γ, $\gamma_{i,n} \leq \gamma < g_i(n-1)$

$$l^{M_n^{**}}(\tau_n, \gamma, g^*(n)) = d_n(i);$$

(b) $\gamma_{i,n} < f_{\alpha_i}(n) < g_i(n-1)$ and $d_n(i) < f_{\alpha_i}(n) = l^{M_n^{**}}(\tau_n, f_{\alpha_i}(n-1), g^*(n))$.

Contradiction. □ Claim 22

Finally we are going to rule out the last possibility, i.e., (2) of Claim 21. The proof will be similar to that of Claim 22.

Claim 23. *There is no increasing sequence $\langle \delta_i \mid i < (2^{\aleph_0})^+ \rangle$ of ordinals below κ^{++} of cofinality κ^+ so that for every $i < (2^{\aleph_0})^+$ there is a least upper bound g_i of $\langle f_\alpha \mid \alpha < \delta_i \rangle$ satisfying the condition* (2) *of Claim* 21.

Proof. Suppose otherwise. Let $\langle \delta_i \mid i < (2^{\aleph_0})^+ \rangle$ be witnessing this. Let N be a model containing $\langle f_\alpha \mid \alpha < \kappa^{++} \rangle$, $\langle \delta_i \mid i < (2^{\aleph_0})^+ \rangle$ and $\langle g_i(n) \mid i < 2^{\aleph_0}, n < \omega \rangle$. For $i < (2^{\aleph_0})^+$, let $A_i = \{n < \omega \mid g_i(n)$ is an accumulation point in $N\}$. As in Claim 22, using the Erdös–Rado theorem and shrinking $\langle g_i \mid i < 2^{\omega_1} \rangle$, we can assume that each $A_i = \omega$ and for every $i < j < \omega_1$, $g_i(n) < g_j(n)$.

For every $n<\omega$, define τ_n, M_n as in Claim 22. Then, starting with some $i_n<\omega_1$, $g_i(n)$ is an indiscernible for τ_n with $\beta^{M_n}(g_i(n))=\beta^N(g_i(n))$ and $g_i(n)=a^{M_n}(\tau_n,\beta_{in},\gamma_{in})$ for some β_{in}, γ_{in} with $\beta_{in}>\beta^{M_n}(g_i(n))$. W.l.o.g. $\gamma_{in}\geqslant g^*(n)$, since otherwise we can just replace it by $g^*(n)$. Also we can assume that there is no indiscernible c_i in M_n for τ_n such that $\gamma_{in}<c_i<g_i(n)$ and $\beta^{M_n}(c_i)\geqslant\beta_{in}$, since otherwise $g_i(n)$ would be at least a $\beta_{in}+1$-accumulation point. Now, using the argument of Claim 7, it is not hard to see that for every $M'_n\supseteq M_n$, $|M'_n|=|M_n|$, ${}^{\omega}M'_n\subseteq M_n$ containing $\langle g_i(n)\mid i<\omega_1\rangle$, $\langle\beta^{M_n}(g_i(n))\mid i<\omega_1\rangle$, $\langle\beta_{in}\mid i<\omega_1\rangle$ and $\langle\gamma_{in}\mid i<\omega_1\rangle$ for all but boundedly many i's there would be no indiscernible c_i for τ_n such that $\gamma_{in}<c_i<g_i(n)$ and $\beta^{M'_n}(c_i)\geqslant\beta_{in}$.

For every $i\geqslant i_n$ and $\gamma<\beta_{in}$ let $d(i,\gamma)=l^{M_n}(\tau_n,\gamma,\gamma_n)$. By the definition of (τ_n,β_{in})-accumulation point, $d(i,\gamma)<g_i(n)$. Define models M^*_n and i^*_n's as in Claim 22. For $i\geqslant i^*_n$ set $d_n(i)=\bigcup\{d(i,\gamma)\mid\gamma<\beta_{in}\}$. Then $d_n(i)\leqslant g_i(n)$. By Fact 2(5), $\beta^{M^*_n}(d_n(i))\geqslant\beta_{in}>\beta^{M_n}(g_i(n))=\beta^{M^*_n}(g_i(n))$. So $d_n(i)<g_i(n)$. But, as it was pointed out above, it is impossible to have indiscernibles c_i for τ_n with $\beta^{M^*_n}(c_i)\geqslant\beta_{in}$ and $\gamma_{in}<c_i<g_i(n)$ for all but boundedly many i's. Contradiction. □ Claim 23 □ Theorem A

The present ideas together with an appropriate generalization of the Mitchell Covering Lemma to hypermeasures, will imply further results. Thus the strength will depend on particular κ and on the gap between κ and 2^κ. Namely, the exact strength of "κ is a strong limit of cofinality $\aleph_0$ and $2^\kappa=\lambda$" is "$o(\kappa)=\lambda$", when $\lambda\neq\rho^+$ for ρ of cofinality $\aleph_0$ and "$o(\kappa)=\rho$", when $\lambda=\rho^+$, $\operatorname{cf}\rho=\aleph_0$.

2. Some forcing constructions

In this section we shall construct two models which are related to cases (1) and (2) of Claim 21. The first model would have an unfaithful diagonal sequence which is a least upper bound of κ^+ faithful diagonal sequences, and the second a sequence of accumulation points. The second construction answers a question of Mitchell [14] about the existence of accumulation points.

Model 1

Let κ be a cardinal of cofinality ω in $\mathcal{K}(\mathcal{F})=V$ so that $\{o^{\mathcal{F}}(\alpha)\mid\alpha<\kappa\}$ is unbounded in it. Pick a cofinal sequence $\langle\kappa_n\mid n<\omega\rangle$ so that $o^{\mathcal{F}}(\kappa_{n+1})=\kappa_n^+$, for each $n<\omega$.

We first force with a tree Prikry forcing and add indiscernibles γ_n, $\kappa_{n-1}<\gamma_n<\kappa_n$ for $\mathcal{F}(\kappa_n,\gamma_{n-1})$ for every $n<\omega$. Let V_1 denote such a generic extension. Clearly, $\langle\gamma_n\mid n<\omega\rangle$ forms an unfaithful diagonal sequence in V_1. We shall define a further extension in which the sequence $\langle\gamma_n\mid n<\omega\rangle$ would be also a limit of κ^+ faithful diagonal sequences.

Let us force over V_1 with the forcing used in [3] and introduce a maximal sequence of indiscernibles to each $\delta \in \kappa \setminus \{\gamma_n \mid n < \omega\}$. Denote by V_2 such extension of V_1. Define V_3 to be an extension of V_2 by forcing of [3] which adds maximal sequences to each γ_n $(n \geq 1)$. Let

$$\mathcal{P} = \mathcal{P}(\gamma_1, \gamma_0) * \mathcal{P}(\gamma_2, \gamma_1) * \cdots * \mathcal{P}(\gamma_{n+1}, \gamma_n) * \cdots$$

be this forcing. Denote $\mathcal{P}(\gamma_1, \gamma_0) * \cdots * \mathcal{P}(\gamma_{n-1}, \gamma_{n-2})$ by $\mathcal{P}_{<n}$ and $\mathcal{P}(\gamma_n, \gamma_{n-1}) * \mathcal{P}(\gamma_{n+1}, \gamma_n) * \cdots$ by $\mathcal{P}_{\geq n}$. Since the filters used in $\mathcal{P}(\gamma_n, \gamma_{n-1})$ are γ_n-complete, $\mathcal{P}$ can be viewed as $\mathcal{P}_{<n} \times \mathcal{P}_{\geq n}$.

For $n < \omega$, let Q_n be the usual forcing over V_2 for shooting a club through the set of regular cardinals of V below γ_n, i.e. $Q_n = \{c \in V_2 \mid c \subseteq \gamma_n,\ c$ is closed and $|c|^{V_2} < \gamma_n\}$, c_1 is stronger than c_2 if c_1 is an end extension of c_2.

Claim 2.1. *For every $n \geq 1$, Q_n over V_2 does not add new sequences of length κ_{n-1}.*

Proof. Since $o^{\mathcal{F}}(\kappa_n) = \kappa_{n-1}^+$, w.l.o.g. assume that there are stationary many δ's, $\kappa_{n-1} < \delta < \gamma_n$ with $o^{\mathcal{F}}(\delta) = \kappa_n$. In V_2, such δ has a cofinal closed subset of the type κ_n consisting of regular in V cardinals. The rest is standard. □ Claim 2.1

Let now Q be the product of Q_n's over V_3, i.e.

$$Q = \{p \in V_3 \mid \operatorname{dom} p = \omega,\ p(n) \in Q_n \text{ for every } n\}.$$

Set $Q_{<n} = \{p \upharpoonright n \mid p \in Q\}$ and $Q_{\geq n} = \{p \upharpoonright (\omega \setminus n) \mid p \in Q\}$.

Claim 2.2. *Let $n \geq 1$. The forcing $\mathcal{P}_{\geq n} \times \mathcal{P}_{\geq n}$ over V_2 does not add new bounded subsets to γ_{n-1} and γ_{n-1} remains regular.*

Proof. By [3], $\mathcal{P}_{\geq n}$ does not add new bounded subsets to γ_{n-1}. For each $m \geq n$, Q_m has in $V_2^{\mathcal{P}(\gamma_m, \gamma_{m-1})}$ a dense γ_{m-1}-closed subset, namely $\{c \in Q_m \mid \max(c)$ is in the generic sequence for $\gamma_m\}$. Now use the fact that $\mathcal{P}_{\geq n}$ does not add new bounded subsets of κ of size less than γ_{n-1}. □ Claim 2.2

Let G be a V_3-generic subset of Q. Denote $G \cap Q_n$ by G_n, $G \cap Q_{<n}$ by $G_{<n}$ and $G \cap Q_{\geq n}$ by $G_{\geq n}$. Our final model will be $V_4 = V_2[G]$.

Claim 2.3. *For every $n < \omega$, κ_n is a regular cardinal in V_4 and $\mathrm{cf}^{V_4} \gamma_{n+1} \geq \kappa_n$.*

Proof. Let $n < \omega$ be fixed. Recall that $\kappa_n < \gamma_{n+1} < \kappa_{n+1} < \gamma_{n+2}$. View V_4 as

$$V_2[G_{\geq n+2}][G_{n+1}][G_{<n+1}].$$

Then, by Claim 2.2, γ_{n+1} remains regular and no new bounded subsets are added to it in $V_2[G_{\geq n+2}]$. Using Claim 2.1, we obtain that κ_n remains regular and

$\mathrm{cf}^{V_2[G_{\geq n+2}][G_{n+1}]}(\gamma_{n+1}) \geq \kappa_n$. Now the last forcing which adds $G_{<n+1}$ cannot effect cardinals $\geq \kappa_n$, since it is of cardinality less than κ_n. □ Claim 2.3

So, $\langle \gamma_n \mid n<\omega \rangle$ is an unfaithful diagonal sequence of indiscernibles in V_4. Work in V_4. Let us show that it is a least upper bound of κ^+ faithful diagonal sequences. First note that $\mathrm{cf}(\prod_{n<\omega} \gamma_n, <) = \kappa^+$, since $\mathrm{cf}\, \gamma_n \geq \kappa_n$ and κ_n is regular for every $n<\omega$. Let $f \in \prod_{n<\omega} \gamma_n$. In V_3, $\mathrm{cf}\, \gamma_n = \mathrm{cf}\, \gamma_0 = \gamma_0 > \aleph_0$ for every n. Also, by Claim 2.2, γ_0 remains regular in $V_3[G]$. Hence there are $\delta_0 < \gamma_0$ and a faithful sequence of indiscernibles $\langle \delta_n \mid n<\omega \rangle$ in V_3 so that $\delta_n > f(n)$ for every n. Using density arguments it is not hard to find such a sequence inside V_4.

It is possible to push everything down to $\aleph_\omega$ by collapsing κ_n's to be $\aleph_n$'s.

Model 2

Suppose that κ is a measurable cardinal in $\mathcal{K}(\mathcal{F}) = V$ and $\{o^{\mathcal{F}}(\beta) \mid \beta<\kappa\}$ is unbounded in κ. We whall construct a generic extension of V in which κ would be a cardinal of cofinality $\aleph_0$ and there would be a sequence of accumulation points. At the first stage a Rudin–Kiesler increasing commutative sequence of ultrafilters over κ of the length κ^+ will be constructed. Then the direct limit of their ultrapowers will be taken. We shall use the first ultrafilter to move the direct limit a little bit further. Next, a κ^+-Cohen subset will be added to κ in the Backward Easton style. Changing one value of each such function in the ultrapower will make all the ultrafilters isomorphic. Finally we shall force with the Prikry forcing in order to change cofinality of κ to $\aleph_0$. This would introduce Prikry sequences for each ultrafilter used in the direct limit. Also there would be a sequence of accumulation points. Similar ideas were used in [4, 5] and we refer to these papers for detailed presentation of the techniques used below.

By [4, 4.2], there exists a generic extension V_1 of V having a Rudin–Kiesler increasing commutative sequence of ultrafilters over κ of length κ^+. The argument is as follows:

Let α^* denote the least $\beta>\alpha$ with $o^{\mathcal{F}}(\beta) = \alpha^+ + 1$, for every $\alpha<\kappa$. Pick $A \in \mathcal{F}(\kappa, 0)$ so that for every $\alpha \in A$, $\alpha^* < \min(A \setminus (\alpha+1))$. Using the forcing of [3] force in every interval (α, α^*) ($\alpha \in A$) a Rudin–Kiesler increasing commutative sequence $\langle U_{\alpha,\beta} \mid \beta < \alpha^+ \rangle$ over α^*. Denote this generic extension of V by V_1. Let U be a normal extension of $\mathcal{F}(\kappa, 0)$ after such forcing.

Let $f: \kappa \to \kappa$, $f \in V$ be so that $f(\alpha) < \alpha^+$ for every $\alpha<\kappa$. Define the ultrafilter U_f over κ as follows: $X \in U_f$ iff $\{\alpha \in A \mid X \cap \alpha^* \in U_{\alpha, f(\alpha)}\} \in U$. Then $[f_1]_{\mathcal{F}(\kappa,0)} < [f_2]_{\mathcal{F}(\kappa,0)}$ implies $U_{f_1} <_{\mathrm{RK}} U_{f_2}$. Also the order type of $\{[f]_{\mathcal{F}(\kappa,0)} \mid f: \kappa \to \kappa,\ f \in V$ and for every $\alpha<\kappa,\ f(\alpha)<\alpha^+\}$ is κ^+. Hence $\langle U_f \mid f \in V, f: \kappa \to \kappa, f(\alpha) < \alpha^+$ for every $\alpha<\kappa \rangle$ is a Rudin–Kiesler increasing commutative sequence of length κ^+.

Let $j: V_1 \to N \simeq V_1^\kappa/U$ be the elementary embedding of V_1 into the ultrapower of V_1 by U. In N, then there will be a Rudin–Kiesler sequence $\langle U_{\kappa\beta} \mid \beta<\kappa^+ \rangle$ of ultrafilters over κ^*, where κ^* and $\langle U_{\kappa\beta} \mid \beta < \kappa^+ \rangle$ are represented in N by

$[\langle \alpha^* \mid \alpha<\kappa\rangle]_U$ and $[\langle U_{\alpha\beta} \mid \beta<\alpha^+\rangle]_U$. Note that the ultrapower of V_1 by U_f is isomorphic to the ultrapower of N by $U_{\kappa,[f]_U}$. Also the direct limit of ultrapowers of V_1 by U_f's, is isomorphic to the direct limit of ultrapowers of N by $U_{\kappa,\beta}$'s. Denote this direct limit by N_1. Let $j^*: V_1 \to N^*$ be its associated elementary embedding. And also consider the following commutative diagram:

$$\begin{tikzcd}
 & N^* \\
V_1 \arrow[ur, "j^*"] \arrow[r, "j_\beta"] \arrow[dr, "j"'] & N_\beta \arrow[u, "i^*_\beta"'] \\
 & N \arrow[u, "i_\beta"'] \arrow[uu, bend right, "i"']
\end{tikzcd}$$

where $\beta<\kappa^+$, j_β is the embedding by U_f for f representing β in N, i_β is the embedding of N by $U_{\kappa,\beta}$ and i^*_β is the direct limit of embeddings of N_β into ultrapowers of it by the ultrafilters in $i_\beta(\langle U_{\kappa,\gamma} \mid \gamma>\beta\rangle)$.

Note that κ^* is the critical point of i_β's. Denote $i_\beta(\kappa^*)$ by κ_β and $i(\kappa^*)$ by κ^{**}. It is not hard to see that N^* is closed under κ-sequences of its elements and ${}^{\kappa^*}N^* \cap N \subseteq N^*$.

Let us form one more ultrapower. Namely, the ultrapower of N^* by $i(U_{\kappa,0})$. Denote this ultrapower by N^{**} and let

$$\begin{tikzcd}
V \arrow[r, "j^*"] \arrow[rr, bend left, "j^{**}"] & N^* \arrow[r, "i^*"] & N^{**}
\end{tikzcd}$$

be the corresponding embeddings.

Note that κ_β $(\beta<\kappa^+)$ is an 'indiscernible' for U_{f_β} (where $[f_\beta]_U=\beta$) and κ^{**} for U_{f_0}, i.e.,

$$X \in U_{f_\beta} \quad \text{iff} \quad \kappa_\beta \in j^{**}(X)$$

and

$$X \in U_{f_0} \quad \text{iff} \quad \kappa^{**} \in j^{**}(X).$$

Force now over V_1 using Backward Easton forcing α^+ Cohen functions from α into α for every inaccessible $\alpha \leqslant \kappa$. Let G be a generic set for this forcing. Denote by $G_{<\kappa}$ the part of G below κ, by G_κ over κ and by $G_{\kappa\beta}$ the β-th generic function in G over κ. Using standard arguments extend j^{**} to an embedding

$$l: V_1[G_{<\kappa}, \langle G_{\kappa\beta} \mid \beta<\kappa^+\rangle] \to N^{**}[G_{<j^{**}(\kappa)}, \langle G_{j^{**}(\kappa)\beta} \mid \beta<j^{**}(\kappa^+)\rangle],$$

where $G_{<j^{**}(\kappa)} \restriction \kappa+1 = G$ and $G_{j^{**}(\kappa),j(\beta)} \restriction \kappa = G_{\kappa,\beta}$ for every $\beta<\kappa^+$. Now let us change the value of $G_{j^{**}(\kappa),j(\beta)}(\kappa^{**})$ to κ_β for every $\beta<\kappa^+$ and $G_{j^{**}(\kappa),0}(\kappa_0)$ to κ^{**}. Denote such changed $G_{j^{**}(\kappa)}$ by $G'_{j^{**}(\kappa)}$. Using $j^{**}(\kappa^+)$-c.c. of the forcing in N^{**}, it is not hard to see that the changed $G'_{j^{**}(\kappa)}$ is still N^{**}-generic. Then it is possible to define an embedding

$$l': V_1[G_{<\kappa}, G_\kappa] \to N^{**}[G_{<j^{**}(\kappa)}, G'_{j^{**}(\kappa)}].$$

For every $\beta < \kappa^+$ define an ultrafilter U_β over κ as follows: $X \in U_\beta$ iff $\kappa_\beta \in l'(X)$. Then $U_\beta \supseteq U_{f_\beta}$ and all U_β's are isomorphic to U_0 since $X \in U_0$ iff $\kappa_0 \in l'(X)$ iff $G_{j^{**}(\kappa),0}(\kappa_0) \in G''_{j^{**}(\kappa),0}(l'(X))$ iff $\kappa^{**} = G_{j^{**}(\kappa),0}(\kappa_0) \in l'(G''_{\kappa,0}(X))$ iff $G_{j^{**}(\kappa),j(\beta)}(\kappa^{**}) \in G''_{j^{**}(\kappa),j(\beta)}(l'(G''_{\kappa,0}(X)))$ iff $\kappa_\beta \in l'((G_{\kappa,\beta} \circ G_{\kappa,0})''X))$ iff $(G_{\kappa,\beta} \circ G_{\kappa,0})''(X) \in U_\beta$.

Finally, force over $V_1[G]$ a Prikry sequence $\langle c_n \mid n < \omega \rangle$ for U_0. Denote $V_1[G][\langle c_n \mid n < \omega \rangle]$ by V_2. It is not hard to see that, for every $\beta < \kappa^+$, $\langle G_{\kappa,\beta}(G_{\kappa,0}(c_n)) \mid n < \omega \rangle$ will be a Prikry sequence for U_β. Let us show that $\langle G_{\kappa,0}(c_n) \mid n < \omega \rangle$ is a sequence of accumulation points. Denote $G_{\kappa,0}(c_n)$ by δ_n. Let τ_n be the minimal $\alpha < \delta_n$ so that $\alpha^* > \beta_n$. Then $\langle \tau_n \mid n < \omega \rangle$ forms a Prikry sequence for U. Let N be a model as in the Covering Lemma containing $\langle \delta_n \mid n < \omega \rangle$, $\langle \tau_n \mid n < \omega \rangle$. Then for all but finitely many n's, τ_n is an indiscernible for $\mathcal{F}(\kappa, 0)$ and δ_n is an indiscernible for $\mathcal{F}(\tau_n^*, 0)$. Suppose that for an infinite set $A \subseteq \omega$ for every $n \in A$, δ_n is not a (τ_n^*, τ_n^+)-accumulation point in N. Then for every $n \in A$ there are $\nu_n < \delta_n$ and $\xi_n < \tau_n^+$ in N so that for every c, β if $\nu_n < c < \delta_n$ and $\xi_n \leqslant \beta < \tau_n^+$ then c is not indiscernible for $\mathcal{F}(\tau_n^*, \beta)$. Suppose for simplicity that $A = \omega$. Returning to $V_1[G]$ pick names $\langle \boldsymbol{\nu}, \boldsymbol{\xi}_n \mid n < \omega \rangle$ for these two sequences. Let $\langle \emptyset, T \rangle$ force that $\langle \boldsymbol{\nu}_n \mid n < \omega \rangle$ and $\langle \boldsymbol{\xi}_n \mid n < \omega \rangle$ are such sequences, where $T \subseteq [\kappa]^{<\omega}$ is a tree in $V[G]$ with all the splittings belonging to U_0. Using standard arguments shrink T to some tree T^* so that there are functions $\{f_s, r_s \mid s \in T^*\}$ such that $\langle s^\frown\alpha, T^*_{s^\frown\alpha} \rangle \Vdash f_s^\vee(\alpha) = \boldsymbol{\nu}_n \wedge r_s^\vee(\alpha) = \boldsymbol{\xi}_n$ where $T_s^* = \{t \in T^* \mid t \,_{T^*}\!\!> s\}$ and $s \in T^*$. Now, recall that G_κ adds κ^+ functions and the forcing for adding it satisfies κ^+-c.c. So for some $\alpha < \kappa^+$, T^* and $\langle f_s, r_s \mid s \in T^* \rangle$ are in $V_1[G_{<\kappa}, G_\kappa \restriction \alpha]$. Since $|T^*| = \kappa$, the number of f_s, r_s is small and so there is $\rho < \kappa^+$, $\rho > \alpha$ so that in the ultrapower of $V_1[G]$ by U_0, $G_{\kappa,\rho}$ represents an ordinal above all the ordinals represented by f_s's ($s \in T^*$) and ρ is above all the ordinals represented by r_s's ($s \in T^*$). Let $t: \kappa \to \kappa$ represent ρ in $V_1[G]^\kappa/U_0$. Then $\{\delta < \kappa \mid f_s(\delta) < G_{\kappa\rho}(\delta) \text{ and } r_s(\delta) < t(\delta)\} \in U_0$.

So we can shrink the condition $\langle \emptyset, T^* \rangle$ level by level to a condition $\langle \emptyset, T^{**} \rangle$ forcing for every $n < \omega$

$$\text{“}\boldsymbol{\nu}_n < G_{\kappa,\rho}(\mathbf{c}_n) < \boldsymbol{\delta}_n \wedge \boldsymbol{\xi}_n < t(\mathbf{c}_n) < \boldsymbol{\tau}_n^+\text{”}.$$

Without loss of generality, assume that $\langle \emptyset, T^{**} \rangle$ belongs to the generic set. Then for all but finitely many n's, $G_{\kappa,\rho}(n)$ is an indiscernible for $\mathcal{F}(\tau_n^*, t(c_n))$ but $\nu_n < G_{\kappa\rho}(c_n) < \delta_n$ and $\xi_n < t(c_n) < \tau_n^+$, which is impossible. Contradiction. So for all but finitely many n's, δ_n is a (τ_n^*, τ_n^+)-accumulation point.

By [14], $\{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in κ, if there is an unbounded in κ sequence of accumulation points. Such a measurable κ was used in the construction above. We do not know if it is possible to remove the measurability of κ. Also we do not know if it is possible to have a sequence of accumulation points which is a least upper bound of diagonal sequences of indiscernibles.

Note that the model with accumulation points for measures over κ can be constructed starting from $o(\kappa) = \delta$ for any δ of cofinality $>\kappa$. Just use a simplified version of the above construction dealing only with measures over κ itself.

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