Note

A characterization of planar partial cubes

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Abstract

Partial cubes as well as planar graphs have been extensively investigated. In this note we introduce an additional topological kind of condition to the Chepoi’s expansion procedure that characterizes planar partial cubes. As a consequence we obtain a characterization of some other planar subclasses of partial cubes.

Keywords: Partial cubes; Planar graphs; Expansion

1. Introduction and preliminaries

Partial cubes are isometric subgraphs of hypercubes and have been largely investigated, see the book [8] and the references therein. The most important subclass of partial cubes are median graphs. Both classes are precisely determined with some expansion procedure. That is, any partial cube can be obtained from $K_1$ by a certain sequence of graph enlargements as shown by Chepoi [4]. The same holds for median graphs (only the rule is different) as proved by Mulder [10,11].

In [12] a topological kind of condition was introduced that ensures – together with Mulder’s expansion theorem – planarity of median graphs. A natural question arose whether a similar condition exists for planar partial cubes. Here we introduce such a condition, that is even more natural than the one in [12]. Surprisingly it gives a characterization – together with Chepoi’s expansion theorem – of planar partial cubes.

The same condition also holds for graph classes that lie between median graphs and partial cubes and can be obtained from $K_1$ by (some) expansion. For additional information on these classes of graphs we recommend [3].

In the remainder of this section we fix the notation. In the next section the main result follows and the discussion of planarity for other graph classes that can be obtained by some expansion procedure.

The distance $d_G(u, v)$ between two vertices $u$ and $v$ in a graph $G$ is defined as the number of edges on a shortest $u, v$-path. A subgraph $H$ of $G$ is called isometric, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$ and $H$ is convex if for every $u, v \in V(H)$ all shortest $u, v$-paths belong to $H$. Convex subgraphs are clearly isometric.

The Cartesian product $G \square H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ where the vertex $(a, x)$ is adjacent to $(b, y)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. Hypercubes or $n$-cubes $Q_n$
are Cartesian products of \( n \) copies of \( K_2 \). Isometric subgraphs of hypercubes are called partial cubes. Trees and even cycles are partial cubes.

Let \( G^1 \) and \( G^2 \) be two isometric subgraphs of a graph \( G \) that form a cover of \( G \) with nonempty intersection \( G^1 \cap G^2 = G' \). Note that there is no edge from \( G^1 \backslash G' \) to \( G^2 \backslash G' \). Graph \( H \) is an expansion of \( G \) with respect to \( G^1 \) and \( G^2 \) as follows. Take disjoint copies of \( G^1 \) and \( G^2 \) and connect every vertex from \( G' \) in \( G^1 \) with the same vertex of \( G' \) in \( G^2 \) with an edge. Such pairs of vertices will be called expansion neighbors. We say that expansion is isometric (connected) if \( G' \) is isometric (connected). It is not hard to see that copies of \( G' \) in \( G^1 \) and in \( G^2 \) and new edges between those two copies form the Cartesian product \( G \square K_2 \).

In [4] Chepoi has shown that \( G \) is a partial cube if and only if it can be obtained from \( K_1 \) by a sequence of expansions.

One of the most useful relations for the investigation of metric properties of graphs in general and partial cubes in particular is the Djoković–Winkler relation \( \Theta \), (cf. [5,13]). Two edges \( e = xy \) and \( f = uv \) of \( G \) are in the relation \( \Theta \) if

\[
d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).
\]

Clearly, \( \Theta \) is reflexive and symmetric, but not transitive in general. Winkler proved in [13] that transitivity of \( \Theta \) is characteristic for partial cubes among bipartite graphs.

Graph \( G \) is planar if it can be drawn in the plane such that any two edges cross only in an endvertex (if they are incident with the same endvertex). Such drawings are called plane drawings of \( G \). Any plane drawing of \( G \) divides the plane into regions which are called faces. One of those faces is unbounded and is called the exterior or the outer face, the others are interior or inner faces. Vertices that lie on an outer face are called outer vertices and others are inner vertices. Note that the boundary of every face of some plane drawing can be the boundary of an outer face of some other plane drawing of the same graph.

A graph \( G \) is outer planar if it is planar and embeddable into the plane so that all vertices lie on the outer face of the embedding. In [1] Behzad and Mahmoodian have shown that \( G \) is outer planar if and only if \( G \square K_2 \) is planar. For more information on planar graphs (or more general graphs on surfaces) we recommend [9].

2. Two-face expansions

Vertex \( u \) of a graph \( G \) is a cut-vertex if \( G - u \) has more components than \( G \), while edge \( e \) is a bridge if \( G - e \) has more components than \( G \). (We remove only the edge \( e \) without endvertices.)

Let \( G \) be a planar graph. We construct the graph \( G^- \) as follows. First delete all bridges from \( G \). Let \( u \) be a cut-vertex in the obtained graph. We delete \( u \), add copies of \( u \) back to all components incident with \( u \) in the natural way and denote this graph with \( G^-_u \). With \( G^- \) we denote the graph that remains from \( G \) after this procedure is executed for all cut-vertices of \( G \). For a tree \( T \) on \( n \) vertices we get the totally disconnected graph on \( n \) vertices for \( T^- \) and if \( G \) is obtained by amalgamating a vertex from cycle \( C_n \) with a vertex of cycle \( C_m \), then \( G^- \) consists of disjoint cycles \( C_n \) and \( C_m \).

Let \( H \) be an expansion of a planar graph \( G \) with respect to \( G^1 \) and \( G^2 \). Then \( H \) is a 2-face expansion of \( G \) if all vertices of \( G' = G^1 \cap G^2 \) are on one face of some plane drawing of \( G^1 \) and on one face of some plane drawing of \( G^2 \). First we need two technical lemmas.

**Lemma 1.** Let \( G \) be a planar 2-connected graph with a subdivision \( S \) of \( K_{2,3} \) and fix planar drawing \( D \). Suppose that there exist vertices \( u_1, u_2, \) and \( u_3 \) of \( S \) that lie pairwise on the same face of \( D \) but not all three on the same face. Then there exists a subdivision \( S' \) of \( K_{2,3} \) where \( \{u_1, u_2, u_3\} \) is one part of a partition of \( S' \).

**Proof.** Let \( \{v_1, v_2\} \) and \( \{w_1, w_2, w_3\} \) be the sets that form a partition of \( S \). Let \( P_1, P_2, \) and \( P_3 \) be the \( v_1, v_2 \)-paths from \( S \). If \( \{w_1, w_2, w_3\} = \{v_1, v_2, u_3\} \), there is nothing to prove. Thus first suppose that \( \{u_1, u_2, u_3\} \in S \) and that they do not form one set of a partition of \( S \).

If all \( u_i \)'s lie on one \( v_1, v_2 \)-path, say \( P_1 \), they have two common faces in \( S \). Suppose that \( u_1 \) is closest to \( v_1 \) on \( P_1 \), \( u_3 \) is closest to \( v_2 \) on \( P_1 \), and \( u_2 \) in between. To ensure that \( \{u_1, u_2, u_3\} \) do not all lie on the same face there must be at least one additional path in \( G \).

If there exists a \( x_1, x_2 \)-path where \( x_1 \) is on \( u_1, u_2 \)-subpath of \( P_1 \) and \( x_2 \) is not on \( u_2, u_3 \)-subpath of \( P_1 \). Then there must also exists a \( u_1, x_3 \)-path where \( x_3 \) is on \( u_2, u_3 \)-subpath of \( P_1 \), otherwise there is a contradiction with the
assumptions. But then \( \{u_1, u_2, u_3\} \) and \( \{x_1, x_3\} \) form a partition of a subdivision \( S' \) of \( K_{2,3} \). (The case when \( x_1 \) is on \( u_2, u_3 \)-subpath of \( P_1 \) and \( x_2 \) is not on \( u_1, u_2 \)-subpath of \( P_1 \) is symmetric.)

If there exists a \( x_1, x_2 \)-path where \( x_1 \) is on \( u_1, u_2 \)-subpath of \( P_1 \) and \( x_2 \) is on \( u_2, u_3 \)-subpath of \( P_1 \). Then there must also exists a \( u_2, x_3 \)-path where \( x_3 \) is on \( w_1, u_1 \)-subpath of \( P_1 \) or on \( u_3, u_2 \)-subpath of \( P_1 \), otherwise we have a contradiction with the assumptions. Note that \( x_1 \) can be \( u_1 \) and \( x_2 \) can be \( u_3 \), but not both at the same time. Suppose that \( x_3 \neq u_3 \). Again \( \{u_1, u_2, u_3\} \) and \( \{x_1, x_3\} \) form a partition of a subdivision \( S' \) of \( K_{2,3} \).

Now let two vertices, say \( u_1 \) and \( u_2 \), be on \( P_1 \) and \( P_2 \). Here \( u_3 \notin \{v_1, v_2\} \) otherwise we have one of the above cases. Now there must exist a \( x_1, x_3 \)-path where \( x_1 \) is on \( u_1, u_2 \)-subpath of \( P_1 \) to avoid a contradiction with the assumptions. Clearly \( \{u_1, u_2, u_3\} \) and \( \{x_1, x_1\} \) form a partition of a subdivision \( S' \) of \( K_{2,3} \).

\[ \square \]

**Lemma 2.** Let \( G \) be a planar 2-connected graph with a subdivision \( S \) of \( K_4 \) and fix planar drawing \( D \). Suppose that there exists vertices \( u_1, u_2, u_3, u_4 \) of \( S \) that every triple lie on the same face of \( D \) but not all four on the same face. Then there exists a subdivision \( S' \) generated by vertices \( U = \{u_1, u_2, u_3, u_4\} \) of \( K_4 \).

**Proof.** Let \( \{w_1, w_2, w_3, w_4\} \) generate a subdivision \( S \). If \( \{w_1, w_2, w_3, w_4\} = U \), there is nothing to prove. Otherwise there exists one \( w_i \), say \( u_1 \), that is different from all \( w_i \). Thus \( u_1 \) lies in exactly two faces \( F_1 \) and \( F_2 \) of \( S \). Suppose that not all vertices \( U \) are on the same face of \( S \). Then we may assume that \( u_2 \) is on \( F_1 \) but not on \( F_2 \) and \( u_3 \) is on \( F_2 \) but not on \( F_1 \). Clearly \( \{u_1, u_2, u_3\} \) do not lie on the same face in \( S \) and thus in \( D \), contrary to the assumption. Hence all vertices from \( U \) lie on the same face of \( S \), say \( F_1 \). Choose the notation so that \( w_1, w_2, \) and \( w_3 \) all lie on \( F_1 \) and that \( u_1, u_2, u_3, u_4 \) lie on \( F \) in such an order.

Suppose now that not all vertices from \( U \) lie on one \( w_j \)-path. Then clearly there must exists either \( u_1, u_2 \)- or \( u_2, u_4 \)-path in \( C \) to satisfy the assumption that all do not lie on one face. (Note that all other paths that separate vertices of \( U \) on \( F_1 \) make even more damage.) But with this we already have a contradiction since either \( \{u_1, u_2, u_4\} \) or \( \{u_2, u_3, u_4\} \) in the first case and \( \{u_1, u_2, u_3\} \) or \( \{u_1, u_3, u_4\} \) in the second case are not on the same face anymore.

Thus all vertices from \( U \) must be on the one \( w_j \)-path in \( S \), say \( w_1, w_2 \)-path. Suppose that they lie in the natural way. There is only one way to satisfy the assumptions: there must be a \( u_1, u_3 \)-path and \( u_2, u_4 \)-path in \( C \). But then \( U \) generates a subdivision of \( K_4 \) with this two paths, \( w_1, w_2 \)-path, \( w_1, w_3 \)-path, and \( w_3, w_2 \)-path.

\[ \square \]

**Theorem 3.** A graph \( G \) is a planar partial cube if and only if \( G \) can be obtained from \( K_1 \) by a sequence of 2-face expansions.

**Proof.** Suppose that \( G \) can be obtained from \( K_1 \) by a sequence of 2-face expansions. Then \( G \) is a partial cube by Chepoi’s expansion theorem. We will show that 2-face expansions preserve planarity by induction on the number of expansions. Let \( H_0 = K_1 \) and denote with \( H_k \) the graph obtained after \( k \) expansions with corresponding subgraphs \( H_k^1 \) and \( H_k^2 \) for the next expansion. By the induction hypothesis \( H_k \) is planar and consequently \( H_k^1 \) and \( H_k^2 \) are planar. Let us draw \( H_k^1 \) and \( H_k^2 \) in such a way, that the face of \( H_k^1 \) and the face of \( H_k^2 \) that correspond to the 2-face expansion are outer faces and that these two drawings have empty intersection. Then \( H_k^1 \) is on outer face of both \( H_k^1 \) and \( H_k^2 \) and \( H_k^1 \square K_2 \) is planar by the result of Behzad and Mahmoodian. Now just connect by an edge every vertex of \( H_k^1 \) in the drawing of \( H_k^1 \) with the same vertex of \( H_k^2 \) in the drawing of \( H_k^2 \). Clearly this can be done so that a new drawing of \( H_k+1 \) is planar. Hence \( G \) is planar.

Now suppose that \( G \) is a planar partial cube. Then \( G \) can be obtained by a sequence of expansions from \( K_1 \) by Chepoi’s theorem. Assume that one of this expansions, say \( H_k \) to \( H_{k+1} \) with respect to \( H_k^1 \) and \( H_k^2 \), is not a 2-face expansion for every drawing of graphs \( H_k^1 \) and \( H_k^2 \). We can assume that the vertices of \( H_k' = H_k^1 \cap H_k^2 \) are not on one face in any drawing of \( H_k^1 \). Choose index \( k \) to be the smallest of all such expansions and fix one drawing \( D \). We will denote with \( u' \) the expansion neighbor of \( u \in H_k^1 \) in the remainder.

We distinguish three cases. The first one occurs when there exists a pair of vertices from \( H_k^1 \) that are not on the same face of \( H_k^1 \). For the other two cases all pairs of vertices from \( H_k^1 \) are mutually on the same face of \( H_k^1 \), but not all on one face. Then note that all vertices must be in the same component \( C \) of \( (H_k^1)^{-} \) and that there is a subdivision of \( K_{2,3} \) or \( K_4 \) in \( C \), since \( C \) is not an outer planar graph.

**Case 1.** There exists a pair of vertices \( u, v \in H_k^1 \) that are not on the same face of \( H_k^1 \) of \( D \).

Let \( u \) and \( v \) be any pair of vertices from \( H_k^1 \) that are not on the same face of this plane drawing of \( H_k^1 \). Note that there exists a \( u', v' \)-path \( P \) in \( H_k^2 \) and the drawing \( D \) of \( H_{k+1} \) is not planar on this drawing. Since this holds for any drawing \( D \) of \( H_{k+1} \), also \( G \) is not planar contrary to the assumption.
Case 2. There exists a component $C$ of $(H_1^k)^-\setminus k$ that contains some vertices from $H_k'$ that are contained in a subdivision of $K_{2,3}$.

There must be at least three vertices $u_1, u_2, u_3 \in H_k'$ in $C$. By Lemma 1 $\{u_1, u_2, u_3\}$ form one set of a partition of a subdivision $S$ of $K_{2,3}$ and let $\{v_1, v_2\}$ be the other set. Furthermore, let $v_3$ be such a vertex form $H_k^2$ such that there exists vertex disjoint (with exception of $v_3$) $v_3, u'_3$-path $P_1, v_3, u'_2$-path $P_2$, and $v_3, u'_3$-path $P_3$ in $H_k^2$. (Note that $v_3$ can be one of $u'_1, u'_2, u'_3$.) Such a vertex exists, since $H_k^2$ is an isometric subgraph of $H_k$. We claim that $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ form a partition of the subdivision of $K_{3,3}$, which is impossible for planar graphs. Obviously $S'$ together with $v_3$ and paths $P_1, P_2, P_3$ form a subdivision of $K_{3,3}$.

Case 3. There exists a component $C$ of $(H_1^k)^-\setminus k$ that contains vertices, some vertices from $H_k'$ that are contained in a subdivision of $K_4$.

Let $W = \{w_1, w_2, w_3, w_4\}$ be vertices that generate the subdivision $S$ of $K_4$. First we will show that if there are only three vertices $u_1, u_2, u_3 \in H_k'$ on $S$, we have Case 2 when $u_1, u_2, u_3$ are pairwise on the same face but not all three on one face of $D$. Indeed, if this holds already on $S$ there exists $u_1$, say $u_1$, that is not in $W$. Also there exist $w_1, w_j \notin \{u_1, u_2, u_3\}, i \neq j, i, j \in \{1, 2, 3, 4\}$, such that only one of $u_1, u_2, u_3$ can be at $w_1, w_j$. Without loss of generality we may assume that $u_1$ is on $w_1, w_2$-path and $u_2$ and $u_3$ are not. But then $\{u_1, u_2, u_3\}$ and $\{w_1, w_2\}$ form a partition of a subdivision of $K_{2,3}$ and we have Case 2.

So let $U = \{u_1, u_2, u_3\}$ be on one face $F$ of $S$. We may assume that $F$ contains $w_1, w_2,$ and $w_3$. Let all vertices from $U$ lie on the same say $w_1, w_2$-path in the natural order. Then there must exists a $x_1, u_3$-path and $u_2, y_1$-path in $C$, where $x_1$ is between $u_1$ and $u_2$ and $y_1$ lies between $u_3$ and $u_2$, or there exists a $u_1, x_2$-path and $y_2, u_3$-path in $C$, where $x_2$ is between $u_2$ and $u_3$ and $y_2$ lies between $u_1$ and $u_2$, or there exists a $x_1, u_2$-path and $y_1, y_3$-path in $C$, where $x_3$ is between $u_1$ and $u_3$ and $y_3$ lies between $u_2$ and $u_3$. In each case $\{u_1, u_2, u_3\}$ and $\{x_1, y_1\}$ form a partition of a subdivision of $K_{2,3}$ and again we end up with Case 2.

If all three vertices are not on one $w_1, w_j$-path, there exists one vertex from $U$, that is the only vertex from $U$ on a path between $w_i$ and $w_j$. Note that if we wish to fulfill the assumptions there must be a path in $C$ from such a vertex to a vertex $x$ that lies between the other two vertices of $U$. Choose the notation so that $u_1$ is the only vertex from $U$ on a $w_1, w_2$-path. But again $\{u_1, u_2, u_3\}$ and $\{w_1, x\}$ form a partition of a subdivision of $K_{2,3}$, that is Case 2.

Thus we must have at least four vertices $u_1, u_2, u_3, u_4 \in H_k'$ on $S$. Even more, every triple of them must lie on the same face (but not all four on one face), otherwise we have Case 2 by the above fact. By Lemma 2 there exists a subdivision $S'$ of $K_4$ of vertices $\{u_1, u_2, u_3, u_4\}$. $S'$ is also a subdivision of wheel $W_3$, since $K_4$ is isomorphic to $W_3$. Choose the notation so that $u_1$ is the center of wheel $W_3$. Let $S'$ contain such $u_1, u_j$-paths $P_{ij}, i \neq j$ and $i, j \in \{1, 2, 3, 4\}$, that their length is as small as possible. We will show that $S'$ is an isometric subgraph of $C$ and with this a partial cube. Indeed, if there exist a shorter path between two different $u_i, u_j$-paths every triple from $\{u_1, u_2, u_3, u_4\}$ is not on the same face anymore and we have Case 1 or Case 2. By Theorem 1 of [6] either $u_2, u_3, u_4$ are neighbors of $u_1$ or all are at distance 2 to $u_1$ ($S'$ is isomorphic to a graph obtained from $K_4$ by subdividing each edge exactly once). Suppose that $v$ is a common neighbor of $u_1$ and $u_2$ that is not in $H_k'$. Then there also exists a common neighbor $z \in H_k^2$ of $u_1$ and $u_2$ that is not in $H_k'$. Now edges $u_1v$ and $u_1z$ are both in relation $\Theta$ with $u_3w$ where $w$ is a common neighbor of $u_2$ and $u_3$, which is impossible for partial cubes.

Thus $u_1$ is a vertex in $H_k' \setminus H_k^2$ with three neighbors $u_2, u_3, u_4 \in H_k'$ in the same component $C$ of $(H_k^1)^-\setminus k$. Denote with $F_{ij}$ the face that contains $u_i$ and $u_j, i \neq j, i, j \in \{2, 3, 4\}$. We claim that $\{u_1, u_2, u_3, u_4, u'_i\}$ form a subdivision of $K_5$ in $H_{k+1}$ — a contradiction with the planarity of $G$.

Vertex $u_1$ is a neighbor of $u_2, u_3, u_4$, and $u'_i$. Paths $u'_i u'_2 u'_2, u'_i u'_3 u'_3$, and $u'_i u'_4 u'_4$ are edge disjoint paths from $u'_i$ to $u_2, u_3$, and $u_4$, respectively. Even more, none of the edges on those paths is in $H_k$. Boundaries of faces $F_{23}, F_{34}$, and $F_{24}$ without paths $u_2u_1u_3, u_3u_1u_4$, and $u_2u_1u_4$, respectively, complete the desired subdivision. 

The proof of the above theorem has structural similarities to the proof of Theorem 3 in [12], however the main difference is that $H_{1,1}^k \cap H_{2,2}^k$ need not be connected in the case of partial cubes.

In [7] Imrich and Klavžar introduced two subclasses of partial cubes: almost-median and semi-median graphs. Almost-median graphs are graphs in which certain subgraphs are isometric and semi-median are graphs for which the same subgraphs are connected. (We do not need the exact definition here.) One motivation for the introduction of these classes of graphs was that almost-median graphs are "clearly" graphs that can be obtained from $K_1$ by isometric expansions and semi-median graphs can be obtained from $K_1$ by connected expansions. This is not true as shown
in [7,2]. However both classes have an expansion characterization with some additional condition, see [2]. Thus graphs that can be obtained from $K_1$ by an isometric (connected) expansion are some other class of partial cubes. For relations between these classes see [3].

If in Theorem 3 we use the above expansions instead of the ordinary expansion, we obtain characterizations of planar almost-median graphs, planar semi-median graphs, planar graphs that can be obtained from $K_1$ by isometric expansions, and planar graphs that can be obtained from $K_1$ by connected expansions. Proofs are the same only that we replace the ordinary expansion with the appropriate other expansions.

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