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A Unifying Theorem for Algebraic Semantics and Dynamic Logics*

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A unified single proof is given which implies theorems in such diverse fields as continuous algebras of algebraic semantics, dynamic algebras of logics of programs, and program verification methods for total correctness. The proof concerns ultraproducts and diagonalization.

1. INTRODUCTION

A widely investigated question today is the problem of finding logics for expressing and proving program properties: various kinds of such logics have been studied, e.g., equational, algorithmic, dynamic, temporal, etc. A common question in all the cases, is whether the logics can be first order, since then it inherits all the nice properties and results known about first-order logics. A way to address this question is to note that models of the logics, in which programs take values, are in a one-to-one correspondence with classes of structures, or algebras; then, the problem translates into: are the corresponding structures first-order definable? E.g., dynamic logics with while loops correspond to classes of continuous dynamic algebras. This raises the question of which classes of continuous dynamic algebras are

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first-order axiomatizable. Now, in general, it is not so easy to prove that a class is first-order axiomatizable because one might have to look rather hard for the axioms; however, it is even more difficult to prove that it is not because then one has to show that no adequate axiom system exists. In this paper, we consider a general tool to prove that algebraic structures are not first-order axiomatizable; we then use this tool to prove properties of very different kinds of structures for models of programs. We show that a class of structures which is first-order axiomatizable is in some sense "bounded"; i.e., roughly speaking, all iterations stop after a finite number of steps. This is done by using the technical tool of ultraproducts and closure under ultraproducts.

This theorem then applies to a wide variety of structures for logics of programs, e.g., to prove (un)boundedness of a class of structures, non-first-order definability, etc. We show the consequences of this theorem in three kinds of very useful classes of models for programs, namely: (1) continuous algebras, which are the fundamentals of both denotational and algebraic semantics and are used in the (in)equational logics for programs; (2) dynamic algebras, which correspond to the dynamic logics of programs; and (3) general computational models of programs and program verification methods. In all three cases, our theorem yields very easy proofs of either non-first-order definability and/or necessary properties for first-order axiomatization and closure under ultraproducts of the classes considered. We stress the fact that the necessary conditions, (un)boundedness of the class, are usually very easy to check. The case of dynamic algebras answers a question by Kozen, namely, which classes of dynamic algebras are closed under ultraproducts.

We deduce as easy corollaries of our theorem proofs of results in (Andréka and Németi, 1976; Courcelle and Guessarian, 1978; Guessarian, 1981; Kfoury and Park, 1975).

The paper is organized as follows: Section 2 is devoted to the proof of the main theorem, and Section 3, to deriving consequences from it.

2. The Main Theorem

Notations. N denotes the set of natural numbers. If $(D, \leqslant)$ is an ordered set, an ω-chain is an increasing sequence $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$, indexed by N. For a class K of algebras, denote by $\mathrm{Up}\, K$ the closure of K under ultraproducts.

Our main theorem will state that, roughly speaking, in a class of algebras closed under ultraproducts, if all chains have a least upper bound, then all chains have a finite number of different elements. To this end we first introduce the technical notion of a chain that we need.

DEFINITION 1. Let D be a poset (partially ordered set), $h=\{h_n \mid n \in N\}$ an ω-chain, and $x \prec_h y$ (abbreviated into $\prec$) the relation "x is the immediate predecessor of y in h," i.e., for some n in N, $x=h_n$ and $y=h_{n+1}$. Let $\varphi(x, y, \pi)$ be a formula, where π is a sequence of variables, possibly of length 0. h is said to be a φ-chain iff there exists a π such that, for all elements x, y of h: $x \prec y \Rightarrow \varphi(x, y, \pi)$.

THEOREM 1 (unifying theorem). *Let K be a class of ordered algebras which is closed under ultraproducts i.e.,* Up $K=K$. *Suppose every algebra in K satisfies the property that every φ-chain has a least upper bound; then, there exists a finite n such that every φ-chain in any algebra of K has at most n distinct elements, i.e., has length at most n.*

Proof. The idea is to proceed by diagonalization. Assuming φ-chains are unbounded, we can find an infinite sequence of φ-chains of strictly increasing length. Then, the corresponding sequence in a suitable ultraproduct has no least upper bound because of a diagonal argument.

Suppose the result is not true and, for any n in N, there exists I_n in K such that I_n has a φ-chain of length at least n, i.e., $\exists i_n: N \to I_n$ $i_n(1) < i_n(2) < \cdots < i_n(n)$ and $\exists \pi(n) \, \forall j \, \varphi(i_n(j), i_n(j+1), \pi(n))$.

Let U be a non-principal ultrafilter on N and let $I=\Pi I_n/U$. Let $(p_n)_{n\in N}$ be the sequence of elements of ΠI_k defined by: $\forall n,k \; p_n(k)=i_k(n)$. The elements of p_n can be represented as a matrix

$$\begin{array}{l} p_1 \qquad p_2 \qquad p_3 \qquad \cdots \\ p_1(1) \leqslant p_2(1) \leqslant p_3(1) \leqslant \cdots \in I_1 \\ p_1(2) < p_2(2) \leqslant p_3(2) \leqslant \cdots \in I_2 \\ p_1(3) < p_2(3) < p_3(3) \leqslant \cdots \in I_3 \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \end{array}$$

For $p=(p(k))_{k\in N}$ in ΠI_k let $\bar{p}=p/U$ be its equivalence class in I. Then for any n and $k>n$, we have $p_n(k)<p_{n+1}(k)$; U being non-principal contains all complements of finite sets, hence $\bar{p}_n < \bar{p}_{n+1}$ for any n. Moreover, if π is the sequence of variables of I defined by $\pi=(\pi(n))_{n\in N}$, $\forall k$, $\forall n$: $\varphi(I_k(n), i_k(n+1), \pi(k)) = \varphi(p_n(k), p_{n+1}(k), \pi(k))$, hence, $\varphi(\bar{p}_n, \bar{p}_{n+1}, \pi)$ and $(\bar{p}_n)_{n\in N}$ is a φ-chain. Let $\bar{q}=\text{lub}\{\bar{p}_n \mid n \in N\}$, in I. Let now p in ΠI_k be defined by $p(n)=p_n(n)$ and let q be in $\bar{q}$,

$\forall n$, $\forall k \geqslant n$: $p_n(k)=i_k(n) \leqslant i_k(k)=p(k)$, hence $\forall n$, $\bar{p}_n \leqslant \bar{p}$, $\bar{p}$ is thus an upper bound for the $\bar{p}_n$'s: $\bar{q}$ being the least upper bound of the $\bar{p}_n$'s, $\bar{q} \leqslant \bar{p}$; hence by the definition of ultraproduct of ordered algebras, $C=\{k \mid q(k) \leqslant i_k(k)=p(k)\}$ is in U. Let $B_2=\{k \mid q(k) \geqslant p_2(k)\}$; for $k \in A=C \cap B_2$, $q(k) \geqslant p_2(k) > i_k(1)$. For each k in A let j_k be the largest integer such that

$j_k \leqslant k$ and $i_k(j_k) < q(k)$. Let q' in ΠI_k be defined by $q'(k) = i_k(j_k)$ for k in A, and $q'(k)$ arbitrary for k not in A. We will show that $\bar{q}'$ is an upper bound of the $\bar{p}_n$'s, a contradiction.

Obviously $\bar{q}' < \bar{q}$; moreover, since for all k in $A \in U$, $q'(k) < q(k)$, $\bar{q}' \neq \bar{q}$. Suppose $\bar{q}'$ is not an upper bound for the $\bar{p}_n$'s . Then for some n in N, $\{k \in A / p_n(k) = i_k(n) \leqslant i_k(j_k)\}$ does not belong to U; hence $B = \{k \in A / i_k(j_k) < i_k(n)\}$ belongs to U. Let $A' = \{k / p_{n+1}(k) = i_k(n+1) \leqslant q(k)\}$; $A' \in U$, let $A'' = B \cap A' \cap \{m / m \geqslant n+1\}$, then $A'' \in U$, $A'' \neq \varnothing$ and $\forall k \in A''$, $i_k(j_k) < i_k(n) < i_k(n+1) \leqslant q(k)$. But this contradicts the definition of j_k. Hence for any n, $\bar{p}_n \leqslant \bar{q}'$ and this shows that $\bar{q}$ is not the lub of the $\bar{p}_n$'s. Hence $\bar{p}_n$ is a φ-chain without lub in I, a contradiction. ∎

Theorem 1 has the following (slightly weaker) corollary:

COROLLARY. *Let K be a first-order definable class of ordered algebras such that every φ-chain in K has a least upper bound; then, there exists a finite n such that every φ-chain in K has length at most n.*

For that corollary, J. Tiuryn gave a direct and short proof, based on the compactness theorem, which proceeds as follows:

Proof of the Corollary. Let T be the first-order axioms defining K. Assume K contains algebras with arbitrarily large φ-chains, then K contains an algebra with a φ-chain having no lub. Extend the language by adding a new unary function symbol F and new constants π, d_0, $d_1, \ldots, d_n, \ldots$.

Define

$$S = T \cup \{(d_i < d_{i+1}) \wedge \varphi(d_i, d_{i+1}, \pi) \mid i \in N\}$$
$$\cup \{\forall x \{d_{i+1} \leqslant x \Rightarrow ((d_i \leqslant f(x)) \wedge (f(x) < x))\} / i \in N\}.$$

Every finite $S_n \subset S$ has a model. Assume all the d_i's occurring in S_n have a subscript $i \leqslant n$; take I_n in K having a φ-chain $d_0 < d_1 < \cdots < d_n$. Define $f: I_n \to I_n$ by $f(x) = \textit{if } d_1 \nleqslant x \textit{ then } d_0 \textit{ else } \max\{d_i \mid d_{i+1} \leqslant x,\ 0 \leqslant i < n\}$. Hence, by the compactness theorem, S has a model I whose reduct (obtained by forgetting the d_i's and f) belongs to K. Then, the chain $d_0 < d_1 < \cdots$ is a φ-chain without a lub: if c is any upper bound of the d_i's, then $f(c) < c$ is also an upper bound of the d_i's. ∎

This corollary is sufficient for many applications, but it is not enough to prove Theorem 2, which solves a problem by Kozen.

3. Applications

Let $\Sigma = \langle F, \langle \leqslant \rangle \rangle$ be a signature where the symbols in F are possibly many-sorted operation symbols. Consider Σ-algebraic structures A such that $\leqslant$ is a partial ordering on the underlying carrier; A is said to be complete (with respect to the ordering) iff every chain has a least upper bound, in short a lub.

Definition 2. The height of a Σ-algebraic structure A is

$$h(A) = \begin{cases} \max\{p \mid a_1 \leqslant a_2 \leqslant \cdots \leqslant a_p \text{ and } a_i \neq a_j \forall i \neq j\} & \text{if this max is finite} \\ \omega \quad \text{otherwise.} \end{cases}$$

A class K of Σ-algebraic structures is *n-bounded* if all algebras A in K have height $\leqslant n$. A class K is *bounded* iff it is n-bounded for some n in N.

The height of a Σ-algebraic structure A is thus the maximum length of a chain in A if this maximum is finite, ω otherwise. This notion could be extended to the case where Σ is a many-sorted signature, by taking the maximum height of the various carriers; however, we do not need it here. Then, Theorem 1 has the following consequences:

Theorem 2 (Courcelle and Guessarian, 1978). *If a class of K complete Σ-algebraic structures is closed under ultraproducts, then K is bounded.*

Proof. Apply Theorem 1 with π the empty sequence of variables and ϕ defined by $\varphi(x, y)$: $x \leqslant y$. ∎

Note that, in exactly the same manner, by forgetting the algebra operations in the definitions above, we can derive

Corollary 3. *If a class K of complete partially ordered sets is closed under ultraproducts, then K is bounded.*

Classes of n-bounded algebras are particularly useful in semantics of programming languages. The most widely studied such class is the class D of 2-bounded algebras, also called discrete algebras, because the ordering on their carriers is a discrete, or flat, ordering (Nivat, 1975).

Corollary 4. *Any first order definable class of complete Σ-algebraic structures is bounded.*

Proof. Note that first-order definable classes are closed under ultraproducts. (See Chang and Keisler, 1973, p. 173). ∎

Ordered algebras and complete partially ordered sets are very important in giving semantics to programming languages. Both denotational semantics (Scott, 1976; Plotkin, 1976; and Smyth, 1978) and algebraic semantics (Guessarian 1981; Nivat, 1975; ADJ 1977) use complete or ordered, or most often bounded, algebras to define semantics. Corollary 4 gives one reason why bounded classes occur so often.

We now turn to another area of application of the unifying theorem, namely dynamic algebras, which are equally important in the theory of programming. Dynamic algebras intuitively correspond to the dynamic logic of programs, in the same way as Boolean algebras correspond to classical logics. They were first introduced by Kozen (1982) and Pratt (1982) and studied in (Németi, 1981, 1982a; Reiterman and Trnkova, 1984). For an overview and connections see (Jónsson, 1984; Henkin, Monk, and Tarski, 1985).

Formally, let $\Sigma = \langle \vee, ;, {}^*, \cdot, -, \Diamond \rangle$ be the signature on the two element set of sorts {Bool, Act}, defined as follows:

$\vee$, ;	are of type: Act × Act → Act
$*$	is of type: Act → Act
$\cdot$	is of type: Bool × Bool → Bool
$-$	is of type: Bool → Bool
$\Diamond$	is of type: Act × Bool → Bool
Σ	will be the above defined signature in the rest of the paper.

Intuitively $\langle \text{Bool}, \cdot, - \rangle$ is a boolean algebra whose elements correspond to predicates, e.g., some post-conditions to be satisfied by the programs ; $\cdot$ corresponds to "and" (intersection), and $-$ to negation (complementation). $\langle \text{Act}, \vee, ;, {}^* \rangle$ similarly represents an algebra of actions performed by programs, and $\vee$ corresponds to "or" (union), ; to composition, and * to iteration. $a \Diamond p$ then corresponds to the total correctness of the action performed by the program a with respect to the post-condition p, i.e., $a \Diamond p$ is the set of initial states such that p holds after an execution of a starting with a state in $a \Diamond p$.

DEFINITION 3. An algebra $\mathbf{D} = \langle \mathbf{A}, \mathbf{B}, \Diamond \rangle$ of signature Σ is said to be a *dynamic set algebra*, in short Ds, if there is a set U, called the base of $\mathbf{D}$, such that

(1) $\mathbf{B} = \langle B, \cdot, - \rangle$ is a boolean set algebra of some subsets of U, i.e., $B \subseteq \mathscr{P}(U)$, $\cdot$ is intersection, and $-$ complementation.

(2) $\mathbf{A} = \langle A, \vee, ;, {}^* \rangle$ is a Kleene set algebra with base U, i.e., $A \subseteq \mathscr{P}(U \times U)$, $\vee$ is set-theoretic union, ; is composition of relations, * is the transitive reflexive closure of a relation, namely, $a^* = \cap \{ b \subseteq U \times U \mid a \subseteq b$ and b is a reflexive and transitive relation on $U \}$.

(3) $a \diamond p = \{u \in U \mid \exists v \in p \text{ such that } \langle u, v\rangle \in a\}$, for any a in A and p in B.

Dynamic set algebras are called Kripke structures and denoted by K in (Pratt, 1982).

DEFINITION 4. Let $\mathbf{D} = \langle \mathbf{A}, \mathbf{B}, \diamond \rangle$ be an algebra of signature Σ, and a in A, $0 < n \in N$. Let a^n be defined inductively by

$$a^1 = a$$

$$a^{n+1} = a^n \vee (a; a^n) \qquad \forall n > 0.$$

Let K be a class of algebras of signature Σ. K is said to be strongly n-bounded, in short n-s-bounded, iff $K \models (\forall a),\ a^n = a^{n+1}$. K is *strongly bounded*, or s-bounded, iff it is n-s-bounded for some n.

Recall that for a formula φ, $K \models \varphi$ means that φ holds in all algebras of K.

Note that the above defined sequence a^n is monotone whenever the relation $a \leqslant b \Leftrightarrow a \vee b = b$ defines an ordering on A.

DEFINITION 5. The class Da of *dynamic algebras* is the variety[1] of algebras with signature Σ defined by

1. Equations defining the usual boolean algebras, e.g.,

$$x \cdot y = y \cdot x$$

$$x \cdot -(-y \cdot -z) = -(-(x \cdot y) \cdot -(x \cdot z))$$

$$x \cdot -(y \cdot -y) = x$$

$x + y$ abbreviates $-(-x \cdot -y)$, 0 abbreviates $x \cdot -x$, and $x \leqslant y$ abbreviates $x + y = y$.

2.a $a \diamond 0 = 0$

b $a \diamond (y_1 + y_2) = (a \diamond y_1) + (a \diamond y_2)$

3. $a_1 \vee a_2 \diamond y = (a_1 \diamond y) + (a_2 \diamond y)$
4. $a_1 ; a_2 \diamond y = a_1 \diamond a_2 \diamond y$
5. $y + a \diamond a^* \diamond y \leqslant a^* \diamond y \leqslant y + a^* \diamond (a \diamond y \cdot -y)$.

[1] K is a variety (or equational class) iff K is definable (i.e., axiomatizable) by a set of equations. Varieties are characterized by the theorem (Birkhoff, 1935): K is a variety iff K is closed under homomorphisms, subalgebras, and products. See (Grätzer, 1979) for the theory of varieties.

A dynamic algebra is said to be *continuous* if in addition it satisfies

6. $\forall n \in N \; z \geqslant a^n \diamond y \Rightarrow z \geqslant a^* \diamond y$

The class of continuous dynamic algebras is denoted by Cn.

A dynamic algebra is said to be *strongly continuous* if, in addition,

7. the relation $a \leqslant b \Leftrightarrow a \vee b = b$ is a partial ordering on A, and

8. $\forall a \in A$, $a^* = \text{lub}\{a^n \mid n \in N\}$, and the operation ; is continuous in the sense that it commutes with lub's.

These conditions are expressed by infinitary quasi-equations (or generalized equational implications) similar to 6. Note that such quasi-equations are not first order.

The class of strongly continuous dynamic algebras is denoted by Kn.

Continuous dynamic algebras and the variety Da were introduced by Pratt (1982) and strongly continuous dynamic algebras by Kozen (1982). Intuitively, the continuity assumptions roughly amount to require that a^* be the *least* upper bound of the a^n's. In dynamic set algebras for instance, $a^* = \bigcup\{a^n \mid n \in N\}$, hence $\text{Ds} \subseteq \text{Cn}$ and $\text{Ds} \subseteq \text{Kn}$. Clearly, we have the following inclusions:

$$\text{Ds} \subseteq \text{Kn} \subseteq \text{Cn} \subseteq \text{Da}.$$

The difference between Kn and Cn is that in Cn only the boolean algebra part is required to be ordered and continuous, whereas in Kn, also the action algebras are ordered and continuous.

COROLLARY 5. *Let* $K \subseteq \text{Kn}$ *be a class of strongly continuous dynamic algebras, if the closure* Up K *of* K *under ultraproducts is contained in* Kn, *then* K *is s-bounded.*

The proof uses

LEMMA 6. *If a class* $K \subseteq \text{Kn}$ *of strongly continuous dynamic algebras is closed under ultraproducts, then* K *is s-bounded.*

Proof. We need to prove that the sequence $\{a^n \mid n \in N\}$ is of finite length. So we adjust the definition of φ to make this sequence into a φ-chain; then, theorem 1 can be applied.

We first remark that strongly continuous dynamic algebras are naturally ordered by $a \leqslant b \Leftrightarrow a \vee b = b$. Apply then theorem 1, with φ defined by: $\varphi(x, y, \pi) \Leftrightarrow y = (x \vee (\pi; x))$, for $x, y \in A$ where $\mathbf{D} = \langle \mathbf{A}, \mathbf{B}, \diamond \rangle$ is an algebra in K; π here is the sequence of variables, of length 1.

Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$ be a φ-chain; then we prove that for any n

in N $x_n = x_1 \vee \pi^{n-1}; x_1$. This is true for $n=2$ by the definition of φ. Assume it holds for n; then: $x_n \leqslant x_{n+1} \Rightarrow x_{n+1} = x_n \vee (\pi; x_n)$ by definition of φ.

$$x_{n+1} = x_1 \vee (\pi^n ; x_1) \vee \pi ; (x_1 \vee \pi^n ; x_1)$$
$$= x_1 \vee \pi^n ; x_1 \vee \pi; x_1 \vee \pi; \pi^n ; x_1$$

Now $\pi \leqslant \pi^n$, hence we may drop the component $\pi ; x_1$ and $x_{n+1} = x_1 \vee (\pi^n \vee \pi ; \pi^n) ; x_1 = x_1 \vee \pi^{n+1} ; x_1$.

In strongly continuous dynamic algebras, π^* is the lub of $\{\pi^n \mid n \in N\}$; moreover $\vee$ and ; commute with lub's; hence every φ-chain $h = \{x_1 \vee \pi^n ; x_1 \mid n \in N\}$ has a lub which is $x_1 \vee \pi^* ; x_1$. So, K satisfies the hypotheses of theorem 1, hence, for some n in N, every φ-chain in K is of length at most n. This implies that K is s-bounded: for let $p \in A$ be an action, then $h = \{p^n \mid n \in N\}$ is a φ-chain, hence it is of length n. So $\forall p\ p^n = p^{n+1}$. ∎

Proof of Corollary 5. Assume Up $K \subseteq$ Kn. Let $L =$ Up K. Then Up $L =$ Up (Up K) = Up $K = L \subseteq$ Kn; hence L satisfies the hypotheses of lemma 6; L is thus s-bounded, and so is K since $K \subseteq L$. ∎

COROLLARY 7 (Andréka and Németi, 1976). *Assume $K \subseteq$ Ds is a class of dynamic set algebras which is not s-bounded. Then*: (i) Up $K \not\subseteq$ Kn *and* (ii) Up $K \not\subseteq$ ISP Ds, *where* ISP Ds *is the closure of* Ds *under isomorphisms, subalgebras, and products, or equivalently* (Grätzer, 1979) *the infinitary quasivariety generated by* Ds.

Proof. (i) is Corollary 5, and (ii) stems from the fact that ISP Ds $\subseteq$ Kn; Kn is an infinitary quasi-variety, namely a class defined by generalized equational implications of the form $\bigwedge_{i \in I} e_i \Rightarrow e$, where e, e_i, $i \in I$, are equations, possibly infinitely many; hence (Andréka and Németi, 1976; Grätzer, 1979), Kn is closed under isomorphisms, subalgebras, and products. ∎

Let us comment some more on some useful properties of ISP K. Sain has shown that the smallest infinitary quasivariety containing a class K is ISP K (see Németi and Sain, 1982; Grätzer, 1985). Another important property of ISP K is that it is the smallest class having free algebras (possibly with defining conditions) and containing K. Using these (and related) results, Makowsky and Mahr proved that infinitary quasi-varieties are of central importance for computer science, see (Ehrig and Mahr, 1985). Similar results can be found in (Burstall and Goguen, 1982).

Corollary 7 generalizes to the case where we substitute Kn for Ds. However, neither Corollary 5 nor 7 can be improved by replacing Kn with Cn. Namely, FACT 8. *There is* $K \subseteq \mathrm{Ds}$ *such that* K *is* not *s-bounded but* $\mathrm{Up}\, K \subseteq \mathrm{Cn}$.

Proof. Let Z be the set of integers and $a = \{\langle n, n+1\rangle \mid n \in Z\}$. Let $\mathbf{D} = \langle \mathbf{A}, \mathbf{B}, \Diamond\rangle$ be a Ds with base Z, generated by $\langle\{a\}, \varnothing\rangle$. Then $B = \{\varnothing, Z\}$, is the 2-element boolean algebra, hence $\mathbf{D}$ is in Cn. A, the action set, is generated by a and the operations ;, $\vee$, *; A will contain, e.g., a, a^*, $a;a$,..., a^2,..., $(a\,;a)^2$, etc. Thus, the class $K = \{\mathbf{D}\}$ consisting of the single algebra $\mathbf{D}$ is not s-bounded, since $a^n \neq a^{n+1} \forall n$. Now clearly

$$K \models \forall y (y = 0 \text{ or } y = 1).$$

hence

$$\mathrm{Up}\, K \models \forall y (y = 0 \text{ or } y = 1).$$

thus $\mathrm{Up}\, K \subseteq \mathrm{Cn}$ (since every dynamic algebra with a finite boolean sort is obviously continuous). ∎

But Fact 8 should be compared with Corollary 9 below. We first define the notion of boundedness of a class $K \subseteq \mathrm{Da}$ of (general) dynamic algebras.

DEFINITION 6. A class $K \subseteq \mathrm{Da}$ of dynamic algebras is said to be n-bounded if $K \models \forall (a, y)\ a^{n+1} \Diamond y = a^n \Diamond y$. K is said to be bounded if it is n-bounded for some n.

COROLLARY 9. *Let* $K \subseteq \mathrm{Da}$ *be a class of dynamic algebras. If* $\mathrm{Up}\, K \subseteq \mathrm{Cn}$ *is a class of continuous dynamic algebras, then* K *is bounded.*

Proof. As in Corollary 5 it suffices to prove that $L = \mathrm{Up}\, K$ is bounded; therefrom will follow that $K \subset L$ is also bounded. We will again apply Theorem 1, where φ is now the formula: $\varphi(y_1, y_2, \pi) \Leftrightarrow y_2 = y_1 + \pi \Diamond y_1$ (recall that $+$ is the sum of the Boolean algebra). A φ-chain is an ω-chain $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$ such that $\forall n$, $x_{n+1} = x_n + \pi \Diamond x_n$. By induction on n we prove that: $\forall n$, $x_{n+1} = x_1 + \pi^n \Diamond x_1$. This is obviously true for x_2; assume it holds for x_{n+1}, then

$$\begin{aligned} x_{n+2} &= x_1 + \pi^n \Diamond x_1 + \pi \Diamond (x_1 + \pi^n \Diamond x_1) && \text{by induction} \\ &= x_1 + \pi^n \Diamond x_1 + \pi \Diamond x_1 + \pi \Diamond \pi^n \Diamond x_1 && \text{by 2(b)} \\ &= x_1 + \pi^n \Diamond x_1 + \pi \Diamond x_1 + \pi\,; \pi^n \Diamond x_1 && \text{by 4.} \end{aligned}$$

By induction on n, we check that $\pi \Diamond x \leqslant \pi^n \Diamond x$: this is obvious for $n = 1$, and $\pi^{n+1} \Diamond x = (\pi^n \vee \pi\,; \pi^n) \Diamond x = \pi^n \Diamond x + (\pi\,; \pi^n) \Diamond x \geqslant \pi^n \Diamond x$. Hence

$$\begin{aligned} x_{n+2} &= x_1 + \pi^n \Diamond x_1 + (\pi\,; \pi^n) \Diamond x_1 \\ &= x_1 + (\pi^n \vee \pi\,; \pi^n) \Diamond x_1 && \text{by 2(b)} \\ &= x_1 + \pi^{n+1} \Diamond x_1. \end{aligned}$$

Now, by 6, $\pi^* \diamond x_1 = \text{lub}\{\pi^n \diamond x_1 \mid n \in N\}$, hence, $\{x_1 + \pi^n \diamond x_1 \mid n \in N\}$ has a lub which is equal to $x_1 + \pi^* \diamond x_1$. The hypotheses of Theorem 1 apply, thus ensuring that every φ-chain in K has length at most n. Then for any a, y, $a^n \diamond y$ is obviously a φ-chain of parameter a, hence is of length n, i.e., L is n-bounded. ∎

We now state a result which was known to the fathers of the theory of dynamic algebras (Kozen, 1982; Pratt, 1982), namely that no first-order axiom can express that a^* is the lub of $\{a^n \mid n \in N\}$. The proof given here is a very short one.

COROLLARY 10. (1) *Neither* Cn *nor* Kn *is first-order axiomatizable, and*

(2) *there exists a dynamic algebra D such that, for some action a, a^* is not the* lub *of* $\{a^n \mid n \in N\}$. *Moreover, D is not even continuous.*

Proof. (2) Da, being a variety, is closed under ultraproducts, so, if (2) were false, then Kn = Cn = Da would be closed under ultraproducts, hence would consist only of s-bounded algebras. Now, let $\mathbf{D} = \langle \mathbf{A}, \mathbf{B}, \diamond \rangle$ be any dynamic set algebra on the base set N, with $a \in A$, where a is the relation $a = \{\langle n+1, n\rangle \mid n \in N\}$, and with $y = \{0\} \in B$. Let $n \in N$, then $n+1 \notin a^n \diamond y$ but $n+1 \in a^{n+1} \diamond y$; hence $\mathbf{D}$ is not bounded. This example shows that both inclusions Kn $\subseteq$ Da and Cn $\subseteq$ Da are strict.

(1) Similarly, if $K =$ Kn (resp. $K =$ Cn) were first-order axiomatizable, then K would be closed under ultraproducts, and hence would be bounded, which is not the case. ∎

The next corollary gives necessary and sufficient conditions for a class K of dynamic set algebras to be closed under ultraproducts and first-order axiomatizable.

Recall that the *-free reduct is obtained by forgetting the * operation.

COROLLARY 11. *Let* $K \subseteq$ Kn. *Then*

(i) *K is first-order axiomatizable iff: the *-free reduct of K is first-order axiomatizable and K is s-bounded*

(ii) *K is closed under ultraproducts iff: the *-free reduct of K is closed under ultraproducts and K is s-bounded.*

Proof. (ii) $\Rightarrow$ obvious.

$\Leftarrow$ assume K is s-bounded and the *-free reduct of K is closed under ultraproducts. Then, there is an n such that $K \models x^* = x^n$. So x^* is first-order definable in K, hence, by our assumption, K is closed under ultraproducts.

(i) Assume K is first-order axiomatizable. Then, by Corollary 5, K is n-s-bounded for some n, hence $K \models x^* = x^n$. Let Ax be a set of first-order axioms defining K. Let Ax' be obtained from Ax by replacing x^* everywhere with x^n. Then Ax' is a first-order axiomatization for the $*$-free reduct of K. To prove the other direction, assume Ax' defines the $*$-free reduct of K and that K is n-s-bounded. Then, the set $Ax = Ax' \cup \{x^* = x^n\}$ defines K, which is thus first-order axiomatized. ∎

THEOREM 12. *A class* $K \subset \mathrm{Cn}$, *or* $K \subset \mathrm{I\,Ds}$, *where* I Ds *denotes the closure of* Ds *under isomorphisms, is closed under ultraproducts iff the* $*$*-free reduct of K is closed under ultraproducts and K is polynomially equivalent with its* $*$*-free reduct* (*i.e.,* $*$ *is term-definable from the rest of the operations in K*).

Proof. Immediate from Corollary 11. ∎

It might be of interest to note that, though ultraproducts in the standard sense lead out from the class of dynamic set algebras because of the above results, the category-theoretic version of the ultraproduct notion of Sain (Sain, 1983; Sain and Hien, in press) does exist in the categories of both dynamic algebras and continuous dynamic algebras. The ultraproduct constructions in (Henkin, et al., 1981, ultraproduct sections) could also be adapted to the present situation so that dynamic set algebras would be closed under them.

Let us show another application of Corollary 5.

DEFINITION 7 (Tarski and Ng, 1977). A transitive relation algebra, in short Tr, is an algebraic structure $\langle A, \vee, ;, * \rangle$, where $A \subseteq \mathscr{P}(U \times U)$ for some base set U. $\vee$ and ; are the usual union and composition of relations, and a^* is the transitive closure of a for any a in A.

The definition of boundedness is the same as the definition of strongly bounded dynamic algebras (Definition 4).

COROLLARY 13. *Let* $K \subseteq \mathrm{Tr}$. *If K is not bounded, then* $\mathrm{Up}\, K \not\subseteq \mathrm{Tr}$.

Proof. The same as that of Lemma 6. ∎

We note that the above corollary immediately yields similar results for Ng–Tarski relation algebras and also for Kleene algebras. The reason for this is as follows: both of the quoted kinds of algebras differ from Tr by adding more operations only. Such additions do not change the meaning of ultraproducts on the original operations ($\vee$, ; , and $*$).

The following corollary is a celebrated result which has influenced the development of program verification theory decisively (Kfoury and Park, 1975).

COROLLARY 14 (Kfoury–Park, 1975). *Let K be an axiomatizable class of models of some signature Σ and let p be a blockdiagram program of signature Σ. Assume that p is* not *equivalent in K to a loop-free program. Then p diverges in K. (In more detail there is $A \in K$ such that p does not terminate in A for some input.)*

Proof. Let $K = \mathrm{Up}\, K$ be a class of models of similarity type d and p be a program of type d not equivalent to a loop-free one in K. Now we shall use the notation of (Andréka *et al.*, 1982; or Németi, 1982b). In particular, td denotes a many sorted similarity type obtained from d (roughly by adding two new sorts: time and intensions), M_{td} denotes the class of all (many-sorted) structures of similarity type td. To denotes the axioms of ordering for sort "time" and $\mathrm{Mod(To)} \subseteq M_{\mathrm{td}}$ denotes the class of all models of To. Define $M = \{\langle \mathbf{T}, \mathbf{D}, I, \mathrm{ext}\rangle \in \mathrm{Mod(To)} \mid \mathbf{D} \in K\}$. K is closed under ultraproducts, hence M is also closed under ultraproducts. Thus $M = \mathrm{Up}\, M$ by $K = \mathrm{Up}\, K$. Let $\mathbf{M} = \langle \mathbf{T}, \mathbf{D}, I, \mathrm{ext}\rangle \in M$. Without loss of generality we may assume that p is monadic, that is, that a trace of p is a single element of I instead of a tuple of such elements. Let $y_i \in I$ and $z_i \in T$; we define $(y, z) < (y_1, z_1) \Leftrightarrow [z \leqslant z_1$ and $(\forall i \leqslant z)\ y(i) = y_1(i)$ and (y_1, z_1) is a partial trace of p in $\mathbf{M}$ (i.e., y_1 is a trace until z_1)], where $y(z)$ abbreviates $\mathrm{ext}(y, z)$. Intuitively, (y, z) means the restriction of y to $\{z_1 \in T \mid z_1 \leqslant z\}$. Clearly, $<$ is a partial ordering. Since p is not equivalent in K with a loop-free program, there are arbitrarily long finite execution sequences of p in K hence there are arbitrarily long chains $\langle (y_i, z_i) \mid i \leqslant n\rangle$ in M. By Theorem 1 then there is $\mathbf{M} = \langle \mathbf{T}, \mathbf{D}, I, \mathrm{ext}\rangle \in M$ and an infinitely long strictly increasing chain $\langle (y_i, z_i) \mid i \in N\rangle$ in $\mathbf{M}$. But then $\langle y_n(n) \mid n \in N\rangle$ is a nonterminating execution-sequence of p in $\mathbf{D}$.

So far we proved the conclusion of the corollary for all classes closed under ultraproducts. This is slightly stronger than the corollary. Since every first-order axiomatizable class is closed under ultraproducts, the corollary follows. ∎

APPENDIX: A FEW BASICS ABOUT ULTRAPRODUCTS (CHANG AND KEISLER, 1973).

An ultrafilter over a set I is a set U of subsets of I such that

(i) $I \in U$

(ii) if $X, Y \in U$ then $X \cap Y \in U$

(iii) if $X \in U$ and $X \subset Y \subset I$ then $Y \in U$

(iv) $X \in U \Leftrightarrow I - X \notin U$.

Every ultrafilter U has the finite intersection property, namely the intersection of any finite number of elements of U is non-empty.

An ultrafilter is principal if it is of the form

$$U = \{X \mid x_0 \in X\} \qquad \text{for some} \quad x_0 \quad \text{in } I.$$

Equivalently, an ultrafilter is principal iff it contains a finite set. Consequently, a non-principal ultrafilter contains no finite set, hence contains all cofinite sets.

Let U be an ultrafilter on N, and let $\{I_n \mid n \in N\}$ be a family of posets. The ultraproduct $I = \Pi I_n / U$ is the quotient of ΠI_n by the congruence $\sim_U$ defined by: for $p = (p(n))_{n \in N}$ and $q = (q(n))_{n \in N}$, $p \sim_U q$ iff $\{n \mid p(n) = q(n)\} \in U$.

More generally, a formula φ holds in I iff $\{n \mid \varphi \text{ holds in } I_n\} \in U$.

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