Factorization of Operators II:
A Nonlinear Volterra Method for
Numerical Solution of Linear
Fredholm Equations*

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Factorization of integral operators is studied as a method for solving Fredholm equations. It is concluded that, by factorization, linear Fredholm equations can be effectively replaced by nonlinear Volterra equations. As a byproduct of the investigation, constructive proofs are obtained for existence and uniqueness of solutions to certain nonlinear systems of Volterra equations in two independent variables. Numerical examples are given.

1. INTRODUCTION

In an important and definitive paper [1], Gohberg and Krein developed a theory of factorization of operators. Their point of view was “functional analytic” and was based on the abstract triangular representation of linear operators. In [2] we reconsidered their results, in combination with the idea of “imbedding” of operators (cf. [3]), and were able to provide an entirely algebraic (and simpler) setting for the theory. In this paper we are concerned with factorization as a method for solving Fredholm equations. As in [3], we approach the problem from the “nonlinear” point of view. Our main conclusion is that, by factorization, linear Fredholm equations can be effectively replaced by nonlinear Volterra equations. This formalism is thus

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a generalization of the practice of trading linear boundary value problems for nonlinear initial value problems (cf. [4] and [11]).

We introduce the following notation. Given any region $\Gamma$ in the $(x, y)$ plane, a function designated by $t^+(x, y)$ on $\Gamma$ is assumed to vanish for $x < y$. Similarly, a function $t^-(x, y)$ vanishes for $x > y$.

A function $k(x, y)$ defined on $x, y \geq 0$, is said to admit Volterra factorization, if there exist functions $s^\pm(x, y)$ on $x, y \geq 0$ such that

$$s^+(x, y) = k^+(x, y) + \int_0^y s^+(x, \theta) s^-(\theta, y) \, d\theta, \quad y \leq x,$$

$$s^-(x, y) = k^-(x, y) + \int_0^x s^+(x, \theta) s^-(\theta, y) \, d\theta, \quad x \leq y.$$  \hfill (1.1)

(For additional background on this concept see [2].)

We now consider the linear Fredholm equation

$$u(x) = f(x) + \int_0^a k(x, y) u(y) \, dy, \quad x \in [0, a].$$ \hfill (1.2)

In the case that $k$ admits the Volterra factorization (1.1), the solution $u(x)$ of (1.2) is determined by the pair of Volterra equations

$$w(x) = f(x) + \int_0^x s^+(x, \theta) w(\theta) \, d\theta,$$ \hfill (1.3)

$$u(x) = w(x) + \int_x^a s^-(x, \theta) u(\theta) \, d\theta.$$ \hfill (1.4)

Thus, the solution $u(x)$ of the linear Fredholm equation (1.2) is alternatively given as the solution of the nonlinear system of four Volterra equations (1.1), (1.3), (1.4).

It is our purpose to demonstrate the practicality of this formalism. The Volterra system is solved numerically by means of an approximating set of matrix equations. The convergence of the method is proved for both continuous kernels and discontinuous kernels with certain integrable singularities. As a byproduct, we obtain a constructive proof of existence and uniqueness for solutions to nonlinear Volterra systems which may be of independent interest.

A brief outline of the paper is as follows: Our main results on factorization and the solutions of nonlinear Volterra systems are found in Sections 2 and 3. Proofs are given in Sections 4, 5, and 6. Finally, in Section 7, some numerical examples are presented.
2. Volterra Factors

In [2] we treated the problem of Volterra factorization from a purely algebraic point of view. Here we adopt a different approach—namely, the Volterra factors are considered as solutions to a nonlinear system of Volterra equations.

We consider complex-valued kernels \( k(x, y) \) defined for all nonnegative real numbers \( x \) and \( y \). Certain integrable singularities are allowed. Specifically, we assume that there exist measurable functions \( \sigma^\pm(x, y) \), defined on \( x, y \geq 0 \), with the following properties:

2.1. For every \( a > 0 \) and \( x, y \geq 0 \) we have

(i) \[
\int_0^a |\sigma^+(x, \theta)| \, d\theta < \infty, \quad \int_0^a |\sigma^-(\theta, y)| \, d\theta < \infty,
\]

(ii) the expressions

\[
\int_0^a |\sigma^+(x, \theta) - \sigma^+(x', \theta)| \, d\theta, \quad \int_0^a |\sigma^-(\theta, y) - \sigma^-(\theta, y')| \, d\theta
\]

tend to zero as \( x' \to x \) and \( y' \to y \), and

(iii) the functions \( q^\pm(x, y) \) defined by

\[
q^+(x, y) = k^+(x, y) - \sigma^+(x, y) + \int_0^a \sigma^+(x, \theta) \sigma^-(\theta, y) \, d\theta,
\]

are continuous on \( x \geq y, x \leq y \), respectively.

Remark. If \( k(x, y) \) satisfies the hypotheses of ([5], p. 487, Theorem 1), then condition (2.1) is met with \( \sigma^\pm = k^\pm \).

Now let \( r^\pm \) be defined by the equations

\[
r^+(x, y) = s^+(x, y) - \sigma^+(x, y),
\]

\[
r^-(x, y) = s^-(x, y) - \sigma^-(x, y).
\]

Then the system (1.1), (1.3), (1.4) is equivalent to

\[
r^+(x, y) = q^+(x, y) + \int_0^y \left[ r^+(x, \theta) \sigma^-(\theta, y) + \sigma^+(x, \theta) r^-(\theta, y) \right] \, d\theta, \quad 0 \leq y \leq x,
\]

\[
r^-(x, y) = q^-(x, y) + \int_0^y \left[ r^+(x, \theta) \sigma^-(\theta, y) + \sigma^+(x, \theta) r^-(\theta, y) \right] \, d\theta, \quad 0 \leq x \leq y,
\]

\[
w(x) = f(x) + \int_0^x \left[ r^+(x, \theta) + \sigma^+(x, \theta) \right] w(\theta) \, d\theta,
\]

\[
u(x) = w(x) + \int_0^x \left[ r^-(x, \theta) + \sigma^-(x, \theta) \right] u(\theta) \, d\theta.
\]
The solution to the system (2.4), (2.5) exists wherever the \( r^\pm \) functions are defined. Thus we first concentrate on equations (2.3) which, independently, form a closed system. We will adjoin (2.4) and (2.5) later.

To provide a numerical solution to (2.3), it is necessary to replace the integrals by suitable quadrature formulas. With this in mind, we consider the functions \( R_{h}^+(m, n) \) \((h \geq 0)\) defined for all nonnegative integers \( m, n \) by the recursive relations

\[
R_{h}^+(m, n) = Q_{h}^+(m, n) + \sum_{j=0}^{n-1} [K_{h}^1(m, n, j) R_{h}^+(m, j) + K_{h}^2(m, n, j) R_{h}^-(j, n)], \quad 1 \leq n \leq m, \tag{2.6}
\]

\[
R_{h}^-(m, n) = Q_{h}^-(m, n) + \sum_{j=0}^{m-1} [K_{h}^1(m, n, j) R_{h}^+(m, j) + K_{h}^2(m, n, j) R_{h}^-(j, n)] + K_{h}^3(m, n, j) R_{h}^+(m, j) R_{h}^-(j, n), \quad 1 \leq m \leq n,
\]

where \( Q_{h}^\pm(m, n) = q^\pm(mh, nh) \), and where the weights \( K_{h}^i(m, n, j) \) are compatible with the properties of the kernels \( \sigma^\pm \) and the integration scheme. Precisely, we assume the following:

2.7. (i) For every compact set \( \Gamma \) in \( 0 \leq x \leq y \), there is a constant \( K \) such that

\[
|K_{h}^1(m, n, j)| \leq K \int_{jh}^{(j+1)h} |\sigma^-(\theta, y)| \, d\theta,
\]

\[
|K_{h}^2(m, n, j)| \leq K \int_{jh}^{(j+1)h} |\sigma^+(x, \theta)| \, d\theta,
\]

\[
|K_{h}^3(m, n, j)| \leq Kh,
\]

for all \((x, y) = (mh, nh) \in \Gamma\); and

(ii) for every continuous function \( t(\theta) \) and every \( \epsilon > 0 \), there is an \( h_1 \) such that if \( h < h_1 \) then

\[
\left| \sum_{j=0}^{n-1} K_{h}^1(m, n, j) t(jh) - \int_{0}^{y} \sigma^-(\theta, y) t(\theta) \, d\theta \right| < \epsilon,
\]

\[
\left| \sum_{j=0}^{m-1} K_{h}^2(m, n, j) t(jh) - \int_{0}^{x} \sigma^+(x, \theta) t(\theta) \, d\theta \right| < \epsilon,
\]

\[
\left| \sum_{j=0}^{N} K_{h}^3(m, n, j) t(jh) - \int_{0}^{a} t(\theta) \, d\theta \right| < \epsilon,
\]

for all \( 0 \leq m, n, j \leq N \), \((x, y) = (mh, nh) \in \Gamma\).
Remark. We immediately see that the "Reimann weights" defined by

\[
K_h^1(m, n, j) = \int_{\frac{j}{h}}^{\frac{(j+1)}{h}} \sigma^-(\theta, nh) \, d\theta,
\]
\[
K_h^2(m, n, j) = \int_{\frac{j}{h}}^{\frac{(j+1)}{h}} \sigma^+(mh, \theta) \, d\theta,
\]
\[
K_h^3(m, n, j) = h,
\]
satisfy condition (2.7), where in property (i) we may take \( K = 1 \). In addition one can verify that an "open-ended" version of the generalized quadrature formulae of Atkinson (cf. [6]) also satisfy condition (2.7). The initial conditions for (2.6) are given by

\[
R_h^+(m, 0) = q^+(mh, 0), \quad R_h^-(0, n) = q^-(0, nh).
\]

Our main result is the following:

**Theorem 1.**

(i) If \( k(x, y) \) satisfies conditions (2.1), then the Eqs. (2.3) have unique continuous solutions \( r^\pm(x, y) \) defined on the respective regions

\[
Q_b^+: \ 0 \leq y \leq x < \infty, \ y < b \leq \infty,
\]
\[
Q_b^-: \ 0 \leq x \leq y < \infty, \ x < b \leq \infty,
\]
which are maximal in the sense that either \( b = \infty \) or

\[
\sup\{|r^+(x, x)|, |r^-(x, x)| : 0 \leq x < b\} = \infty.
\]

(ii) For every compact subset \( Q^\pm \subset Q_b^\pm \) the family of functions \( \{r_h^\pm(x, y)\} \) defined by

\[
r_h^\pm(x, y) = R_h^\pm(m, n), \quad mh \leq x < (m + 1)h, \ nh \leq y < (n + 1)h,
\]

converge uniformly to \( r^\pm(x, y) \) on \( Q^\pm \) as \( h \) tends to zero.

We now return to Eqs. (2.4) and (2.5). As before, along with these equations, we consider the functions \( W_h(m), U_h(m) (h > 0) \) defined for all nonnegative integers \( m \) by the recursive relations

\[
W_h(m) = F_h(m) + \sum_{j=0}^{m-1} [K_h^3(m, j) R_h^+(m, j) + K_h^3(m, j)] W_h(j), \quad (2.8)
\]
\[
U_h(N - m) = W_h(N - m) + \sum_{j=0}^{m-1} [K_h^3(N - m, j) R_h^-(N - m, N - j)
+ K_h^3(N - m, j)] U_h(N - j), \quad (2.9)
\]
where \( N = \text{integer part of} \ a/h \), \( F_h(m) = f(mh) \), \( K_h^*(m, j) = K_h^*(m, m, j) \), and where \( K_h^*(m, j) \) satisfies conditions akin to (2.7), namely

\[
|K_h^*(N - m, j)| \leq K \int_{jh}^{(j+1)h} |\sigma^*[(N - m)h, Nh - \theta]| \, d\theta,
\]

\[
\sum_{j=0}^{m-1} K_h^*(N - m, j) t(jh) - \int_0^{mh} \sigma^*[(N - m)h, Nh - \theta] t(\theta) \, d\theta < \epsilon, \quad h < h_1,
\]

where the various parameters are quantified exactly as in (2.7). The initial condition for (2.8) is given by \( W_h(0) = f(0) \). Equation (2.8) is iterated until \( m = N \), at which point the starting condition \( U_h(N) = W_h(N) \) is obtained for (2.9).

**Theorem 2.** Let \( b, k, \) and \( r \) be as defined in Theorem 1 and suppose \( 0 \leq a < b \). Then the following conclusions are valid on the interval \([0, a]\):

(i) Equations (2.4) and (2.5) have unique continuous solutions \( w(x), u(x) \).

(ii) The functions \( w_h(x), u_h(x) \) defined by

\[
W_h(x) = W_h(m), \quad u_h(x) = U_h(m), \quad mh \leq x < (m + 1)h
\]

converge uniformly to \( w(x) \) and \( u(x) \).

(iii) The function \( u(x) \) satisfies the original Fredholm equation (1.2).

(iv) If \( b < \infty \), then there exists a nontrivial continuous function \( \varphi(x) \) defined on \([0, b]\) which satisfies

\[
\varphi(x) = \int_0^b k(x, y) \varphi(y) \, dy.
\]

**Remarks.** (a) By virtue of the compactness of the ker \( k \) and the Fredholm alternative, part (iv) of Theorem 2 gives the following characterization of the number \( b \):

\[
b = \sup\{c : I - k \text{ is invertible in the ring of bounded linear operators on the space of continuous functions on } [0, a], \text{for all } a < c\}.
\]

(b) The proof of part (iv) has some constructive merit and could possibly be used as a method of computing eigenfunctions.

### 3. Nonlinear Volterra Equations

In this section we extend Theorem 1 to more general systems of Volterra equations. We first note that Eqs. (2.3) can be conveniently written in vector form as

\[
r(x, y) = f(x, y) + \int_0^x k(x, y, \theta) g(r(\theta, x), r(\theta, y), \theta) \, d\theta, \quad 0 \leq x \leq y, \quad (3.1)
\]
where
\[ r(x, y) = [r^+(y, x), r^-(x, y)], \quad f(x, y) = [g(y, x), g(x, y)'], \]
\[ g[r(\theta, x), r(\theta, y)] = [r^+(\theta, x), r^+(\theta, y), r^-(\theta, x), r^-(\theta, y), r^+(x, \theta), r^+(y, \theta), r^-(x, \theta), r^-(y, \theta)'], \]
and \( k(x, y, \theta) \) is the \( 2 \times 6 \) matrix whose rows are
\[
[0, \sigma(\theta, x), \sigma(y, \theta), 0, 0, 1],
[\sigma(\theta, y), 0, 0, \sigma(x, \theta), 1, 0].
\]

Equation (3.1) turns out to be just as easy to treat in a more general format. Specifically, the following hypotheses on the ingredients of (3.1) are made:

3.2. \( r \) and \( f \) are complex-valued \( n \)-dimensional vector functions; \( g \) is a complex-valued \( m \)-dimensional vector function; and \( k \) is a complex-valued \( n \times m \) matrix function.

Associated with the values of these functions, we have conveniently chosen norms \( \{\| \cdot \|\} \), which are assumed to satisfy the inequality
\[ kg \| \leq k \| g \|. \]

In addition,
\[ f(x, y) \] is continuous on \( x, y \geq 0 \).

**Property 3.4.**

(i) \( g(u, v, \theta) \) is defined and continuous for all \( n \)-vectors \( u, v \) and numbers \( \theta \geq 0 \).

(ii) for every \( a, B > 0 \), there exists an \( L > 0 \) such that
\[ |g(u_0, v_0, \theta) - g(u, v, \theta)| \leq L(|u_0 - u| + |v_0 - v|), \]
for all \( 0 \leq \theta \leq a, \ |u_0|, |u|, |v_0|, |v| \leq B \).

3.5. \( k(x, y, \theta) \) is measurable in \( \theta \) for all \( x, y \geq 0 \) and \( k(x, y, \theta) \equiv 0 \) unless \( 0 \leq \theta \leq x \leq y \); further, for every \( a > 0 \) and \( x, y \geq 0 \), we have

(i) \[ \int_0^a |k(x, y, \theta)| \, d\theta < \infty, \]
and

(ii) \[ \int_0^a |k(x, y, \theta) - k(x', y', \theta)| \, d\theta \to 0, \]
as \( x' \to x, y' \to y \).

As in the last section, we consider, along with Eq. (3.1), the vector functions \( R_h(m, n) \) \( (h > 0) \) defined for all nonnegative integers \( m, n \) by the recursive relations
\[ R_h(m, n) = F_h(m, n) + \sum_{j=0}^{m-1} K_h(m, n, j) g(R_h(j, m), R_h(j, n), jh), \quad 1 \leq m \leq n, \quad (3.6) \]
where $F_h(m, n) = f(mh, nh)$ and where the weights $\{K_h(m, n, j)\}$ are compatible with the properties of the ker $k(x, y, \theta)$. Precisely, we assume the following:

3.7. $K_h(m, n, j) \equiv 0$ unless $0 \leq j \leq m - 1 \leq n - 1$; further, for every compact set $\Gamma$ in $0 \leq x \leq y$, there exists a constant $K$ such that for all $x = mh$, $y = nh \in \Gamma$,

(i) $|K_h(m, n, j)| \leq K \int_{j h}^{(j+1)h} |k(x, y, \theta)| \, d\theta,$

and

(ii) for every continuous $m$-dimension vector function $t(\theta)$ and every $\epsilon > 0$, there is an $h_1$ such that if $h < h_1$, then

$$\left| \sum_{j=0}^{m-1} K_h(m, n, j) t(jh) - \int_0^x k(x, y, \theta) t(\theta) \, d\theta \right| < \epsilon,$$

for all $x = mh$, $y = nh \in \Gamma$.

The initial conditions for (3.6) are given by $R_h(0, n) = f(0, nh)$.

The following results generalize Theorem 1; the constructive nature of their proof may be of independent interest.

**Theorem 3.** (i) If $f$, $g$, and $k$ satisfy conditions (3.3)–(3.5), then equation (3.1) has a unique continuous solution $r(x, y)$ defined on the region

$$\Delta_b : 0 \leq x \leq y < b,$$

which is maximal in the sense that either $b = \infty$ or

$$\sup\{|r(x, y)| : (x, y) \in \Delta_b\} = \infty.$$

(ii) For every compact $\Delta \subset \Delta_b$, the family of functions $\{r_h(x, y)\}$ defined by

$$r_h(x, y) = R_h(m, n), \quad mh \leq x < (m+1)h, \quad nh \leq y < (n+1)h,$$

converge uniformly to $r(x, y)$ on $\Delta$ as $h$ tends to zero.

**Remarks.** (a) Equation (3.1) should be considered as a Volterra equation in two variables ($y$ is not a parameter). Certain Volterra equations in several variables have been studied by Walter (cf. [7]) and results similar to part (i) of Theorem 3 have been obtained. However, the type of equations considered in [7] do not appear to cover equation (3.1). Further, the methods of proof are different.

(b) In proving part (ii) it is observed that the hypotheses on $k$ can be weakened with an analogous result still holding; namely: If (3.5.ii) is replaced by the condition

$$\int_0^t |k(x, y, \theta)| \, d\theta \quad \text{tends to} \quad \int_0^t |k(x', y', \theta)| \, d\theta,$$

(3.5ii)'
as \( x' \to x, y' \to y \), uniformly for \( t \) in \([0, a]\), then the functions \( \{r_n(x, y)\} \) converge uniformly to the solution \( r(x, y) \) of (3.1) on any compact \( \mathcal{A} \) in which \( r \) is continuous.

Part (i) of Theorem 3 shows that conditions which give \textit{a priori} bounds also imply existence. Such conditions are obtained by strengthening hypothesis (3.4.ii).

**Theorem 4.** Let \( f, g, h, \) and \( \Delta_b \) be as defined in Theorem 3. In addition, suppose for given numbers \( a, B, C > 0 \), there corresponds a number \( L_0 = L_0(a, B, C) > 0 \) such that

\[
|g(u, v, \theta)| \leq L_0\{|u| + |v| + 1\},
\]

for all \( 0 \leq \theta \leq a, |u| \leq B, |v| \leq C \).

(i) If \( L_0 \) in (3.8) is independent of \( B \) and \( C \), then there is a global solution to equation (3.1), i.e., \( b = \infty \).

(ii) If \( L_0 \) in (3.8) is independent of \( C \) only, then the solution to (3.1) on \( \Delta_b \) can be extended to a region of the form \( 0 \leq x \leq y < \infty, x < b \).

**Remark.** Theorem 3 in conjunction with part (ii) of Theorem 4 provide the aforementioned generalization of Theorem 1.

4. **Proof of Theorem 3.**

We begin with a preliminary result concerning the mean continuity of our kernels. This is followed by a lemma giving bounds on functions satisfying a Volterra like inequality. This is a discrete two-dimensional analogue of Gronwall's inequality. The proof of part (ii) is carried out in Lemma 3. The rest of the section is concerned with the proof of part (i). Here we use as a vehicle the solution to the discrete equation (3.6) for conveniently chosen weights. The method of proof is based on \textit{a priori} estimates (Lemmas 4, 5 and 6) and an extension argument (Lemma 7 in conjunction with Lemma 3).

4a. **Uniform Behavior of Kernels**

The following lemma explicitly states three aspects of uniform behavior for kernels satisfying condition (3.5).

**Lemma 1.** Let \( k(x, y, \theta) \) satisfy condition (3.5). Then for all given numbers \( a, \epsilon > 0 \) there are numbers \( M, \delta > 0 \) such that for every \( 0 \leq x, x', y, y', t \leq a \) we have

\[
\int_0^a |k(x, y, \theta) - k(x', y', \theta)| \, d\theta < \epsilon,
\]

(i)
whenever $|x' - x|, |y' - y| < \delta$,

$$\int_0^a |k(x, y, \theta)| \, d\theta \leq M, \quad (ii)$$

and

$$\int_{t-\delta}^t |k(x, y, \theta)| \, d\theta < \epsilon. \quad (iii)$$

**Proof.** For parts (i) and (ii) the conclusion follows immediately from condition (3.5) and the compactness of $[0, a]$.

For part (iii) it is convenient to consider the functions $\{a_\beta(t)\}$, where $\beta = (x, y)$ and $t \in [0, a]$ defined by

$$a_\beta(t) = \int_0^t |k(x, y, \theta)| \, d\theta.$$

Condition (3.5.i) and the absolute continuity of the Lebesgue integral show that $a_\beta(t)$ is a continuous function of $t$ for each $\beta$. Thus the family of functions $A = \{a_\beta(t) : \beta = (x, y), 0 \leq x \leq y \leq a\}$ is a subset of the metric space of continuous functions on $[0, a]$. Part (i) of this Lemma then implies that this subset is sequentially compact. [The same conclusion is also guaranteed by condition (3.5.ii)]. The standard Ascoli-type argument in turn implies that the family $A$ is equicontinuous. This is precisely the conclusion of part (iii). Q.E.D.

4b. A Volterra Inequality

**Lemma 2.** Let $A$ and $a$ be given positive numbers and suppose a function $v_h(m, n)$ satisfies the inequality

$$|v_h(m, n)| \leq A + \sum_{j=0}^{m-1} |K_h(m, n, j)|(|v_h(j, m)| + |v_h(j, n)|),$$

for all $m, n$ such that $0 \leq mh \leq nh \leq a$, where the weights $K_h(m, n, j)$ satisfy conditions (3.7). Then there exists a constant $B = B(a)$, independent of $A$ and $h$, such that

$$|v_h(m, n)| \leq BA,$$

for all $0 \leq mh \leq nh \leq a$, and all sufficiently small $h$.

**Proof.** For any given $a$, Lemma 1 provides a constant $\delta$ for which

$$\int_{t}^{t+\delta} |k(x, y, \theta)| \, d\theta < 1/4K,$$

for all $0 \leq x, y, t, t + \delta \leq a$; the constant $K$ is that describing the properties of the weights $K_h(m, n, j)$ in (3.7) with $\Gamma = \{0 \leq x \leq y \leq a\}$.
For any \( h < \delta \), let \( M \) be the largest integer for which \( Mh \leq \delta \), so that \( Mh \geq \delta/2 \). If \( N \) is the largest integer such that \( NMh \leq a \), then \( N \) is less than the integer part of \( 2a/\delta \) or \( N_1 \), say. It is important to note that \( N_1 \) is independent of \( A \) and \( h \).

Now let \( W_h(i) \) be the greatest value of \( |v_h(j,n)| \) for \( nh \leq a \) and \( j \leq iMh \). The given inequality for \( |v_h(m,n)| \) implies

\[
W_h(i) \leq A + \sum_{\ell=1}^{i} \sum_{j=\ell M}^{i M-1} |K_h(m,n,j)| 2W_h(\ell),
\]

and by (3.7), for \( x = mh \), \( y = nh \),

\[
\sum_{j=\ell M}^{i M-1} |K_h(m,n,j)| \leq K \int_{\ell Mh}^{(\ell-1)Mh+\delta} |h(x,y,\theta)| d\theta \leq 1/4.
\]

Thus,

\[
W_h(i) \leq A + \sum_{\ell=1}^{i} \frac{1}{2} W_h(\ell) \leq 2A + \sum_{\ell=1}^{i-1} W_h(\ell) \leq 2^i A,
\]

and hence \( |v_h(m,n)| \leq BA \), where the number \( B = 2^{N_1} \) is independent of \( A \) and \( h \).

Q.E.D.

4c. Proof of Part (ii)

In this subsection we prove a lemma which implies the conclusion of part (ii).

The following notation will be useful. For given positive numbers \( a \) and \( M \), set

\[
G(a, M) = \sup \{|g(u,v,\theta)| : |u|, |v| \leq M, 0 \leq \theta \leq a\},
\]

and define \( L(a, M) \) to be the infimum of all numbers \( \alpha \) such that

\[
|g(u_0, v_0, \theta) - g(u,v,\theta)| \leq \alpha |u_0 - u| + |v_0 - v|
\]

for all \( |u_0|, |u|, |v_0|, |v| \leq M, 0 \leq \theta \leq a \). Next, denote by \( \Delta(a) \) the closed triangle: \( 0 \leq x \leq y \leq a \), and for \( c \) such that \( 0 \leq c \leq a \), let \( Q(c, a) \) denote the closed region: \( 0 \leq x \leq y \leq a \), \( x \leq c \). Finally, let \( \Delta_h(a) \) and \( Q_h(c, a) \) denote the collection of lattice points given by

\[
\Delta_h(a) = \{(m,n) : (mh,nh) \in \Delta(a)\},
\]

\[
Q_h(c, a) = \{(m,n) : (mh,nh) \in Q(c, a)\}.
\]
Lemma 3. If \( r(x, y) \) is a continuous solution of (3.1) in \( Q = Q(c, a) \), then the solutions \( R_h(m, n) \) of (3.6) are uniformly bounded in \( Q \). Further the functions \( r_h(x, y) \) (defined in Theorem 3) converge uniformly to \( r(x, y) \) in \( Q \) as \( h \) tends to zero.

Proof. Let \( x = mh, y = nh \in Q_h \). From (3.1) and (3.6) we have

\[
| r(x, y) - R_h(m, n) | \leq \left| \int_0^\pi k(x, y, \theta) g[r(\theta, x), r(\theta, y), \theta] - \sum_{j=0}^{m-1} K_h(m, n, j) g[r(jh, x), r(jh, y), jh] \right| + \left| \sum_{j=0}^{m-1} K_h(m, n, j) [g[r(jh, x), r(jh, y), jh] - g(R_h(j, m), R_h(j, n), jh)] \right|.
\]

Let \( A \) be the supremum of the first norm on the right side of (4.1) and let \( M \) be the upper bound of \( | r(x, y) | \) in \( Q \). If \( | R_h(j, m)|, | R_h(j, n)| \leq 2M \), then

\[
| g(r(jh, x), r(jh, y), jh) - g(R_h(j, m), R_h(j, n), jh) | \leq L(a, 2M) | r(jh, x) - R_h(j, m) | + | r(jh, y) - R_h(j, n) |.
\]

Given \( m \), suppose

\[
| R_h(j, n)| \leq 2M,
\]

for \( j = 0, 1, \ldots, m - 1 \), for all \( n \) with \((m, n) \in Q_h \). From (4.1), we have

\[
| r(x, y) - R_h(m, n) | \leq A + L \sum_{j=0}^{m-1} | K_h(m, n, j) | | r(jh, x) - R_h(j, m) | + | r(jh, y) - R_h(j, n) |.
\]

and from Lemma 2, \( | r(x, y) - R_h(m, n)| \leq AB \), where \( B \) is independent of \( A \) and \( h \). Given arbitrary \( \epsilon, 0 < \epsilon < M \), by property (3.7.ii) we can choose \( h_1 \) so that \( h < h_1 \) implies \( A \leq \epsilon/B \). Hence,

\[
| r(x, y) - R_h(m, n) | < \epsilon,
\]

which in turn implies \( | R_h(m, n)| < 2M \). Since \( | R_h(0, n)| \leq M < 2M \) for all \( n \), by induction on \( j \), inequality (4.2) holds for all \((j, n) \in Q_h \).

This completes the first part of the Lemma. Given now that \( | R_h(m, n)| \leq 2M \) in \( Q_h \) we have that (4.3) is also valid in \( Q_h \). Let \( x, y \) be any point in \( Q \). Since \( r(x, y) \) is continuous in the compact \( Q \), there exists a \( \delta \) such that \( | x - x' | + | y - y' | < \delta \).
implies \( |r(x, y) - r(x', y')| < \varepsilon \). Thus for \( h_1 < \delta \) and \((x, y)\) in \( Q \), \( mh \leq x < (n + 1)h \), \( nh \leq y < (n + 1)h \), we have, for all \( h < h_1 \),

\[
| r(x, y) - r_h(x, y) | \leq | r(x, y) - r(mh, nh) | + | r(mh, nh) - R_h(m, n) | \leq 2\varepsilon.
\]

Q.E.D.

4d. A priori Estimates

We shall use the solutions of the discrete equation (3.6) as a vehicle for proving part (i). Consequently, we are at liberty to choose any set of weights satisfying (3.7). A convenient choice is given by the "Riemann weights"

\[
K_h(m, n, j) = \int_{j}^{j+h} k(mh, nh, \theta) \, d\theta.
\]  

Throughout the remainder of Section 4, the weights are given by (4.4).

In this subsection, we prove certain consequences of an assumed bound on the functions \( R_h(m, n) \).

**Lemma 4.** If \( | R_h(m, n) | \leq M \), for all \((m, n)\) in \( Q_h = Q_h(c, a) \), then for any \( \delta > 0 \) there is a number \( C = C(a, M) \) independent of \( h \), such that

\[
\max \{ | R_h(m, n) - R_h(m', n') | : | m - m' | h, | n - n' | h \leq \delta, \\
(m - 1, n), (m' - 1, n') \in Q_h \} \leq C(\omega(f, \delta) + \mu_1(k, \delta) + \mu_2(k, \delta)),
\]

where

\[
\omega(f, \delta) = \sup \{ | f(x, y) - f(x', y') | : | x - x' |, | y - y' | \leq \delta \},
\]

\[
\mu_1(k, \delta) = \sup \left\{ \int_{a'}^x | k(x, y, \theta) | \, d\theta : | x - x' | \leq \delta \right\},
\]

\[
\mu_2(k, \delta) = \sup \left\{ \int_0^x | k(x, y, \theta) - k(x', y', \theta) | \, d\theta : | x - x' |, | y - y' | \leq \delta \right\},
\]

and where the above suprema are restricted to points in \( Q = Q(c, a) \).

**Proof.** For fixed \((m, n) \in Q_h \), set

\[
M_h(m, n) = \max \{ | R_h(m, n) - R_h(m^0, n^0) | : | m - m^0 | h \leq \delta, | n - n^0 | h \leq \delta, \\
(m - 1, n), (m^0 - 1, n^0) \in Q_h \}.
\]

We may assume, without loss of generality, that \( m \geq m' \). It then follows from (3.6),
property (3.4), and the definitions of $G = G(a, M)$ and $L = L(a, M)$, given in subsection 4c, that

$$| R_h(m, n) - R_h(m', n') | \leq | F_h(m, n) - F_h(m', n') | + G \sum_{j=m'}^{m-1} | K_h(m, n, j) |$$

$$+ G \sum_{j=0}^{m'-1} | K_h(m, n, j) - K_h(m', n', j) |$$

$$+ L \sum_{j=0}^{m'-1} | K_h(m, n, j) | (| R_h(j, m) - R_h(j, m') | + | R_h(j, n) - R_h(j, n') |).$$

By virtue of property (3.7), this in turn implies

$$M_h(m, n) \leq \omega(f, \delta) + G[\mu_1(k, \delta) + \mu_2(k, \delta)]$$

$$+ L \sum_{j=0}^{m-1} | K_h(m, n, j) | (M_h(j, m) + M_h(j, n)).$$

From Lemma 2, we obtain a constant $B$, independent of $h$, for which

$$M_h(m, n) \leq BG'[\omega(f, \delta) + \mu_1(k, \delta) + \mu_2(k, \delta)],$$

where $G' = \max(G, 1)$. The desired inequality follows, since the constant $C = BG'$ is independent of $m$, $n$, and $h$. Q.E.D.

We now linearly interpolate the values $R_h(m, n)$ to form a continuous function $p_h(x, y)$ in $Q$. This is accomplished as follows. Let $h < c$ and define $m_h, n_h$ as the largest integers with $m_h h \leq c$ and $n_h h \leq a$. If $(x, y) \in Q$ and $x \leq m_h h, y \leq n_h h$, then either $(x, y)$ lies in a closed triangle with vertices $(m_h h, n_h h)$, $(m_h h, (n + 1)h)$, and $[(m + 1)h, (n + 1)h]$ say, which is contained in $Q$, or $(x, y)$ lies in the reflection of such a triangle about its diagonal. (A picture helps here!) In the first case the vector function $p_h(x, y)$ is defined by the condition that, for each component $p_h^i(x, y)$, the point $[p_h^i(x, y), x, y]$ lies in the plane through the points $[R_h^i(m, n), m_h h, n_h h]$, $[R_h^i(m, n + 1), m_h h, (n + 1)h]$, and $[R_h^i(m + 1, n + 1), (m + 1)h, (n + 1)h]$. An analogous definition also applies in the second case.

Finally, in the remaining portions of $Q$ the following definition holds:

$$p_h(x, y) = p_h(m_h h, y), \quad m_h h < x \leq c, \quad x \leq y \leq n_h h,$$

$$= p_h(x, n_h h), \quad n_h h < y \leq a, \quad x \leq m_h h,$$

$$= p_h(m_h h, n_h h), \quad m_h h < x \leq c, \quad n_h h < y \leq a.$$
LEMMA 5. If \( |R_h(m, n)| \leq M \) for all \( (m, n) \in Q(a, c) \), then there is a sequence \( \{h_n\} \) tending to zero such that the functions \( p_{h_n}(x, y) \) converge uniformly on \( Q = Q(c, a) \) and the limit function is a continuous solution to (3.1).

Proof. By virtue of Lemma 4, it is verified that the family of continuous functions \( \{p_n(x, y)\} \) is equicontinuous on \( Q \). Ascoli's theorem then provides the first part of our conclusion. The fact that the limit function is a solution to (3.1) on \( Q \) follows from property (3.7.ii) and a continuity argument. Q.E.D.

Now it will be useful to have a certain uniform increment \( \delta \). This is defined as follows. For given positive numbers \( a, M \), let \( \delta(a, M) \) be the supremum of all numbers \( \beta \) such that

\[
C(a, M)(\omega(f, \beta, a) + \mu_1(k, \beta, a) + \mu_2(k, \beta, A)) \leq M,
\]

where the elements on the left side are defined in Lemma 4.

LEMMA 6. If \( |R_h(m, n)| \leq M \) in \( \Delta_h(a) \), then \( |R_h(m, n)| \leq 2M \) in \( \Delta_h(a + \delta) \), for all sufficiently small \( h \), where \( \delta = \delta(2a, 2M) \).

Proof. Since \( f(0, nh) = R_h(0, n) \), we have that \( |f(0, y)| \leq M \) for all \( 0 \leq y \leq a \). Note \( C(2a, 2M) \geq 1 \) so that our definition of \( \delta(2a) \) implies that \( |f(0, y)| \leq 2M \) for \( 0 \leq y \leq a + \delta \). In turn this implies

\[
|R_h(0, n)| \leq 2M, \quad \text{for all } nh \leq a + \delta. \tag{4.5}
\]

Take \( h < \delta \) and suppose \( nh \leq a < (n + 1)h \). Consider the set \( S \) of all \( (m, n) \) in \( \Delta_h(a + \delta) \) such that \( |R_h(m, n)| > 2M \). Let \( m_0 \) be the smallest first coordinate of elements in \( S \) \( [m_0 \geq 1 \) by (4.5)\)], and let \( n_0 \) be the smallest second coordinate of points in \( S \) of the form \( (m_0, n) \). We can find an integer \( p \) such that \( (m_0 - p, n_0) \in \Delta_h(a) \) and \( ph \leq 28 \). In fact, \( p \) can always be chosen in the range \( 1 \leq p \leq n_0 - n_a \). Consider the region \( Q((m_0 - 1)h, a + \delta) \). Lemma 4 and the definition of \( \delta \) imply that

\[
|R_h(m_0, n_0) - R_h(m_0 - p, n_0)| \leq M,
\]

and consequently

\[
|R_h(m_0, n_0)| \leq 2M,
\]

since \( |R_h(m_0 - p, n_0)| \leq M \). Thus \( S \) is empty. Q.E.D.

4e. Proof of Part (i)

The following extension cycle for solutions is immediate from Lemmas 3, 5 and 6.

LEMMA 7. If \( r(x, y) \) is a continuous bounded solution to (3.1) on an open triangle \( \Delta^0(a) \), then this solution can be extended to a continuous solution on a (closed) triangle \( \Delta(a') \), where \( a' > a \).
All the ingredients are now in place. The fact that the solution exists on a nontrivial triangle follows from the uniform bound on $R_h$ in the triangle $\Delta(0)$ (i.e., $x = 0, y = 0$) and Lemmas 6 and 5, in that order. Equation (3.1) cannot have two solutions on any triangle $\Delta(a)$ by virtue of the fact that the functions $r_h(x, y)$ are uniquely defined globally and by Lemma 3 would have to converge to both solutions uniformly on $\Delta(a)$. Finally, Lemma 7 shows that the maximal triangle of existence has the required properties.

5. Proof of Theorem 4

Theorem 3 shows that the only way in which a solution to (3.1) can fail to exist is by becoming unbounded. This is equivalent to the family $\{R_h\}$ becoming unbounded. The proof of part (i) is a consequence of this fact and the following lemma.

**Lemma 8.** Let $f, g, k$ satisfy the conditions of part (i) of Theorem 4. Then for every number $b > 0$, there exists a number $M = M(b)$ such that

$$\sup\{|R_h(m, n)| : (m, n) \in \Delta_h(b)\} \leq M.$$ 

*Proof.* Given $b$, by hypothesis, there exists an $L_0$ such that

$$|g(u, v, \theta)| \leq L_0(|u| + |v| + 1)$$

for all $\theta \in [0, b]$ and all $u, v$. If $N = \lfloor b/h \rfloor + 1$, we have from (3.6)

$$|R_h(m, n)| \leq |f(mh, nh)| + L_0 \sum_{j=0}^{N} |K_h(m, n, j)| \leq |f(mh, nh)| + L_0 \sum_{j=0}^{N} |K_h(m, n, j)|$$

for all $0 \leq mh \leq nh \leq b$. By Lemma 2, $|R_h(m, n)| \leq M$, a constant depending only on $L_0$, $b$, and bounds on $\int_0^b |K(x, y, \theta)| d\theta$ and $|f(x, y)|$ in $\Delta(b)$. Q.E.D.

The proof for part (ii) of this theorem is based on the same principle. Lemma 8 requires a slight modification; however, the details are similar and will be omitted.

6. Proofs of Theorems 1 and 2.

6a. **Proof of Theorem 1.** We first note that the hypotheses of Theorem 1 imply those of Theorems 3 and 4(ii). Thus, parts (i) and (ii) follow. In particular, Theorems 3 tells us that $b = \infty$ or

$$\sup\{|r^+(x, y)|, |r^-(x, y)| : 0 \leq x, y < b\} = \infty.$$
In the latter case it remains to show the indicated singularity occurs on the diagonal \( x = y \). This last statement is a direct consequence of part (ii) of Theorem 4.

6b. Proof of Theorem 2. Part (i). Assume \( r^\pm \) to be defined on \( \mathbb{Q}_b^\pm \). We first note that Theorem 3 applied individually to Eqs. (2.4) and (2.5) implies the existence and uniqueness of continuous solutions \( w(x), u(x) \) on \([0, a]\). This gives part (i).

Part (ii). Next note that Eq. (2.4) can be combined with Eq. (2.3) in the general vector formulation of (3.1). In this case, the vector function \( r(x, y) \) is given by \( r(x, y) = [r^+(y, x), r^-(x, y), w(x)] \) and the other elements \( f, g, k \) are defined appropriately. Theorem 4(ii) then yields one-half of part (ii), i.e., that part which concerns the \( w \) functions.

The other half is now considered. Let \( U_h(m) \) be the solution of (2.9). We define a family \( u_h(x) \) of continuous functions as follows:

(i) \( u_h(x) = U_h(m) \) at the grid points \( x = mh \);

(ii) \( u_h(x) \) is extended to the rest of \([0, a]\) by linear interpolation. We show that \( u_h(x) \), so defined, converges uniformly to \( u(x) \) on \([0, a]\), as \( h \to 0 \).

Since \( w(x) \) is continuous on \([0, a]\) and the kernel \( \kappa^-(x, y) = r^-(x, y) + \sigma^-(x, y) \) satisfies condition (3.5), Theorem 3 shows that the solution \( u(x) \) is the uniform limit of functions \( \tilde{u}_h(x) \) whose values at grid points \( x = mh \) in \([0, a]\) satisfy the equation

\[
\tilde{U}_h(N - m) = w[(N - m) h] + \sum_{j=0}^{m-1} \kappa_h^-(N - m, j) \tilde{U}_h(N - j), \quad N = \text{integer part of} \ a/h,
\]

where the weights \( \kappa_h^-(N - m, j) \) can be chosen to be

\[
K_h^c(N - m, j) R_h^c(N - m, N - j) + K_h^s(N - m, j).
\]

On comparing this equation with (2.9) and defining

\[
Y_h(m) = U_h(N - m) - \tilde{U}_h(N - m)
\]

we have

\[
Y_h(m) = \{W_h(N - m) - w[(N - m) h]\} + \sum_{j=0}^{m-1} \kappa_h^-(N - m, j) Y_h(j). \quad (6.1)
\]

Given any \( \epsilon > 0 \), we can choose \( h \) small enough to ensure that the first term in the right-hand side of (6.1) is less than \( \epsilon \) for all integers \( m \) with \( mh \in [0, a] \). Therefore, by Lemma 2, there is a constant \( B = B(a) \), independent of \( \epsilon \) and \( h \), such that \( |Y_h(m)| < Be \) for all \( m \) with \( mh \in [0, a] \). It must be concluded that \( u_h(x) \) tends uniformly to the same limit as \( \tilde{u}_h(x) \) as \( h \to 0 \). This proves part (ii).

Part (iii). Next we show \( u(x) \) satisfies Eq. (1.2) on \([0, a]\). We now have functions
\[ s^\pm(x, y) = r^\pm(x, y) + o^\pm(x, y), \quad w(x), \quad \text{and} \quad u(x) \text{ defined on } 0 \leq x \leq y \leq a \text{ satisfying}
\text{equations (1.1), (1.3), (1.4). From Eq. (1.4),}
\[
\omega(x) = u(x) - \int_x^a s^-(x, \theta) u(\theta) \, d\theta,
\]
and so from Eq. (1.3),
\[
u(x) = f(x) + \int_x^a s^-(x, \theta) u(\theta) \, d\theta + \int_0^a s^+(x, \theta) \left[ u(\theta) - \int_\theta^a s^-(\theta, \varphi) u(\varphi) \, d\varphi \right] \, d\theta
\]
This gives part (iii).

Part (iv). Suppose \( b < \infty \), so that \( r^+(x, y) \) and \( r^-(x, y) \) are defined and continuous on the triangles \( A^+(b) : 0 \leq y \leq x < b, \ A^-(b) : 0 \leq x \leq y < b \). By the definition of \( b \), we have that at least one of the two functions \(| r^+(x, x) | \) or \(| r^-(x, x) | \) becomes unbounded as \( x \) tends to \( b \). We first consider the case where
\[
\sup \{| r^-(x, x) | : 0 \leq x < b \} = \infty. \quad (6.2)
\]
In this case let \( v_y(x) \) be defined by the equation
\[
v_y(x) = r^-(x, y) + \int_x^y \left[ r^-(x, \theta) \sigma^-(\theta, y) + \sigma^-(x, \theta) \sigma^-(\theta, y) \right] \, d\theta
\]
Existence, uniqueness and continuity of \( v_y(x) \) as a function of \( x \) are assured by viewing (6.3) as a Volterra equation for each fixed \( y \) and applying Theorems 3 and 4. It is important to note that
\[
v_y(y) = r^-(y, y). \quad (6.4)
\]
It follows from Eq. (6.3) in conjunction with the system (2.3) and some straightforward algebra that \( v_y(x) \) also satisfies
\[
v_y(x) = \left[ k^-(x, y) - \sigma^-(x, y) + \int_0^y k(x, \theta) \sigma^-(\theta, y) \, d\theta \right]
\]
Let \( k_1(x, y) \) denote the term in brackets in the right side of (6.5). From properties (2.1) we see that \( k_1(x, y) \) is continuous for all \( 0 \leq x \leq y < \infty \) [not just on \( A^-(b) \)].
From (6.2) and (6.4), there exists a sequence of points $y_n$ tending to $b$ with $|v_{yn}(y_n)|$ tending to $\infty$. Hence, there is an integer $N$, such that $|v_{yn}(y_n)| \geq 1$ for all $n \geq N$. Let

$$w_n = \sup\{|v_{yn}(x)| : 0 \leq x \leq y_n\},$$

so that $w_n \geq 1$ for $n \geq N$. Now define the functions $\varphi_n(x)$, for $x \in [0, b]$ and $n \geq N$, by the formula

$$\varphi_n(x) = \begin{cases} 
\frac{v_{yn}(x)}{w_n}, & 0 \leq x \leq y_n, \\
\frac{v_{yn}(y_n)}{w_n}, & y_n \leq x \leq b.
\end{cases}$$

It follows that $|\varphi_n(x)| \leq 1$, for all $n \geq N$. Furthermore, for each $n \geq N$, $\varphi_n$ is a continuous function of $x$ and from (6.5) is a solution to the equation

$$\varphi_n(x) = k_1(x, y_n)/w_n + \int_0^{y_n} k(x, \theta) \varphi_n(\theta) d\theta, \quad 0 \leq x \leq y_n. \quad (6.6)$$

If $x, x' \leq y_n$ then

$$|\varphi_n(x) - \varphi_n(x')| \leq \left| \frac{k_1(x, y_n) - k_1(x', y_n)}{w_n} \right| + \int_0^{y_n} |k(x, \theta) - k(x', \theta)| d\theta; \quad (6.7)$$

if $x > y_n$ and $x' < y_n$ then $|\varphi_n(x) - \varphi_n(x')|$ is less than the right-hand side of (6.7) with $x = y_n$; if $x, x' \geq y_n$ then $|\varphi_n(x) - \varphi_n(x')| \equiv 0$. Hence, the family $\{\varphi_n(x)\}$, $n \geq N$ is equicontinuous on $[0, b]$ and by Ascoli’s theorem, there is a subsequence $\{n'\} \subset \{n\}$ such that $\varphi_{n'}(x)$ converges uniformly to a continuous function $\varphi(x)$ which is not identically zero on $[0, b]$. From (6.6) and the fact that $w_{n'}$ tends to $\infty$ as $n' \to \infty$, we have

$$\varphi(x) = \int_0^b k(x, \theta) \varphi(\theta) d\theta. \quad (6.8)$$

In the case where $\sup\{|r^+(x, x)| : 0 \leq x < b\} = \infty$ an analogous argument gives the existence of a nontrivial continuous function $\psi(x)$ on $[0, b]$ which satisfies

$$\psi(x) = \int_0^b \psi(\theta) k(\theta, x) d\theta. \quad (6.9)$$

However, the compactness of the ker $k$ as an operator on the space of continuous functions on $[0, b]$ insures that the equation adjoint to (6.9), namely (6.8), has a nontrivial continuous solution.

Q.E.D.
7. Numerical Examples

We present here results of three numerical experiments for the solution of Fredholm integral equations. These involve kernels which admit Volterra factorization in the rectangle \(0 \leq x, y \leq 1\). The first \(k_1(x, y) = \frac{1}{2}e^{-x-y}\) was chosen to be analytic in the rectangle; the second, \(k_2(x, y) = \frac{1}{2}e^{-|x-y|}\) analytic in the two triangular subregions separated by the diagonal and discontinuous in derivatives along the diagonal. To test the full force of our theory, the third ker

\[
\left( \frac{1}{2} E_n(|x - y|) \right) = -\frac{1}{2} \left( \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n |x - y|^n}{nn!} + \log |x - y| \right),
\]

\(\gamma = \) Euler-Mascheroni constant, was chosen to have a logarithmic singularity along \(x = y\).

Since these kernels are symmetric, their factors satisfy the relation

\[
s^-(x, y) = s^+(y, x),
\]

and hence,

\[
s^+(x, y) = k(x, y) + s^+(x, 0) s^+(y, 0) dO, x \leq y.
\]

For the two nonsingular examples, the matrices \(S_h^+(m, n)\) were computed from Eqs. (2.6) using both the trapezoidal rule and Simpson's rule to evaluate the integrals. In both cases the forcing function \(f(x)\) was chosen so as to generate a solution \(u(x) \equiv 1\). The trapezoidal and Simpson's integration formulae introduce an aspect not present in the recursive system (2.6), i.e., the equations are not strictly iterative. The diagonal elements satisfy a quadratic equation, and the nondiagonal elements satisfy a linear equation. Since the additional terms are of order \(h\), a solution is guaranteed for \(h\) small enough. Thus, in the trapezoidal program,

\[
S_h^+(m, m) = K_h^+(m, m) + \frac{h}{2} \sum_{i=0}^{m-1} \left( [S_h^+(i, m)]^2 + [S_h^+(i + 1, m)]^2 \right),
\]

and \(S_h^+(m, m) = x\), say, is given as a solution of a quadratic equation

\[
h/2x^2 - x + A = 0.
\]

The required solution is \(2A/[1 + \sqrt{1 - Ah/2}]\).

6a. Example 1

Table I shows the factors \(s_h^+(x, y)\) [and by symmetry \(s_h^-(x, y)\)] for the ker \(\frac{1}{2}e^{-x-y}\). The row \(u\) is the calculated solution for forcing function \(1 + (e^{-x} - 1)/2x\) which theoretically generates a solution \(u \equiv 1\). All integrals were evaluated by the trapezoidal
FACTORIZATION OF OPERATORS

TABLE I

<table>
<thead>
<tr>
<th>y</th>
<th>x</th>
<th>u</th>
<th>ul</th>
<th>u2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000006</td>
<td>1.000000017</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
</tr>
<tr>
<td>0.25</td>
<td>1.000000017</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
</tr>
<tr>
<td>0.5</td>
<td>1.00000018</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
</tr>
<tr>
<td>0.75</td>
<td>1.000010</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
</tr>
<tr>
<td>1.0</td>
<td>1.000001</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
</tr>
</tbody>
</table>

Values of $s_{h}(x, y)$, $u_{h}(x)$, $u_{1}(x)$, and $u_{2}(x)$, $h = 1/32$, for the kernel $e^{-x}$ using the trapezoidal rule and the forcing function $1 + (e^{-x} - 1)/2x$.

A sequence of step sizes $h = 1/2^{r}$, where $r$ ranged from zero to five, gave six progressively more accurate results. The last results for $h = 1/32$ are given in the table.

Romberg’s extrapolation procedure [8] was applied to the computed values of $u(x)$ for step sizes corresponding to $r = 3, 4, 5$. Denote by $u^{(r)}(x)$ the value of the computed solution $u$ at $x$ when $h = 1/2^{r}$. The values $u_{1}(x)$ in Table I were calculated from the formula

$$u_{1} = u^{(5)} + (1/3)(u^{(5)} - u^{(4)})$$

on the assumption that the results are of order $h^{2}$ accuracy. The last row $u_{2}$ of the table results from the formula

$$u_{2} = u_{1}^{(5)} + (1/15)(u_{1}^{(5)} - u_{1}^{(4)})$$

now on the assumption that the $u_{1}$ results are of order $h^{4}$ accuracy.

Table II shows results for the same kernel and forcing function but with all integrals evaluated by Simpson’s rule. A refined network was used to obtain starting values. The results were a little inferior to $u_{2}(x)$ obtained by Romberg’s procedure from the trapezoidal results.
TABLE II

<table>
<thead>
<tr>
<th>x</th>
<th>1.0</th>
<th>0.75</th>
<th>0.5</th>
<th>0.25</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
</tr>
<tr>
<td>y</td>
<td>0.0</td>
<td>0.25</td>
<td>0.5</td>
<td>0.75</td>
<td>1.0</td>
</tr>
<tr>
<td>u1</td>
<td>1.000062</td>
<td>1.000103</td>
<td>1.000132</td>
<td>1.000149</td>
<td>1.000153</td>
</tr>
<tr>
<td>u2</td>
<td>1.00000000</td>
<td>1.00000001</td>
<td>1.00000002</td>
<td>1.00000002</td>
<td>1.00000003</td>
</tr>
</tbody>
</table>

Values of $s_h(x, y)$, $u_h(x)$, $h = 1/32$; for the ker $1/2 e^{-y}$ using Simpson's rule and the forcing function $1 + (e^{-x} - 1)/2x$.

6b. Example 2

The calculations of the previous example were repeated for the kernel $1/2 e^{-|x-y|}$ and forcing function $(1/2)(e^{-x} + e^{-y})$. The results in Tables III and IV show the loss of one figure in accuracy over those for the first example.

TABLE III

<table>
<thead>
<tr>
<th>x</th>
<th>1.0</th>
<th>0.75</th>
<th>0.5</th>
<th>0.25</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>1.0000000</td>
<td>0.63646</td>
<td>0.49567</td>
<td>0.36396</td>
<td>0.23618</td>
</tr>
<tr>
<td>y</td>
<td>0.0</td>
<td>0.25</td>
<td>0.5</td>
<td>0.75</td>
<td>1.0</td>
</tr>
<tr>
<td>u1</td>
<td>0.99999998</td>
<td>0.99999990</td>
<td>0.99999982</td>
<td>0.99999975</td>
<td>0.99999971</td>
</tr>
<tr>
<td>u2</td>
<td>1.00000000</td>
<td>1.00000001</td>
<td>1.00000002</td>
<td>1.00000002</td>
<td>1.00000003</td>
</tr>
</tbody>
</table>

Values of $s_h(x, y)$, $u_h(x)$, $u1(x)$, and $u2(x)$, $h = 1/32$ for the ker $1/2 e^{-|x-y|}$ using the trapezoidal rule and the forcing function $1/2 (e^{-x} + e^{-y})$. 
Values of $s_h^+(x, y)$, $u_h(x)$; $h = 1/32$, for the ker $e^{-|x-y|}$ using Simpson's rule and the forcing function $\frac{1}{2} (e^{x} + e^{y})$.

Example 3

Table V shows results for the singular ker $k(x, y) = \psi(y - x)$, which is of interest in radiative transfer theory [9]. The forcing function was chosen as $(1/4)e^{2(x-1)}$ so that the solution $u(x)$ is the function $J(1, x, 1/4)$, [10]. The ker $E_1(z)$ has a logarithmic singularity at $z = 0$,

$$E_1(z) = f(z) - \log z = -z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!} - \log z,$$

which was subtracted off from its factors. Since $k(x, y)$ is symmetric,

$$r^+(x, y) = s^+(x, y) + \frac{1}{2} \log(x - y)$$

satisfies the equation

$$r^+(x, y) = \frac{1}{2} f(|x - y|) + \frac{1}{2} \int_{0}^{\psi} \log(x - \theta) \log(y - \theta) d\theta$$

$$+ \int_{0}^{\psi} [r^+(x, \theta) r^+(y, \theta) - \frac{1}{2} \log(x - \theta) r^+(y, \theta)$$

$$- \frac{1}{2} \log(y - \theta) r^+(x, \theta)] d\theta.$$
## TABLE V

<table>
<thead>
<tr>
<th>y</th>
<th>x</th>
<th>u</th>
<th>ul</th>
<th>J(1, x, (\frac{1}{32}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.12567</td>
<td>0.20734</td>
<td>0.28207</td>
</tr>
<tr>
<td>0.25</td>
<td>0.25</td>
<td>0.1061</td>
<td>-0.0310</td>
<td>0.0125</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>-0.2886</td>
<td>-0.1710</td>
<td>-0.0667</td>
</tr>
<tr>
<td>0.75</td>
<td>0.75</td>
<td>0.3106</td>
<td>0.0771</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>0.3678</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Values of \(r^+_h(x, y) = s_h(x, y) + \frac{1}{3} \log |x - y|, u_h(x)\); \(h = 1/16\) for the \(\ker \frac{1}{h} E(x - y)\) using the trapezoidal rule and forcing function \(\frac{1}{h} e^{-2(h-x)}\).

<table>
<thead>
<tr>
<th>y</th>
<th>x</th>
<th>u</th>
<th>ul</th>
<th>J(1, x, (\frac{1}{32}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.12521</td>
<td>0.20608</td>
<td>0.28036</td>
</tr>
<tr>
<td>0.25</td>
<td>0.25</td>
<td>0.1019</td>
<td>-0.0319</td>
<td>0.0121</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>-0.2886</td>
<td>-0.1710</td>
<td>-0.0667</td>
</tr>
<tr>
<td>0.75</td>
<td>0.75</td>
<td>0.3036</td>
<td>0.0751</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>0.3598</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Values of \(r^+_h(x, y), u_n(x), u_1(x), h = 1/32\) as for table above.

The first set of values in Table V is a selection of the results for a matrix of size \(17 \times 17\) corresponding to a step size \(h = 1/24\); and the second set for the next
refinement of the grid with $h = 1/2^5$. The comparison shows the $r^+(x, y)$ matrix values to be about two per cent in error on the diagonal near $(x, y) = (1, 1)$ and perhaps 2 in $10^5$ in error near $(0, 1)$.

The value of $u(0)$ is a convenient indicator of the accuracy of the calculations, since $J(1, 0, \frac{1}{2}) = 0.125$ exactly. Since $u1$ values show a significant improvement over those of $u$, the error term is still nearly order $h^2$. $u2$ however produces no further improvement. Aitken’s convergence process applied to $u^{(5)}(0), u^{(4)}(0),$ and $u^{(3)}(0)$ gives $0.125016$.

The reason for the poor results from Romberg’s procedure is the presence of a singular term of the form $-\frac{1}{3}(x - y) \log^2(x - y)$ in $s^+(x, y)$. While other singular terms like $y \log y$ present in $u, w,$ and $(x - y) \log(x - y)$ in $s^+$ give order $h^2$ accuracy with the trapezoidal rule, the former does not. Table VI shows results for the same kernel and forcing function where an attempt was made to introduce correction terms for this singularity. The results are significantly improved and appear to extrapolate better, suggesting they are now indeed in error of order $h^2$. These final values $u1(x)$ agree to within 2 in $10^5$ with values computed by a different method.

<table>
<thead>
<tr>
<th>TABLE VI</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
</tr>
<tr>
<td>u</td>
</tr>
<tr>
<td>u1</td>
</tr>
</tbody>
</table>

Values of $r^+(x, y), u_4(x)$ and $u1(x); h = 1/32$ for the kernel $\frac{1}{4}E_l(|x - y|)$ and forcing function $\frac{1}{4}e^{-3|x-y|}$ using the trapezoidal rule with correction for certain singular terms to attain order $h^2$ accuracy.
REFERENCES


