Journal of Pure and Applied Algebra 15 (1979) 23–40

THE MAPPING CYLINDER RESOLUTION FOR A GROUPNET DIAGRAM*

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Communicated by M. Kelly
Received 10 October 1977

1. Introduction

The work of G. Higman, A. Karrass, Hanna Neumann, B. H. Neumann, D. Solitar and many others has drawn attention to groups which are graph products – free products with amalgamation, HNN groups, tree products and the like. They are fundamental groups of graphs of groups in the terminology of Bass and Serre. The theory of groupnets (Brandt groupoids) lends itself neatly to the study of such groups.

The bridge between topology and combinatorial group theory provided by the groupnet has been used more frequently since the appearance of Higgins' formalisation [7] of the theorems of Grushko, Kurosh, Neilsen and Schreier in terms of groupnets. The category $\mathcal{G}rpnet$ of groupnets contains as a full subcategory the category $\mathcal{G}rp$ of groups, but also contains algebraically-determined constructs – homotopies, fibrations and 'unit intervals' – which are either undefined or vacuous in $\mathcal{G}rp$ yet correspond closely to the topological definitions through the forgetful functor from $\mathcal{G}rpnet$ to the category of directed graphs. Restriction of the object set to abelian groupnets determines the category $\mathcal{A}bnet$ which contains the category $\mathcal{A}b$ of abelian groups as full subcategory.

The term groupnet is preferred to groupoid in order to emphasise the graph underlying each groupnet, to harmonise with more general definitions of partial product net and pregroupnet [4] and to avoid ambiguity with the algebraist's groupoid (a set with a binary operation). All the groupnet theory used here is found in [4], as is a description of the close relationship between the graph product of a diagram of groups with monic edge morphisms and the fundamental group of a graph of groups. The graph product approach does, however, appear to be the more natural.

* The work reported in this paper is part of a dissertation presented in October 1976 for a Ph.D. degree in Pure Mathematics at the Australian National University. I wish to acknowledge the advice and encouragement of my supervisor, Dr. N. Smythe.

In this paper, the ringoids of Mitchell [9] are extended to form a category of ringnets, which is employed to construct a 'mapping cylinder' complex for the homotopy colimit G of each diagram of groupnets (D, A), given any diagram of complexes corresponding to (D, A), with the subsequent proof (Corollary 5.6) that

if G is a graph product, and each vertex complex is a free resolution of its trivial module, then the G-mapping cylinder is a free resolution of its trivial module.

In the course of proof the notion of chain homotopy is extended to a form strongly motivated by the topological definition of homotopy.

This mapping cylinder complex determines Mayer-Vietoris sequences for graph products which extend the results of Bieri [1] for HNN groups and Lyndon and Swan [11] for free products with amalgamation. The sequences are in turn used to generalise results on cohomological dimension and duality groups. Several of these results are shown to be equivalent to those of Chiswell [3] (see also Dicks [5]), Bieri and Eckmann [1, 2] derived by other means.

General categorical notation employed is as follows: $|\mathscr{C}|$ denotes the object set of a small category $\mathscr{C}$, $f \in \mathscr{C}$ means f is a morphism of $\mathscr{C}$, an object C of $\mathscr{C}$ may at any time represent its identity morphism 1_C, and all diagrams denote the statement that they commute wherever possible. The *unit interval groupnet* will be denoted $I = \{0, 1, *, *^{-1}\}$.

2. The category of ringnets

Groupnets form a wider class than groups, extending the category of small monoids for which every morphism is an isomorphism to the category of small categories for which every morphism is an isomorphism. The analogous extension from unital rings to ringoids; that is, from the category $\mathscr{R}ng$ of small preadditive monoids to the category $\mathscr{R}ngoid$ of small preadditive categories, has been dealt with in depth by Mitchell [9]. The category $\mathscr{R}ngnet$ described below forms an even wider class than $\mathscr{R}ngoid$, with full subcategory inclusion functors $\mathscr{R}ng \to \mathscr{R}ngoid \to \mathscr{R}ngnet$.

2.1. Definition. A category $\mathscr{C}$ is *partially preadditive* if it admits an abelian groupnet structure on hom sets. A functor between partially preadditive categories is *partially additive* if it is an abelian groupnet morphism on each hom set.

For instance, $\mathscr{A}bnet$ is partially preadditive.

2.2. Definition. A *ringnet* is a small partially preadditive category. The category of all ringnets and the partially additive covariant functors between them is called $\mathscr{R}ngnet$.

The set of morphisms $R = \bigvee_{p \in zR} R(p)$ of a ringnet $\mathcal{R}$ is algebraically an abelian groupnet whose additive identities form an associative product net with identities, the *set of zeros* zR of R. Composition $r^* \circ r$ in $\mathcal{R}$ will be written as a product rr^* in R throughout, while $|\mathcal{R}|$ is identifiable with either $\mathrm{Id}\, zR$ or the *set of identities* $\mathrm{Id}\, R = \{1_i : i \in \mathrm{Id}\, zR\}$ of R. The *zero map* $z : R \to zR$ is given for r in $R(p)$ by $zr = p$.

In these terms, a covariant partially additive functor is a *ringnet morphism* $f : R \to S$; that is, a set map satisfying

(i) $f : zR \to zS$ is a morphism of partial product nets with identities,
(ii) for each p in zR, $f : R(p) \to S(f(p))$ is an abelian group morphism,
(iii) if $i \in \mathrm{Id}\, zR$, then $f(1_i) = 1_{f(i)}$, and
(iv) if $rr^* \in R$, then $f(rr^*) = f(r)f(r^*)$.

2.3. Definition. The *groupringnet* $\mathbf{Z}A$ of a groupnet A has as zero set the groupnet $z\mathbf{Z}A = \{(i, j) \in \mathrm{Id}\, A \times \mathrm{Id}\, A : A(i, j) \neq \emptyset\}$ with $\mathrm{Id}\, z\mathbf{Z}A \cong \mathrm{Id}\, A$, $\lambda(i, j) = i$, $\rho(i, j) = j$, and partial product $(i, j)(j, k) = (i, k)$. For each (i, j), $\mathbf{Z}A(i, j)$ is the free abelian group on $\{[a] : a \in A(i, j)\}$, while $1_i = [i]$ for i in $\mathrm{Id}\, A$ and the product is extended linearly from that of A.

The *trivial groupringnet* for A is $\mathbf{Z}(\mathrm{Id}\, A)$: it is a disjoint union of copies of the ring of integers $\mathbf{Z}$, one for each identity of A, and extends the description of $\mathbf{Z}$ as the groupring of the trivial group. Any groupnet morphism extends linearly to a groupringnet morphism.

The *tensor product* $R \otimes S$ of two ringnets has $z(R \otimes S) = zR \times zS$, $R \otimes S(p, q) = R(p) \otimes_{\mathbf{Z}} S(q)$, and all actions defined by co-ordinate. For groupnets A and B, $\mathbf{Z}(A \times B) \cong \mathbf{Z}A \otimes \mathbf{Z}B$.

Functorial natural equivalence becomes homotopy of ringnet morphisms, described as follows: two ringnet morphisms $f, g : R \to S$ are *homotopic* ($f \simeq g$) if there is a ringnet morphism $F : \mathbf{Z}I \otimes R \to S$ satisfying

(i) $F([0], r) = f(r)\ \forall r \in R$, and
(ii) $F([1], r) = g(r)\ \forall r \in R$.

Such a *homotopy* F is entirely defined by f and

$$\{F([*], 1_i), F([*^{-1}], 1_i) : i \in \mathrm{Id}\, zR\}.$$

Any ringnet morphism $f : R \to S$ induces a *constant homotopy* $\chi(f) : f \simeq f$. Homotopic groupnet morphisms induce homotopic groupringnet morphisms.

2.4. Definition. A *ringnet diagram* (D, R) is a functor R from the free category on the directed graph D to $\mathcal{R}ngnet$. It consists of a collection of ringnets $\{R_v : v \in D\}$ and a collection of ringnet morphisms $\{R_e : R_{\lambda e} \to R_{\rho e}, e \in D\}$.

Each ringnet diagram has a homotopy colimit but its construction is not of concern here. A rather less free object is of interest.

2.5. Definition. A *representation* $\sigma : (D, R) \to \sigma(D, R)$ of a ringnet diagram (D, R) comprises

(i) a ringnet $\sigma(D, R)$,

(ii) a ringnet morphism $\sigma_v : R_v \to \sigma(D, R)$, $\forall v \in D$, and

(iii) a ringnet homotopy $\sigma_e : \sigma_{\lambda e} \simeq \sigma_{\rho e} \circ R_e$, $\forall e \in D$.

Each groupnet diagram (D, A) induces a groupringnet diagram $(D, \mathbf{Z}A)$ in an obvious manner.

3. Modules over ringnets

For a ring $\mathcal{K}$, a left (right) unitary $\mathcal{K}$-module is an additive covariant (contravariant) functor $\mathcal{K} \to \mathcal{A}b$. For a ringoid $\mathcal{C}$, a left (right) unitary $\mathcal{C}$-module is an additive covariant (contravariant) functor $\mathcal{C} \to \mathcal{A}b$. A covariant functor describes a left module in terms of the categories concerned, but a right module in terms of the internal algebraic structure.

3.1. Definition. If $\mathcal{R}$ is a ringnet, a unitary *R-module* is a partially additive functor $\mathcal{R} \to \mathcal{A}bnet$. If A is a groupnet, a $\mathbf{Z}A$-module is referred to as an *A-module*. For any R-module $\mathcal{M}$ the abelian groupnets $\{\mathcal{M}(i) : i \in \mathrm{Id}\, zR\}$ will be assumed disjoint as there always exists a functor naturally isomorphic to $\mathcal{M}$ for which this is true. A covariant partially additive functor $\mathcal{M} : \mathcal{R} \to \mathcal{A}bnet$ has the internal algebraic structure of a *right R-module M*. Explicitly, if zM, the *set of zeros* of M, is the set of additive identities of the abelian groupnet $M = \bigvee_{i \in \mathrm{Id} zR} \mathcal{M}(i)$, $M(z)$ is the abelian group in $\mathcal{M}(i)$ with additive identity z, and the *right map* $\rho : M \twoheadrightarrow \mathrm{Id}\, zR$ is extended by component from $\rho z = i$, $z \in zM \cap \mathcal{M}(i)$, then the following rules are obtained (cf Mitchell [9, p. 17]):

(i) $(m + m^*)r = mr + m^*r$, $\rho m = \rho m^* = \lambda r$,

(ii) $m(r + r^*) = mr + mr^*$, $\rho m = \lambda r^* = \lambda r$,

(iii) $m(rr^*) = (mr)r^*$, $\rho m = \lambda r$, $\rho r = \lambda r^*$,

(iv) $m 1_{\rho m} = m$.

Similarly, a contravariant functor with *left map* λ determines a *left R-module.*

It proves necessary to extend the definition of a bimodule from the usual one; namely, a bifunctor partially additive in both arguments, contravariant in one and covariant in the other.

3.2. Definition. An abelian groupnet M is an *R-S bimodule* if it is both a left R-module and a right S-module such that, for $r \in R$, $s \in S$ and $m \in M$, if either of $(rm)s$ or $r(ms)$ is defined, both are and are equal.

If $\sigma : \mathcal{R} \to \mathcal{S}$ is a partially additive covariant functor and $\mathcal{N} : \mathcal{R} \to \mathcal{A}bnet$ and $\mathcal{M} : \mathcal{S} \to \mathcal{A}bnet$ are both covariant, then a natural transformation $\mathcal{N} \to \mathcal{M} \circ \sigma$ is internally a *σ-morphism* $f : N \to M$ of right modules and is a groupnet morphism $f : N(i) \to M(\sigma(i))$ for each i in $\mathrm{Id}\, zR$, such that $f(nr) = f(n)\sigma(r)$ whenever nr is defined. When σ is the identity on R, f is known as an *R-morphism.* Composed functor $\mathcal{M}\circ\sigma$ is called the (right) *pullback M^σ of M along σ* and has

$$zM^\sigma = \{(z, i) \in zM \times \mathrm{Id}\, zR : \sigma(i) = \rho z\},$$

$M^\sigma(z, i) = M(z) \times \{i\}$, $\rho(z, i) = i$, and $(m, \lambda r)r = (m\sigma(r), \rho r)$. The canonical σ-morphism $\sigma^* : M^\sigma \to M$ is called *pullback projection.* The category of right (left) R-modules and R-morphisms is called R-$\mathcal{M}od^{\mathrm{r}}$ (R-$\mathcal{M}od^{\mathrm{l}}$) but usually the superscript is suppressed. An R-module Z is a *zero module* if $Z = zZ$, and each (right) zero R-module Z determines an abelian subcategory R-$\mathcal{M}od(Z)$ of R-$\mathcal{M}od$, called the category of *standard* R-modules and R-morphisms *over* Z in which

(i) $|R\text{-}\mathcal{M}od(Z)|$ is the class of all (right) R-modules M for which $zM = Z$, and

(ii) morphisms are those (right) R-morphisms which are the identity morphism on Z.

Results in R-$\mathcal{M}od(Z)$ are usually obtained by imitating the proof of the corresponding result in $\mathcal{A}b$ 'pointwise on Z'. For instance R-$\mathcal{M}od(Z)$ admits arbitrary products and coproducts, with direct sum

$$\left(\bigoplus_{j=1}^{n} M_j\right)(z) = \bigoplus_{j=1}^{n} M_j(z) \quad \forall z \in Z.$$

If $z = \mathrm{Id}\, R$, then R-$\mathcal{M}od(Z)$ is known as the category R-$\mathcal{M}odreg$ of *regular* (right) R-modules and R-morphisms.

For example, the trivial groupringnet $\mathbf{Z}(\mathrm{Id}\, A)$ for a groupnet A is regular as either a left or right A-module and is called the *trivial A-module TA.* The pullback of a regular module is regular and in fact if the groupnet morphism $\sigma : A \to B$ is extended to groupringnets then $(TB)^\sigma \cong TA$ as A-modules.

3.3. Definition. If M is a right R-module and N is a left R-module their *tensor product $M \otimes_R N$ over R* is an abelian groupnet with

$$\mathrm{Id}\,(M \otimes_R N) =$$
$$= \{(u,v) \in zM \times zN : \rho u = \lambda v\}/\langle(up, v) \sim (u, pv)\ \forall p \in zR : \lambda p = \rho u, \rho p = \lambda v\rangle$$

and

$$M \otimes_R N(\alpha) = \coprod_{(u,v)\in\alpha} M(u) \otimes_{\mathbf{Z}} N(v)/\langle(mr, n) = (m, rn) \text{ whenever defined}\rangle.$$

Should M be an S-R bimodule then $M \otimes_R N$ inherits a left S-module structure. The tensor product associates and satisfies the identities $M \otimes_R R \cong M$ and $M \otimes_S (S^\sigma) \cong M^\sigma$ for a ringnet morphism $\sigma : R \to S$.

For two left R-modules M and N the set $\hom_R(M, N)$ of R-morphisms has an abelian groupnet structure with $\mathrm{Id} \hom_R(M, N) = \hom_R(zM, zN)$ and $\hom_R(M, N)(h) = \{f \in \hom_R(M, N) : f|_{zM} = h\}$. The hom functor $\hom_S(M, -)$ so defined is no longer right adjoint to the tensor functor $M \otimes_R -$ for an S-R bimodule M, so that the straightfoward extension of classical theory so far generally employed must be used with caution when dealing with cohomology theory. However, for a ringnet morphism $\sigma : R \to S$ the identity $\hom_S(S^\sigma \otimes_R M, L) \cong \hom_R(M, {}^\sigma L)$ holds for left modules M and L.

3.4. Definition. Let R be a ringnet, Z be a (left) zero R-module, $\mathscr{X}$ be a set diagram $X \overset{z}{\twoheadrightarrow} zX \subseteq Z$ and let $\mathscr{U} : R\text{-}\mathscr{M}od(Z) \to \mathscr{S}et$ be the forgetful functor. A standard R-module M over Z is *free with basis* $\mathscr{X}$ if

(i) X is a subset of $\mathscr{U}M$,

(ii) $zX = \{y \in Z : y = zx, x \in X\}$ and

(iii) for any N in $|R\text{-}\mathscr{M}od(Z)|$ and $\mathscr{S}et$ diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[d, "z"'] & & \mathscr{U}N \arrow[d, "\mathscr{U}z"] \\ zX \arrow[rr, "zg"] \arrow[dr, "\lambda"'] & & \mathscr{U}zN \arrow[dl, "\lambda"] \\ & \mathrm{Id}\, R & \end{tikzcd} \qquad \text{(D3.1)}$$

there is a unique extension $\bar{g} : M \to N$ of g in $R\text{-}\mathscr{M}od(Z)$. In fact $M \cong F\mathscr{X}$ constructed as

(i) $\quad F\mathscr{X}(y) = \coprod_{\substack{p \in zR \\ x \in X \\ p \cdot zx = y}} R(p) \times \{x\} \quad \forall y \in Z,$ with

(ii) R-action $r^*(r, x) = (r^* r, x)$ defined whenever $(r, x) \in F\mathscr{X}(y)$ and $\rho r^* = \lambda y$.

If $R = \mathbf{Z}A$ for a *group* A, and Z is the zero A-module $\{0\}$, a standard free A-module with basis $\mathscr{X}$ over Z is precisely the classical free A-module with basis X. For each monomorphism $m : B \rightarrowtail A$ in $\mathscr{G}rpnet$, the (right) pullback $\mathbf{Z}A^m$ of A along $m : \mathbf{Z}B \rightarrowtail \mathbf{Z}A$ is a free (right) B-module.

The structural requirements of each generating diagram $\mathscr{X}$ and each set diagram (D3.1) imply that each set X may determine more than one free module over Z. Hence a free module in $R\text{-}\mathscr{M}od(Z)$ is not necessarily a free object of that category, though most of the properties associated with free objects are preserved [8, 2.2.20].

4. Homotopy

The category of *standard R-chain complexes over Z* is called $R\text{-}\mathscr{C}omp(Z)$ and consists of those R-chain complexes (C, ∂) for which $C_n \in |R\text{-}\mathscr{M}od(Z)|$ and $\partial_n \in R\text{-}\mathscr{M}od(Z)$ for n in $\mathbf{Z}$, and those R-chain maps $f : C \to D$ for which $f_n \in R\text{-}\mathscr{M}od(Z)(C_n, D_n)$ for n in $\mathbf{Z}$. It is, of course, abelian. When $Z = \mathrm{Id}\, R$ it is called $R\text{-}\mathscr{C}ompreg$, the category of *regular* complexes and morphisms. If $\sigma : R \to S$ is a ringnet morphism, C is an R-complex and D is an S-complex, chain map $f : C \to D$ is a *σ-chain map* if f_n is a σ-morphism for all n. In future the only complexes considered will be standard over some zero module.

If M is an R-module, a *complex over M* is a standard chain complex

$$\cdots \to C_n \to \cdots \to C_0 \xrightarrow{\varepsilon} M \to zM$$

with *augmentation map* $\varepsilon : C_0 \to M$. If $\sigma : R \to S$ is a ringnet morphism and C is an S-complex then the *pullback complex* ${}^{\sigma}C$ consists of the pullback modules ${}^{\sigma}C_n$ together with the pullback morphisms ${}^{\sigma}\partial_n : (i, c) \mapsto (i, \partial c)$. It follows that the pullback of a standard/regular/exact complex is standard/regular/exact; the pullback of a resolution of M is a resolution of the pullback of M, and thus that if $\sigma : A \to B$ is a groupnet morphism and C is a resolution of TB then ${}^{\sigma}C$ is a resolution of TA.

4.1. Example. For each ringnet R, the *unit R-complex* $\mathscr{I}$ is defined as $\mathscr{I} = TI \otimes_{\mathbf{Z}} \mathbf{R}$ where

$$\mathbf{R} = zR \rightarrowtail \mathbf{R}_1 \xrightarrow[\partial_1]{} \mathbf{R}_0 \twoheadrightarrow zR$$

is the element of $R\text{-}\mathscr{C}omp(zR)$ defined by

(i) $\mathbf{R}_1 \cong R$ is the free (right) R-module over zR with basis $\{\gamma\} \times \mathrm{Id}\, R \twoheadrightarrow \mathrm{Id}\, R$ with $z(\gamma, i) = i$,

(ii) $\mathbf{R}_0 \cong R \oplus R$ is the free (right) R-module over $\mathbf{Z}R$ with basis $\{\alpha, \beta\} \times \mathrm{Id}\, R \twoheadrightarrow \mathrm{Id}\, R$ with $z(\alpha, i) = z(\beta, i) = i$, and

(iii) $\partial_1(\gamma, r) = (\alpha, r) - (\beta, r)\ \forall r \in R$.

In the homotopy theory for complexes, $\mathscr{I}$ serves in a fashion comparable to the way $\mathbf{Z}I$, I and $[0, 1]$ serve for ringnets, groupnets and topological spaces, respectively. It is a left $\mathbf{Z}I \otimes R$-complex with left map $1 \times \lambda : z\mathscr{I} \twoheadrightarrow \mathrm{Id}\, I \times \mathrm{Id}\, R$ and left action defined by coordinate: $([i], r)(\rho i, \xi, r^*) = (\lambda i, \xi, rr^*)$ for i in I, rr^* in R and ξ in $\{\alpha, \beta, \gamma\}$. It is also a complex of free right $\mathbf{Z}(\mathrm{Id}\, I) \odot R$-modules over $z\mathscr{I}$, with generators denoted as $(1, \alpha, i) = (\mathbf{a}, i)$, $(1, \beta, i) = (\mathbf{b}, i)$, $(1, \gamma, i) = (\mathbf{c}, i)$, $(0, \alpha, i) = (*\mathbf{a}, i)$, $(0, \beta, i) = (*\mathbf{b}, i)$ and $(0, \gamma, i) = (*\mathbf{c}, i)$. In these terms $\partial_1(\mathbf{c}, r) = (\mathbf{a}, r) - (\mathbf{b}, r)$, $\partial_1(*\mathbf{c}, r) = (*\mathbf{a}, r) - (*\mathbf{b}, r)$, and the $\mathbf{Z}I \otimes R$-action becomes

$$([*], r)(\mathbf{a}, r^*) = (*\mathbf{a}, rr^*),$$

$$([*^{-1}], r)(*\mathbf{a}, r^*) = (\mathbf{a}, rr^*) \quad \text{and so on.}$$

The *tensor product* $C \otimes_R D$ of a standard left R-complex D and a standard right R-complex C is the obvious extension of that for complexes over a ring.

If C is a left standard R-complex, then $\mathscr{I} \otimes_R C$ is a left standard $\mathbf{Z}I \otimes R$-complex, which since $R \otimes_R C_n \cong C_n$ will be henceforth written as

$$z(\mathscr{I} \otimes_R C) = \mathrm{Id}\, I \times zC,$$

$$(\mathscr{I} \otimes C)_n(1, z) = \{\mathbf{a}\} \times C_n(z) \oplus \{\mathbf{b}\} \times C_n(z) \oplus \{\mathbf{c}\} \times C_{n-1}(z),$$

$$(\mathscr{I} \otimes C)_n(0, z) = \{*\mathbf{a}\} \times C_n(z) \oplus \{*\mathbf{b}\} \times C_n(z) \oplus \{*\mathbf{c}\} \times C_{n-1}(z),$$

$$\partial_n((\mathbf{a}, c) + (\mathbf{b}, c') + (\mathbf{c}, c'')) = (\mathbf{a}, \partial c + c'') + (\mathbf{b}, \partial c' - c'') - (\mathbf{c}, \partial c''),$$

$$\partial_n((*\mathbf{a}, c) + (*\mathbf{b}, c') + (*\mathbf{c}, c'')) = (*\mathbf{a}, \partial c + c'') + (*\mathbf{b}, \partial c' - c'') - (*\mathbf{c}, \partial c'').$$

Homotopy of two chain maps between standard complexes is only possible if the images of the zeros under the second map are acted on by specific ringnet elements to give the images of the zeros under the first map. This condition is automatically satisfied for regular chain complexes, including complexes over a ring.

4.2. Definition. Let C be a standard R-complex, D be a standard S-complex, $\sigma, \tau : R \to S$ be homotopic ringnet morphisms with homotopy $\nu : \sigma \simeq \tau$, and $f, g : C \to D$ be σ, τ-chain maps respectively for which $f(z) = \nu([*], \lambda z)g(z)$ for z in zC. A *ν-homotopy $F : f \simeq g$ between f and g* is a ν-chain map $F : \mathcal{I} \otimes_R C \to D$ satisfying

(i) $F_n(*\mathbf{a}, c) = f_n(c)$ and

(ii) $F_n(\mathbf{b}, c) = g_n(c)$ for all c in C_n and n in $\mathbf{Z}$.

It is thus completely determined by f, g and $\{F_n(*\mathbf{c}, c') : c' \in C_{n-1}, n \in \mathbf{Z}\}$.

In fact, ν-homotopy extends the usual definition for complexes over a ring.

4.3. Lemma [8, 3.2.3]. *Morphism f is ν-homotopic to g if and only if there is a set $G = \{G_n : n \in \mathbf{Z}\}$ of σ-morphisms $G_n : C_n \to D_{n+1}$ satisfying*

(i) *$G_n(z) = f(z)$ $\forall z \in zC$, and*

(ii) *$(\partial G + G\partial)(c) = f(c) - \nu([*], \lambda c)g(c)$ $\forall c \in C_n$, $n \in \mathbf{Z}$.*

Chain maps between complexes over rings are chain homotopic in the classical sense if and only if they are $\chi(1)$-homotopic.

5. The mapping cylinder

5.1. Definition. A *complex diagram* (D, R, C) consists of a directed graph D, a ringnet diagram (D, R), an R_v-complex C^v for each v in D and an R_e-morphism $C^e : C^{\lambda e} \to C^{\rho e}$ for each e in D. It is a *standard/regular/projective/free/exact* complex diagram when C^v is a standard/regular/projective/free/exact complex for each v in D and, in the first two cases, when C^e is a standard/regular chain map for each e in D.

5.2. Definition. A *$\sigma(D, R)$-mapping cylinder* $\mu : (D, R, C) \to \mu(D, R, C)$ for a complex diagram (D, R, C) comprises

(i) a representation $\sigma : (D, R) \to \sigma(D, R)$ of (D, R),

(ii) a $\sigma(D, R)$-complex $\mu(D, R, C)$,

(iii) a σ_v-chain map $\mu^v : C^v \to \mu(D, R, C)$ for v in D, and.

(iv) a σ_e-homotopy $\mu^e : \mu^{\lambda e} \simeq \mu^{\rho e} \circ C^e$ for e in D, which is

(v) universal with respect to all constructions satisfying conditions (ii)–(iv).

Once σ has been prescribed, the mapping cylinder may be considered as a 'homotopy colimit with respect to σ'.

The following construction of the mapping cylinder corresponds to the process in $\mathcal{T}op$ of adding a handle to the union of vertex spaces for each edge of the directed graph, identifying its initial boundary with the source complex and its terminal boundary with the sink complex.

5.3. Theorem. *For each representation $\sigma:(D, R)\to S$ for a standard complex diagram (D, R, C), there exists an S-mapping cylinder $\mu:(D, R, C)\to M$; moreover, M is a standard S-complex.*

Proof. Some notation is first necessary. For each e in D, $\mathcal{I}_e$ represents a subscripted copy of the unit $R_{\lambda e}$-complex. In $S^v\otimes_v C^v_n$ – the R_v-tensor product of the pullback S^v of S along σ_v with C^v_n for each v in D – the element $((s, \lambda c), c)$ with $\rho s=\sigma_v(\lambda c)$ will be written (s, c). In the $\mathbf{Z}I\otimes R_{\lambda e}$-tensor product $S^e\otimes_e(\mathcal{I}_e\otimes C^{\lambda e})_n$ of the pullback S^e of S along σ_e with $(\mathcal{I}_e\otimes C^{\lambda e})_n$, for each e in D, the element $((s, ([1], \lambda c)_e), (\mathbf{d}, c))$ with $\rho s=\sigma_{\rho e}R_e(\lambda c)$ will be written $(s, \mathbf{d}, c, e)$, and the element $((s, ([0], \lambda c)_e), (*\mathbf{d}, c))$ with $\rho s=\sigma_{\lambda e}(\lambda c)$ will be written $(s, *\mathbf{d}, c, e)$ for $\mathbf{d}$ in $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Thus, in this terminology,

$$(s, *\mathbf{d}, c, e)=(s\sigma_e([*], \lambda c), (\mathbf{d}, c, e). \tag{D5.1}$$

(i) For each e in D, $z(S^e\otimes_e(\mathcal{I}_e\otimes C^{\lambda e}))\cong z(S^{\lambda e}\otimes_{\lambda e} C^{\lambda e})$. Define an abelian groupnet M_n to have identity set

$$zM=\Big(\bigvee_{v\in D} z(S^v\otimes_v C^v)\Big)\Big/\Big\langle\begin{matrix}(q, z)\sim(q\sigma_e([*], \lambda z), C^e(z)),\\ \forall(q, z)\in z(S^{\lambda e}\otimes_{\lambda e} C^{\lambda e}),\ e\in D\end{matrix}\Big\rangle.$$

With $W_v=W\cap z(S^v\otimes_v C^v)$ for each set equivalence class W of zM and v in D, let

$$M_n(W)=\coprod_{\substack{w\in W_v\\ v\in D}}(S^v\otimes_v C^v_n)(w)\oplus\coprod_{\substack{w\in W_{\lambda e}\\ e\in D}}(S^e\otimes_e(\mathcal{I}_e\otimes C^{\lambda e})_n(w)$$

$$\Big/\Big\langle\begin{matrix}(s, c)=(s, *\mathbf{a}, c, e),\ \forall(s, c)\in S^{\lambda e}\otimes_{\lambda e} C^{\lambda e}_n,\ e\in D\\ (\bar{s}, C^e_n(c))=(\bar{s}, \mathbf{b}, c, e),\ \forall(\bar{s}, C^e_n(c))\in S^{\rho e}\otimes_{\rho e} C^{\rho e}_n,\ e\in D\end{matrix}\Big\rangle. \tag{D5.2}$$

This equation may be simplified using (D5.1) to

$$M_n(W)=\coprod_{\substack{w\in W_v\\ v\in D}}(S^v\otimes_v C^v_n)(w)\oplus\coprod_{\substack{w\in W_{\lambda e}\\ e\in D}}(S^{\lambda e}\otimes_e C^{\lambda e}_{n-1})(w)\times\{e\}. \tag{D5.3}$$

The left S-action on $S^v\otimes_v C^v$ and $S^e\otimes_e(\mathcal{I}_e\otimes C^{\lambda e})$ is compatible with the relations in (D5.2) so that M_n is an S-module.

(ii) Boundary map $\partial_n: M_n\to M_{n-1}$ is induced from the boundary maps on the direct summands of (D5.2) so that, using (D5.3), $\partial_n(s, c)=(s, \partial_n c)$ and $\partial_n(s, c, e)=(s, c)-(s\sigma_e([*], \lambda c), C^e_{n-1}(c))-(s, \partial c, e)$. Routine calculation shows that ∂_n is a well-defined S-morphism and that (M, ∂) is a standard S-complex.

(iii) In terms of (D5.2) the σ_v-morphism $\mu^v : C^v \to M$ is given by $\mu^v_n(c) = (\sigma_v(\lambda c), c)$ for v in D and all n, and the σ_e-homotopy $\mu^e : \mathscr{I} \otimes C^{\lambda e} \to M$ is given by

$$\mu^e_n(*\mathbf{d}, c) = (\sigma_e([0], \lambda c), *\mathbf{d}, c, e)$$

and

$$\mu^e_n(\mathbf{d}, c) = (\sigma_e([1], \lambda c), (\mathbf{d}, c, e)$$

for $\mathbf{d}$ in $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, e in D and all n.

(v) If $\nu : (D, R, C) \to N$ satisfies (5.2. ii–iv), the unique map $\theta : M \to N$ proving the universality of μ is given (in terms of (D5.3)) by $\theta_n(s, c) = s.\ \nu^v_n(c)$ for c in C^v_n, and $\theta_n(s, c', e) = s.\ \nu^e_n(*\mathbf{c}, c')$ for c' in $C^{\lambda e}_{n-1}$.

5.4. Lemma. *The mapping cylinder is a mapping cone.*

Proof. Consider (D5.3). Define standard S-complexes (K, ∂) and (K', ∂') over zM by

$$K_n(W) = \coprod_{\substack{w \in W_{\lambda e} \\ e \in D}} (S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_n)(w) \times \{e\},$$

$$K'_n(W) = \coprod_{\substack{w \in W_v \\ v \in D}} (S^v \otimes_v C^v_n)(w),$$

for all W in zM, $\partial_n : (s, c, e) \mapsto (s, \partial c, e)$ and $\partial'_n : (s, c) \mapsto (s, \partial c)$. Define S-chain map $\partial^* : K \to K'$ by $\partial^*_n(s, c, e) = (s, c) - (s\sigma_e([*], \lambda c), C^e_n(c))$. Then $M_n = K'_n \oplus K_{n-1}$ and

$$\partial((s', c'), (s, c, e)) = (\partial'(s', c') + \partial^*(s, c, e),\ -\partial(s, c, e)).$$

5.5. Corollary. *Let (D, A) be a groupnet diagram with mapping cylinder $m : (D, A) \to G$ and induced representation $m : (D, \mathbf{Z}A) \to \mathbf{Z}G$ of $(D, \mathbf{Z}A)$. Let $(D, \mathbf{Z}A, C)$ be a standard complex diagram with G-mapping cylinder $\mu : (D, \mathbf{Z}A, C) \to M$. Then*

(i) *if $(D, \mathbf{Z}A, C)$ is regular, M is regular,*

(ii) *if $(D, \mathbf{Z}A, C)$ is free, M is free,*

(iii) *if G is a graph product and $(D, \mathbf{Z}A, C)$ is exact, M is exact, and*

(iv) *if G is a graph product and C^v is a resolution of TA_v for each v in D, then M is a resolution of TG.*

Proof. (i) An isomorphism $zM \cong \operatorname{Id} G$ is required. As

$$zM = \{(i, k) \in z\mathbf{Z}G : k \in \operatorname{Id} A_v,\ v \in D\} \Big/ \left\langle \begin{array}{l} (i, k) \sim (i, l)\ \forall A_v(k, l) \neq \emptyset,\ v \in D, \\ (i, j) \sim (i, A_e(j))\ \forall j \in \operatorname{Id} A_{\lambda e},\ e \in D \end{array} \right\rangle,$$

the map $\zeta : zM \to \operatorname{Id} G$ induced from $\zeta(i, k) = i$ is a well-defined surjection. If $\zeta(i, i) = \zeta(i, j)$ then there is an element $g = \prod_{l=1}^n p_l$ of $G(i, j)$, where each p_l has one of the forms

$$p_l = \begin{cases} m_{v_l}(a), & a \in A_{v_l}, v_l \in D, \\ m_{e_l}([*], [q]), & q \in \mathrm{Id}\, A_{\lambda e_l}, e_l \in D, \\ m_{e_l}([*^{-1}], [q]), & q \in \mathrm{Id}\, A_{\lambda e_l}, e_l \in D. \end{cases} \tag{D5.4}$$

Considered as an element of $z\mathbf{Z}G$,

$$(i, j) = (i, i)(\lambda p_1, \rho p_1) \cdots (\lambda p_n, \rho p_n);$$

hence in zM, $(i, j) = (i, \rho p_n) \sim (i, \lambda p_n) = (i, \rho p_{n-1}) \sim (i, i)$ by induction, and ζ is an isomorphism.

(ii) If C_n^v is the free left A_v-module with basis $\mathscr{X}_n^v = X_n^v \twoheadrightarrow zX_n^v$, then M_n is isomorphic to the free left G-module over zM with basis

$$\mathscr{X}_n = X_n \overset{z}{\twoheadrightarrow} zX_n,$$

with

$$X_n = \Big(\bigvee_{v \in D} X_n^v\Big) \vee \Big(\bigvee_{e \in D} \{[x|e]: x \in X_{n-1}^{\lambda e}\}\Big),$$

$$zX_n = \Big(\bigcup_{v \in D} \{W \in zM: ((\lambda y, \lambda y), y) \in W, y \in zX_n^v\}\Big)$$

$$\cup \Big(\bigcup_{e \in D} \{W \in zM: ((\lambda y, \lambda y), y) \in W, y \in zX_{n-1}^{\lambda e}\}\Big),$$

and the obvious induced map z.

When $(D, \mathbf{Z}A, C)$ is a free regular complex diagram, $F\mathscr{X}_n \cong M_n$ may be written, for each i in $\mathrm{Id}\, G$, as

$$F\mathscr{X}_n(i) = \coprod_{\substack{(i, \lambda zx) \in z\mathbf{Z}G \\ x \in X_n^v \\ v \in D}} \mathbf{Z}G(i, \lambda zx) \times \{x\} \oplus \coprod_{\substack{(i, \lambda zx) \in z\mathbf{Z}G \\ x \in X_{n-1}^{\lambda e} \\ e \in D}} \mathbf{Z}G(i, \lambda zx) \times \{[x|e]\}.$$

For the next two sections of proof it is assumed G is a graph product.

(iii) Since the maps $m_v: A_v \to G$ are embeddings, the right pullback $\mathbf{Z}G^v$ of $\mathbf{Z}G$ along m_v is a free right A_v-module. If $(D, \mathbf{Z}A, C)$ is exact, the complex $(\mathbf{Z}G^v \otimes_v C^v, 1 \otimes \partial)$ is exact for each v in D. Thus (K, ∂) and (K', ∂') are exact. The result follows either by direct computation or from the long exact homology sequence for the short exact sequence $K' \rightarrowtail M \twoheadrightarrow K^+$, where $K_n^+ = K_{n-1}$ and $\partial_n^+ = -\partial_{n-1}$.

(iv) If C^v is a $TA_v = T_v$-resolution for each v in D, then for each e in D,

$$\begin{tikzcd} C_0^{\lambda e} \arrow[r, "\varepsilon"] \arrow[d, "C_0^e"'] & T_{\lambda e} \arrow[d, "A_e"] \\ C_0^{\rho e} \arrow[r, "\varepsilon"] & T_{\rho e} \end{tikzcd} \tag{D5.5}$$

Hence $(D, \mathbf{Z}A, C)$ determines an exact regular complex diagram $(D, \mathbf{Z}A, B)$, where B^v is the augmented complex $C^v \to T_v$ and B^e is the extended chain map determined by (D5.5). If M^* is the G-mapping cylinder of $(D, \mathbf{Z}A, B)$, then M^* is exact and M is exact in all dimensions greater than 1, by (iii) above. If, for each i in Id G,

$$N_0(i) = \coprod_{\substack{(i,j)\in z\mathbf{Z}G\\ j\in \mathrm{Id}A_{\lambda e}\\ e\in D}} \mathbf{Z}G^{\lambda e}(i,j) \otimes_{\lambda e} T_{\lambda e}(j) \times \{e\},$$

then $M_0^*(i) = M_0(i) \oplus N_0(i)$. If $p_0 : M_0^* \to M_0$ is projection of the first coordinate, $M_{-1}(i) = M_{-1}^*(i)/\varepsilon^* N_0(i)$ for i in Id G, and $p_{-1} : M_{-1}^* \to M_{-1}$ is the canonical quotient map, then

$$\begin{tikzcd}
\cdots \arrow[r] & M_n^* \arrow[r] \arrow[d, "="] & \cdots \arrow[r] & M_1^* \arrow[r] \arrow[d, "="] & M_0^* \arrow[r, "\varepsilon^*"'] \arrow[d, "p_0"] & M_{-1}^* \arrow[r] \arrow[d, "p_{-1}"] & \mathrm{Id}\, G \arrow[d, "="] \\
\cdots \arrow[r] & M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r, "\bar{\varepsilon}"'] & M_{-1} \arrow[r] & \mathrm{Id}\, G,
\end{tikzcd} \tag{D5.6}$$

where $\bar{\varepsilon}(m_0) = p_{-1}\varepsilon^*(m_0, 0)$.

But $N_0 \cong \varepsilon^* N_0$, so the bottom row of (D5.6) is exact. The G-morphism $\eta : M_{-1} \to TG$ defined by $(g, [\rho g]) \mapsto [\lambda g]$ for $g \in G$ is an isomorphism, since $g = \prod_{l=1}^n p_l$ (see (D5.4)), so that in M_{-1}

$$\begin{aligned}(g, [\rho g]) &= \left(\prod_{l=1}^{n-1} p_l, [\lambda p_n]\right)\\ &= (\lambda g, [\lambda g]) \quad \text{by induction.}\end{aligned}$$

5.6. Corollary. *Under the conditions of* (5.5), *if G is a graph product and C^v is a free resolution of T_v for v in D then the G-mapping cylinder M is a free resolution of TG. Moreover, if G^* is the loop group at a selected identity i of* Id G, *– the classical case – then $M(i)$ is a free G^*-resolution of $\mathbf{Z}$.*

Proof. If T is a maximal tree in the connected component of G containing i, each $g \in G(i,j)$ for j in Id A_v may be uniquely written as $g = g^* t_{i,j}$ for g^* in G^* and $t_{i,j}$ in $T(i,j)$. The classical result follows from (5.5) when every free generator x of M_n is replaced by element $t_{i,\lambda x}x$ of M_n, and corresponding adjustments are made to the boundary maps.

It is not necessary that G be a graph product for the mapping cylinder of an exact complex diagram to be exact, but neither is exactness of the complex diagram sufficient to ensure exactness of the mapping cylinder.

6. Homology and its applications

If $\nu : \mathbb{Z}I \otimes R \to S$ is a ringnet homotopy and $f, g : C \to D$ are ν-homotopic standard chain maps, their induced homology chain maps differ only by a 'change of base point isomorphism'. For, if $F : f \simeq g$ and $c \in \operatorname{Ker} \partial_n(z)$, then $f(c) + \operatorname{Im} \partial_{n+1}(z) = \nu([*], \lambda c)(g(c) + \operatorname{Im} \partial_{n+1}(z))$. It follows that homotopic chain complexes in R-$\mathscr{C}omp(Z)$ have isomorphic homology modules. It is possible to extend the comparison theorem generally to take into account chain maps between complexes over different ringnets, but only the following partial generalisation will be needed [8, 4.2.4].

6.1. Lemma (Regular Comparison Theorem). *Let C be a projective complex over M in R-$\mathscr{C}ompreg$, let D be a resolution over N in S-$\mathscr{C}ompreg$ and let $\sigma : R \to S$ be a ringnet morphism. Then any σ-morphism $f : M \to N$ lifts to a σ-chain map $g : C \to D$ with $\varepsilon g = f\varepsilon$; any two such lifting maps are $\chi(\sigma)$-homotopic.*

6.2. Definition. Let A be a groupnet, M be a right A-module and N be a left A-module. If C is a projective left A-resolution of TA, the *homology module $H_*(A; M)$ of A with coefficients in M* is the homology module $H(M \otimes_A C)$, and the *cohomology module $H^*(A; N)$ of A with coefficients in N* is the homology module $H(\hom_A(C, N))$.

The definition is independent of the projective left A-resolution C of TA.

In order to determine the Mayer-Vietoris sequences of the next theorem, it must be noted that under the conditions of (5.3), for each left S-module L, each element h of $\hom_S(zM, zL)$ determines a unique collection $\{h_v \in \hom_v(zC^v, z^vL) : v \in D\}$ with $h_v(z) = (z, h(\sigma_v(\lambda z), z))$ for z in zC^v, and vice versa.

6.3. Theorem (Mayer-Vietoris Sequence). *Let $\sigma : (D, R) \to S$ be a ringnet representation and $\mu : (D, R, C) \to M$ be the S-mapping cylinder of the standard complex diagram (D, R, C). Let N be a right S-module and L be a left S-module. The following sequences are exact, for each Y in $z(N \otimes_S M)$ and h in* $\hom_S(zM, zL)$:

$$\cdots \to \coprod_{\substack{y \in Y_{\lambda e} \\ e \in D}} H_m(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e})(y) \times \{e\} \xrightarrow{\partial_m} \coprod_{\substack{y \in Y_v \\ v \in D}} H_m(N^v \otimes_v C^v)(y)$$

$$\xrightarrow{i_m} H_m(N \otimes_S M)(Y) \xrightarrow{p_m} \coprod_{\substack{y \in Y_{\lambda e} \\ e \in D}} H_{m-1}(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e})(y) \times \{e\} \to \cdots, \tag{D6.1}$$

$$\cdots \to H_m(\hom_S(M, L))(h) \xrightarrow{\iota_m} \prod_{v \in D} H_m(\hom_v(C^v, {}^vL))(h_v) \xrightarrow{\delta_m}$$

$$\prod_{e \in D} H_m(\hom_{\lambda e}(C^{\lambda e}, {}^{\lambda e}L))(h_{\lambda e}) \times \{e\} \xrightarrow{\pi_{m+1}} H_{m+1}(\hom_S(M, L))(h) \to \cdots. \tag{D6.2}$$

Proof. (D6.1). Directed graph D determines a groupnet diagram $(D, 1)$ with the trivial group at each vertex and the identity morphism on each edge. The induced ringnet diagram $(D, \mathbf{Z})$ has a representation $j:(D, \mathbf{Z}) \to \mathbf{Z}$ consisting of the obvious identity morphisms and homotopies. The pullback N^v of N along $\sigma_v : R_v \to S$ determines a standard $\mathbf{Z}$-complex $(N \otimes C)^v = N^v \otimes_v C^v$. If

$$\bar{C}^e = (N \otimes C)^e : (N \otimes C)^{\lambda e} \to (N \otimes C)^{\rho e}$$

is given by

$$\bar{C}^e_k(n, c) = (n\sigma_e([*], \lambda c), C^e_k(c))$$

(where the notational conventions are those of (5.3) with S replaced by N), a standard complex diagram $(D, \mathbf{Z}, N \otimes C)$ is determined, whose $j(D, \mathbf{Z})$-mapping cylinder H is isomorphic to $N \otimes_S M$. In other words, the mapping cylinder commutes with tensor products. For $Y \in z(N \otimes_S M)$,

$$H_n(Y) = \coprod_{\substack{y \in Y_v \\ v \in D}} (N^v \otimes_v C^v_n)(y) \oplus \coprod_{\substack{y \in Y_{\lambda e} \\ e \in D}} (N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_{n-1})(y) \times \{e\}$$

so that (5.4) there is a short exact sequence

$$\coprod_{\substack{y \in Y_v \\ v \in D}} (N^v \otimes_v C^v_m)(y) \xrightarrow[i^*_m]{} N \otimes_S M_m(Y) \xrightarrow[p^*_m]{} \coprod_{\substack{y \in Y_{\lambda e} \\ e \in D}} (N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_{m-1})(y) \times \{e\}$$

for each dimension m. As the homology functor preserves arbitrary coproducts,

$$H\Big(\coprod_{\substack{y \in Y_v \\ v \in D}} (N^v \otimes_v C^v_m)(y)\Big) \cong \coprod_{\substack{y \in Y_v \\ v \in D}} H(N^v \otimes_v C^v_m)(y)$$

and the required result is given by the long exact homology sequence corresponding to this short exact sequence, together with these isomorphisms. Map ∂_m is determined from the homology morphism of

$$\partial^*_m : (n, c, e) \mapsto (n, c) - \bar{C}^e_m(n, c).$$

(D6.2). Since $\hom_S(-, L)$ is left exact, the short exact sequence (5.4) $K'_m \rightarrowtail M_m \twoheadrightarrow K^+_m$ induces the left exact sequence

$$\hom_S(K^+_m, L) \rightarrowtail \hom_S(M_m, L) \to \hom_S(K'_m, L)$$

which in fact is short exact. Each $f \in \hom_S(K'_m, L)(h)$ determines a collection $\{f_v \in \hom_v(C^v_m, {}^vL)(h_v) : v \in D\}$ with $f_v(c) = (\lambda c, f(\sigma_v(\lambda c), c))$ and vice versa, to determine an isomorphism

$$\hom_S(K'_m, L)(h) \cong \prod_{v \in D} \hom_v(C^v_m, {}^vL)(h_v)$$

and similarly there is an isomorphism

$$\hom_S(K^+_m, L) \cong \prod_{e \in D} \hom_{\lambda e}(C^{\lambda e}_{m-1}, {}^{\lambda e}L)(h_{\lambda e}) \times \{e\}.$$

Since the homology functor preserves arbitrary products, the required result follows from these isomorphisms together with the long exact homology sequence of the short exact sequence above. Map δ_m is determined from the homology morphism of

$$\delta_m^*:(s, c_m, e)\mapsto(s, c_m)-(s\sigma_e([*], \lambda c_m), C_m^e(c_m)).$$

6.4. Corollary (Mayer-Vietoris Sequence for Graph Products). *Let (D, A) be a groupnet diagram with graph product $m:(D, A)\to G$. Let N be any right G-module and L be any left G-module. The following sequences are exact:*

$$\cdots\to\coprod_{\substack{y\in Y_{\lambda e}\\ e\in D}} H_m(A_{\lambda e}; N^{\lambda e})(y)\times\{e\}\to\coprod_{\substack{y\in Y_v\\ v\in D}} H_m(A_v; N^v)(y)\to H_m(G; N)(Y)$$
$$\to\coprod_{\substack{y\in Y_{\lambda e}\\ e\in D}} H_{m-1}(A_{\lambda e}; N^{\lambda e})(y)\times\{e\}\to\cdots, \tag{D6.3}$$

$$\cdots\to H^m(G; L)\to\prod_{v\in D} H^m(A_v; {}^vL)(h_v)\to\prod_{e\in D} H^m(A_{\lambda e}; {}^{\lambda e}L)(h_{\lambda e})\times\{e\}$$
$$\to H^{m+1}(G; L)(h)\to\cdots. \tag{D.6.4}$$

Proof. There always exists a standard complex diagram $(D, \mathbf{Z}A, C)$ with C^v a free A_v-resolution of T_v, hence (5.6) there always exists a mapping cylinder complex which is a free G-resolution of TG.

Any groupnet diagram (D, A) determines a *derived loop group diagram* (D, A^*) where A_v^* is the collection of loop groups of A_v, at selected identities, determined by a retraction $r_v: A_v\to A_v^*$, and where $A_e^*=r_{\rho e}\circ A_e$. If A_e is a monomorphism, so is A_e^*. If the mapping cylinder of (D, A) is connected, so is that of (D, A^*) and they have isomorphic loop groups [4, Theorem 8.4].

6.5. Corollary (Mayer-Vietoris Sequence for groups with the homotopy type of graph products). *Let (D, A) be a groupnet diagram of connected groupnets with derived loop group diagram (D, A^*) for which each A_e is a monomorphism. Assume the graph product $m:(D, A^*)\to G$ is connected, and let $r: G\to G^*$ be a retraction of G to its loop group G^* at a selected identity. For any regular right G^*-module N and any regular left G^*-module L the following sequences are exact:*

$$\cdots\to\coprod_{e\in D} H_m(A_{\lambda e}^*; N)\times\{e\}\to\coprod_{v\in D} H_m(A_v^*; N)\to H_m(G^*; N)$$
$$\to\coprod_{e\in D} H_{m-1}(A_{\lambda e}^*; N)\times\{e\}\to\cdots, \tag{D6.5}$$

$$\cdots\to H^m(G^*; L)\to\prod_{v\in D} H^m(A_v^*; L)\to\prod_{e\in D} H^m(A_{\lambda e}^*; L)\times\{e\}\to H^{m+1}(G^*; L)$$
$$\to\cdots. \tag{D6.6}$$

Proof. If $t: A \to A^*$ is a retraction of a connected groupnet A onto its loop group at a selected identity, N is any regular right A^*-module, and L, C are any regular left A^*-modules then there are isomorphisms $N \otimes_{A^*} C \cong N^t \otimes_A {}^tC$ and $\hom_{A^*}(C, L) \cong \hom_A({}^tC, {}^tL)$. Since there always exists a free left A^*-resolution of TA^* whose pullback along t is a projective A-resolution of TA, $H_*(A; N^t) \cong H_*(A^*; N)$ and $H^*(A; {}^tL) \cong H^*(A^*; L)$. As $(N^t)^v \cong N$ and ${}^v({}^tL) \cong L$ here, the results follow from (D6.3) and (D6.4) since the regularity of modules ensures that all zeros are singleton sets.

A regular G^*-module is classically a G^*-module. If (D, A) is a group diagram derived from a graph of groups, the Mayer-Vietoris sequences for the graph of groups are found from those of (D, A) by dividing out, in each dimension, one copy of each source vertex (co)homology group and one of the corresponding pair of 'edge' groups. Specifically, the short exact sequences

$$H_m(A^*_{\lambda e}; N) \times \{e\} \to \partial_m(H_m(A^*_{\lambda e}; N) \times \{e\})$$

and

$$H^m(A^*_{\lambda e}; L) \to \delta_m H^m(A^*_{\lambda e}; L)$$

are divided out. The sequences for a graph of groups are due to Lyndon and Swan [11, 2.3] in the case of the free product with amalgamation, to Bieri [1] in the case of the HNN group and also recently to Chiswell [3] and Dicks [5] in the general case.

6.6. Definition. A connected groupnet A is of *finite cohomological dimension* $\mathrm{cd}(A) \leq m$ if $H^k(A; L) = 0$ for every $k > m$ and every regular A-module L. It is of *cohomological dimension* m if $\mathrm{cd}(A) \leq m$ and $\mathrm{cd}(A) \not\leq m-1$. *Homological dimension* $\mathrm{hd}(A)$ is correspondingly defined. If (D, A) is a groupnet diagram with connected vertex groupnets,

$$n^E = \sup\{\mathrm{cd}\, A_{\lambda e} : e \in D\} \leq \infty$$

and

$$n^V = \sup\{\mathrm{cd}\, A_v : v \in D,\ v \neq \lambda e,\ e \in D\} \leq \infty.$$

Numbers n_E and n_V are similarly defined in terms of homological dimension. Both n^V and n_V are assumed to exist.

6.7. Lemma. *If (D, A) has connected vertex groupnets and connected graph product $m: (D, A) \to G$, then*

$$\mathrm{cd}\, G = n^V \quad \text{if } n^E < n^V,$$
$$n^V \leq \mathrm{cd}\, G \leq n^V + 1 \quad \text{if } n^E = n^V.$$

Moreover, the corresponding result holds for homological dimension.

This result is a simple consequence of (6.4). It is proved by Bieri [1, 4.1] for HNN groups and partially proved by Gildenhuys [6, Theorem 2] in the case when D is a tree. The result cd $G \leq 1+\sup(n^E, n^V)$ for the general case is also obtained by Chiswell [3].

6.8. Lemma. *If D is a finite graph, (D, A) is a group diagram, the graph product G of (D, A) is connected, and G^* is the loop group of G at a selected identity, then*
(i) *if A_v is of type* $(\overline{\mathrm{FP}})$ *for all v in D, so is G^*, and*
(ii) *if A_v is of type* (FP) *for all v in D, so is G^*.*

This result is a consequence of (5.5.ii), (5.6) and (6.7). It is also proved by Chiswell [3, Theorem 3] and for amalgamated free products and HNN groups, by Bieri and Eckmann [1, 2].

Since Strebel [10, Theorem, Section 4.4] has shown that all duality groups are necessarily of type (FP), the final results of this paper follow from (6.4).

6.9. Theorem. *Let D be a finite graph, (D, A) be a group diagram with connected graph product G, and G^* be the loop group of G at a selected identity. If*
(i) *A_v is a duality group of dimension $n-1$ for all v in D such that $v = \lambda e$ for some e in D, and*
(ii) *A_v is a duality group of dimension n for all other v in D,*
then G^* is a duality group of dimension n.

6.10. Lemma. *Under the conditions of* (6.9), *if A_v is a duality group of dimension n for all v in D, and if* cd $G^* \leq n$, *then G^* is a duality group of dimension n.*

These results extend those of Bieri and Eckmann [1, 2] for HNN groups and free products with amalgamation.

References

[1] R. Bieri, Mayer-Vietoris sequences for HNN-groups and homological duality, Math. Z. 143 (1975) 123–130.
[2] R. Bieri and B. Eckmann, Amalgamated free products of groups and homological duality, Comment. Math. Helv. 49 (1974) 460–478.
[3] I. M. Chiswell, Exact sequences associated with a graph of groups, J. Pure Appl. Algebra 8 (1976) 63–74.
[4] R. H. Crowell and N. Smythe, The subgroup theorem for amalgamated free products, HNN-constructions and colimits, Proc. Second Internat. Conf. Theory of Groups, Canberra 1973 (Lecture Notes in Math. 372, Springer-Verlag, Berlin, 1974) 241–280.
[5] W. Dicks, A short exact sequence for colimits of rings, Proc. London Math. Soc. 34 (3) (1977) 557–576.
[6] D. Gildenhuys, The cohomology of groups acting on trees, J. Pure Appl. Algebra 6 (1975) 265–274.

[7] P. J. Higgins, Notes on Categories and Groupoids (Van Nostrand Reinhold Math. Studies 32, Van Nostrand, London, 1971).
[8] K. J. Horadam, The homology of groupnets, Ph.D. thesis, Australian National University, Canberra, A.C.T. (1976).
[9] B. Mitchell, Rings with several objects, Advances in Math. 8 (1972) 1–161.
[10] R. Strebel, On homological duality, J. Pure Appl. Algebra 8 (1976) 75–96.
[11] R. G. Swan, Groups of cohomological dimension one, J. Algebra 12 (1969) 585–610.