Backward stochastic differential equations with a uniformly continuous generator and related $g$-expectation

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Abstract

In this paper, we will study a class of backward stochastic differential equations (BSDEs for short), for which the generator (coefficient) $g(t, y, z)$ is Lipschitz continuous with respect to $y$ and uniformly continuous with respect to $z$. We establish several properties for such BSDEs, including comparison and converse comparison theorems, a representation theorem for $g$ and a continuous dependence theorem. Then we introduce a new class of $g$-expectation based on such backward stochastic differential equations, and discuss its properties.

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1. Introduction

The $g$-expectation introduced by Peng [23] in 1997, based on backward stochastic differential equations (BSDEs for short) and studied by many mathematicians and economists, is a kind of nonlinear expectation as a generalization of the Girsanov transformation. As a prelude to this paper, we first recall the notion of Peng’s $g$-expectation. It is well known that the starting point of the development of the general BSDE

$$y_t = X + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_sdW_s, \quad 0 \leq t \leq T, \quad (1)$$

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is the celebrated paper of Pardoux and Peng [22] from 1990. The authors of [22] established the existence and uniqueness of the solution to (1), where $X$, the terminal condition, is an $\mathcal{F}_T$-measurable, square integrable random variable, $W$ is an $\mathbb{R}^d$-valued standard Brownian motion, and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies some reasonable conditions, in which the important assumption is that $g$ is Lipschitz continuous with respect to $(y, z)$.

For under the above assumptions and the additional condition $g(t, y, 0) = 0$ for each $(t, y) \in [0, T] \times \mathbb{R}$, Peng [23] introduced in 1997 the notions of $g$-expectation and conditional $g$-expectation. The $g$-expectation of $X$ is given by

$$\mathcal{E}^g[X] = y_0 : L^2(\mathcal{F}_T) \rightarrow \mathbb{R},$$

and the conditional $g$-expectation of $X$ given $\mathcal{F}_t$ is defined by

$$\mathcal{E}^g[X|\mathcal{F}_t] = y_t : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t),$$

where the pair of processes $(y, z)$ is the solution of (1). Here we call the function $g$ a generator of $g$-expectation. More importantly, Peng proved that almost all properties of expectation hold true for $g$-expectation except the linearity, in other words, $g$-expectation is a kind of nonlinear expectation, and in the case when $g \equiv 0$, $g$-expectation becomes the usual expectation. Hence, the original motivation for studying $g$-expectation comes from the theory of expected utility, which is very significant in economics and finance. Since the notion of $g$-expectation was introduced, many of its properties have been studied by Peng [23,24], El Karoui, Peng and Quenez [9], Chen [4], Jiang [14,15], Briand, Coquet, Hu, Memin and Peng [1], Chen and Epstein [6], Coquet, Hu, Memin and Peng [8] and Rosazza Gianin [26]. Chen and Epstein [6] gave an application of $g$-expectation to recursive utility. Rosazza Gianin [26] introduced some examples of dynamic risk measures via $g$-expectations.

As mentioned before, an essential hypothesis for Peng’s $g$-expectation is that the generator $g$ is assumed to be Lipschitz continuous with respect to $(y, z)$. For this framework, Peng defined such nonlinear expectations, and studied their properties and applications. For convenience, we call this kind of $g$-expectation standard $g$-expectation, and call the associated generator the standard generator. Recently, Jia [13, 2008] proved that if $g$ is uniformly continuous (UC for short) in $z$ and Lipschitz continuous in $y$, then BSDE (1) has a unique solution in the usual sense. It is natural for us to define a new kind of $g$-expectation based on BSDEs with UC generators just like Peng has done. What does the new $g$-expectation look like? What properties does it have? Answering these questions is the aim of this paper. In the first part of this paper, we will establish several properties for such BSDEs, including comparison and converse comparison theorems, a representation theorem for $g$ and a continuous dependence theorem. An interesting result already found in the theory of such BSDEs is that unlike the standard one, it does not have strict monotonicity any longer. Such a phenomenon might be seen in a financial market with many arbitrage opportunities, so these kinds of BSDEs may provide a useful tool for studying a market like that. The second part of this paper is devoted to introducing a new kind of $g$-expectation based on BSDEs with UC generators, and studying its properties. Recently, the notion of $g$-expectation has been extended. In Hu, Ma, Peng and Yao [12], a kind of $g$-expectation based on quadratic BSDEs is studied. Of course there are some important differences between these two kinds of new $g$-expectations, which stem from the differences between the two kinds of BSDEs. (See [19,23,10] for more details about non-UC BSDEs.)

This paper is organized as follows. In the next section, we will recall and show some fundamental results for BSDEs with UC generators, including an existence and uniqueness theorem and a very useful approximation lemma. After this, we apply ourselves to establishing
some important properties of solutions of such BSDEs. Section 4 is devoted to defining and studying our new $g$-expectation. Finally, we will give an interesting remark on Jensen’s inequality for this kind of $g$-expectation.

2. Preliminaries and the uniqueness theorem for BSDEs with UC generators

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t \geq 0}$ a $d$-dimensional Brownian motion in this space. The natural filtration generated by this Brownian motion is denoted by $\{\mathcal{F}_t; 0 \leq t \leq \infty\}$. All processes mentioned in this paper are supposed to be $\mathcal{F}_t$-adapted. And we are interested in the behavior of processes on a given interval $[0, T]$ where $T > 0$ is the terminal time.

We need the following notation. Suppose $p \geq 1$ and let $\tau \leq T$ be a given $\mathcal{F}_\tau$-stopping time.

- The scalar product and norm of the Euclid space $\mathbb{R}^n$ are respectively denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$.
- The set of all rational numbers contained in $\mathbb{R}^d$ is denoted by $\mathbb{Q}^d$.
- The set of all positive integers is denoted by $\mathbb{N}$.
- $L^p(\mathcal{F}_\tau; \mathbb{R}^n) := \{\text{the space of all } \mathbb{R}^n\text{-valued } \mathcal{F}_\tau\text{-measurable random variables } X \text{ such that } E[|X|^p] < \infty\}$.
- $L^p(0, \tau; \mathbb{R}^n) := \{\mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-adapted stochastic processes } \phi \text{ such that } E \int_0^\tau |\phi_t|^p \, dt < \infty\}$.
- $S^p_\mathcal{F}(0, \tau; \mathbb{R}^n) := \{\text{all continuous processes } \phi \text{ such that } E[\sup_{t \in [0, T]} |\phi_t|^p] < \infty\}$.

If $n = 1$, we denote $L^p(\mathcal{F}_\tau; \mathbb{R})$, $L^p(0, \tau; \mathbb{R})$ and $S^p_\mathcal{F}(0, \tau; \mathbb{R})$ by $L^p(\mathcal{F}_\tau)$, $L^p(0, \tau)$ and $S^p_\mathcal{F}(0, \tau)$ respectively.

For $X \in L^2(\mathcal{F}_\tau)$, the solution of (1) is a pair of processes $(y, z) \in S^2_\mathcal{F}(0, T) \times L^2_\mathcal{F}(0, T; \mathbb{R}^d)$ satisfying (1), $P$-a.s. We usually denote the BSDE (1) by $(g, T, X)$ and denote its solution by $(y_{g, T, X}, z_{g, T, X})$.

Here are the conditions for $g$ when we discuss $(g, T, X)$:

(A1) $\{g(t, y, z)\}_{t \in [0, T]} \in L^2_\mathcal{F}(0, T)$ for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$; and $g$ is of linear growth in $(y, z)$, i.e., there exists a constant $L > 0$ such that for each $(y, z)$, $|g(t, y, z)| \leq L(1 + |y| + |z|)$, $dP \times dt$-a.e.

(A2) $g$ is Lipschitz continuous in $y$ with constant $K > 0$, and UC in $z$ with continuous modulus $\phi$, i.e., there exist a constant $K$ and a continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$ such that for any $(y_i, z_i) \in \mathbb{R} \times \mathbb{R}^d$, $i = 1, 2$, one has

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq K |y_1 - y_2| + \phi(|z_1 - z_2|), \quad dP \times dt\text{-a.e.},$$

where $\phi$ is increasing, subadditive, and of linear growth with constant $K$, i.e., $0 \leq \phi(x) \leq K(1 + x)$, $\forall x \in \mathbb{R}_+$.

(A3) $g(\cdot, y, z)|_{z=0} \equiv 0$.

(A4) For each $(y, z)$, $t \mapsto g(t, y, z)$ is continuous.

**Remark 2.1.** Here $\phi$ can be written as

$$\phi(x) = \sup \left\{|g(t, y, z_1) - g(t, y, z_2)| : |z_1 - z_2| \leq x, z_1, z_2 \in \mathbb{R}^d, \forall y\right\}, \quad \forall x \in \mathbb{R}_+.$$

**Remark 2.2.** Clearly, a BSDE with generator $g$ satisfying (A1), (A2) with $\phi(x) = Kx$ becomes the situation Pardoux and Peng [22] studied.

We first recall the existence and uniqueness theorem for BSDEs with UC generators (see [13]).
Theorem 2.3. Suppose that $g$ satisfies (A1) and (A2). Then for each $X \in L^2(\mathcal{F}_T)$, there exists a unique pair of processes $(y, z) \in S^2_T(0, T) \times L^2_T(0, T; \mathbb{R}^d)$ satisfying BSDE (1).

The existence part of the above theorem is due to Lepeltier and San Martin [20]. To prove the uniqueness part, we need the following technical lemma, which comes from Lemma 2 in [13]. For $m \in \mathbb{N}$, we define

$$
\underline{g}_m(t, y, z) := \inf_{v \in \mathbb{Q}^d} \{ g(t, y, v) + m \, |z - v| \}, \quad \forall t, y, z
$$

and

$$
\bar{g}_m(t, y, z) := \sup_{v \in \mathbb{Q}^d} \{ g(t, y, v) - m \, |z - v| \}, \quad \forall t, y, z.
$$

Lemma 2.4. Let $g$ satisfy (A1), (A2); and let $C = \max \{ K, L \}$. Then for any $m \geq C$, one has:
(i) for each $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, the inequalities $-C(|y| + |z| + 1) \leq \underline{g}_m(t, y, z) \leq g(t, y, z) \leq \bar{g}_m(t, y, z) \leq C(|y| + |z| + 1)$ hold true;
(ii) for each $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $g(t, y, z)$ is non-decreasing and $\bar{g}(t, y, z)$ is non-increasing;
(iii) $g_m(t, \cdot, \cdot)$ and $\bar{g}_m(t, \cdot, \cdot)$ are both Lipschitz functions with constant $m$;
(iv) if $(y_m, z_m) \to (y, z)$ as $m \to \infty$, then $g_m(t, y_m, z_m) \to g(t, y, z)$ and $\bar{g}_m(t, y_m, z_m) \to g(t, y, z)$ as $m \to \infty$;
(v) for each $(t, y, z)$, the inequalities $0 \leq g(t, y, z) - \underline{g}_n(t, y, z) \leq \phi(\frac{2C}{n-C})$ and $0 \leq \bar{g}_n(t, y, z) - g(t, y, z) \leq \phi(\frac{2C}{n-C})$ hold true.

Remark 2.5. It is worth noting that we assumed in [13] or (A1) that $|g(\cdot, 0, 0)| \leq L$. But this is not an essential condition; $g(\cdot, 0, 0) \in L^2_T(0, T)$ is enough. Indeed, we take $\tilde{g}(t, y, z) = g(t, y, z) - g(t, 0, 0)$, which satisfies (A1), (A2). Then we write $g$ as $g(t, y, z) = \tilde{g}(t, y, z) + g(t, 0, 0)$, and only transform $\tilde{g}$ by Lemma 2.4. Finally, this is not hard to check by procedures similar to those in the proof of Theorem 5 in [13]. In addition, it is also not hard to check that all results in this paper hold true in the same way. Furthermore, the Lipschitz continuity of $g$ with respect to $y$ can also be relaxed; some monotonicity condition such as that in [21] is enough for the existence and uniqueness of a solution to (1). It is just for simplicity that we assume that $g$ satisfies (A1).

To end this section, we will prove some properties of the solutions of BSDEs with $g$ satisfying (A1), (A2).

Lemma 2.6. Let $g$ satisfy (A1), (A2) and $X \in L^2(\mathcal{F}_T)$. Then for any given $n \geq C$, one has:
(i) there exists a constant $H$ only depending on $K$ and $T$ such that for each $t \in [0, T]$, the inequalities

$$
0 \leq E[\gamma_t^{g, T, X} - \gamma_t^{\underline{g}_n, T, X}] \leq H \phi \left( \frac{2C}{n-C} \right) (T - t) \quad \text{and}
$$

$$
0 \leq E[\gamma_t^{\bar{g}_n, T, X} - \gamma_t^{\bar{g}, T, X}] \leq H \phi \left( \frac{2C}{n-C} \right) (T - t)
$$

hold true;
(ii) there exists a constant $L(n)$ only depending on $n$ and $T$ such that for each $t \in [0, T]$, the inequalities
\[
E \left[ \sup_{s \in [t, T]} (e^{\beta s} |y_s^g, T, X - y_s^g, T, X|^2 + \int_t^T e^{\beta s} |\zeta_s^g, T, X - \zeta_s^g, T, X|^2 \, ds \right] \leq L(n)(T - t)^2,
\]
\[
E \left[ \sup_{s \in [t, T]} (e^{\beta s} |\hat{y}_s^g, T, X - \hat{y}_s^g, T, X|^2 + \int_t^T e^{\beta s} |\hat{\zeta}_s^g, T, X - \hat{\zeta}_s^g, T, X|^2 \, ds \right] \leq L(n)(T - t)^2
\]
hold true, where $\beta = 2(n + n^2)$.

**Proof.** For the sake of simplicity, let $(y_t, z_t)_{t \in [0, T]}$ be the solution of $(g, T, X)$ and $(\hat{y}_t, \hat{z}_t)_{t \in [0, T]}$ the solution of $(g, T, X)$. Note that
\[
y_t - y_t^n = \int_t^T [g(s, y_s, z_s) - g_n(s, y_s^n, z_s^n)] \, ds - \int_t^T (z_s - \hat{z}_s) \, dW_s
\]
\[
= \int_t^T [g(s, y_s, z_s) - g_n(s, y_s, z_s) + g_n(s, y_s, z_s) - g_n(s, y_s^n, z_s^n)] \, ds
\]
\[
- \int_t^T (z_s - \hat{z}_s) \, dW_s.
\]
Set $\hat{y}_t^n := y_t - y_t^n, \hat{z}_t^n := z_t - z_t^n$, and denote by $z_t^{i, n, i}$ the components of $z_t$ and $z_t^n$ respectively. Define $z_t^{i, 0} := z_t, z_t^{i, n, i} := (z_t^{i, 1}, \ldots, z_t^{i, n, 1}, \ldots, z_t^{i, 1}, \ldots, z_t^{i, 1})$ for $1 \leq i \leq d$.
\[
b_t^n := \begin{cases} g_n(t, y_t^n, z_t^{n, i-1}) - g_n(t, y_t^n, z_t^{n, i}) & \text{if } z_t^{i, n, i} \\ 0 & \text{otherwise} \end{cases},
\]
\[
a_t^n := \begin{cases} g_n(t, y_t, z_t) - g_n(t, y_t^n, z_t) & \text{if } y_t \neq y_t^n \\ 0 & \text{otherwise} \end{cases}
\]
and $\hat{g}_s^n = g(s, y_s, z_s) - g_n(s, y_s, z_s)$. So the above equation can be rewritten as
\[
\hat{y}_t^n = \int_t^T [a_t^n \hat{y}_s^n + b_t^n \cdot \hat{z}_s^n + \hat{g}_s^n] \, ds - \int_t^T \hat{z}_s^n \, dW_s.
\]
By Lemma 2.4, one has $0 \leq \hat{g}_s^n \leq \phi(\frac{2C}{n-C}), |a_t^n| \leq K$ and $|b_t^n| \leq n$, and
\[
\hat{y}_t^n = E \left[ \int_t^T \exp \left\{ \int_t^s a_r^n \, dr + \int_t^s b_r^n \, dW_r - \frac{1}{2} \int_t^s |b_r^n|^2 \, dr \right\} \hat{g}_s^n \, ds \bigg| \mathcal{F}_t \right].
\]
Therefore
\[
E \hat{y}_t^n = E \left[ \int_t^T \exp \left\{ \int_t^s a_r^n \, dr + \int_t^s b_r^n \, dW_r - \frac{1}{2} \int_t^s |b_r^n|^2 \, dr \right\} \hat{g}_s^n \, ds \right]
\]
\[
\leq \exp \{KT\} \phi \left( \frac{2C}{n - C} \right) (T - t).
\]
Part (i) of this lemma has been proved. From (i) of this lemma and Proposition 2.2 in [1], (ii) can be obtained immediately. The proof is complete. \qed

In the sequel we always assume $n > C$. 

3. Some properties of the solutions to BSDEs with UC generators

3.1. The comparison theorem and the strict comparison theorem

The comparison theorem due to Peng [25] is a very important result in the theory of BSDEs just like the maximum principle in the theory of PDEs. Here is the comparison theorem in our settings.

**Theorem 3.1 (Comparison Theorem).** Let \( g \) satisfy (A1) and (A2), and let \((y', z')\) be the solution of the following BSDE:

\[
y'_t = X' + \int_t^T g'_s \, ds - \int_t^T z'_s \, dW_s, \quad 0 \leq t \leq T,
\]

where \( g' \in L^2_T(0, T; \mathbb{R}) \) and \( X' \in L^2(\mathcal{F}_T) \) are given such that \( X \geq X' \), \( g(t, y'_t, z'_t) \geq g'_t, \) \( P\)-a.s., \( \forall t \in [0, T] \).

Then for \( t \in [0, T] \), \( y^{g,T}_t \geq y'_t, \) \( P\)-a.s.

**Proof.** As in Lemma 2.4, we define \( \overline{g}_n(t, y, z) = \sup_{v \in \mathbb{R}^d} \{ g(t, y, v) - n |z - v| \} \).

Again by Lemma 2.4, one has \( \overline{g}_n(t, y'_t, z'_t) \geq g(t, y'_t, z'_t) = g'_t, \) \( P\)-a.s.. It follows immediately from the well known comparison theorem for BSDEs with standard generators that for each \( t \in [0, T] \), one has

\[
y^{\overline{g}_n,T}_t \geq y'_t, \quad P\text{-a.s.}
\]

which leads to \( y_t \geq y'_t, \) \( P\)-a.s. \( \square \)

Note that there is an important difference between the above comparison theorem for BSDEs with UC generators and the standard one: the strict comparison theorem does not hold in general whenever \( g \) is UC in \( z \). Here is an example.

**Example 3.2.** Consider the following BSDE:

\[
y_t = X + \int_t^T (3 |z_s|^{2/3}) \, ds - \int_t^T z_s \, dW_s, \quad 0 \leq t \leq T.
\]

where \( W \) is a one-dimensional standard Brownian motion. Clearly, \( g(z) = -3 |z|^{2/3} \) satisfies (A1) and (A2). It is not hard to verify that \((y_t^{g,T,0}, z_t^{g,T,0})_{t \in [0, T]} = (0, 0)\) is the unique solution of \((g, T, 0)\), and \((y_t^{g,T,1/4 W^4_T}, z_t^{g,T,1/4 W^4_T})_{t \in [0, T]} = (W^4_T, W^3_T)_{t \in [0, T]}\) is the unique solution of \((g, T, 1/4 W^4_T)\). Note that \( P(1/4 W^4_T > 0) = 1 > 0\), but \( y_0^{g,T,0} = y_0^{g,T,1/4 W^4_T} = 0\).

Certainly, there do exist some special situations in which the strict comparison theorem still holds. For instance:

**Example 3.3.** Let \( g \) satisfy (A1)–(A3), and \( g(t, y, z) \geq 0, \) \( P\)-a.s., \( \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \). If \( X \geq 0, \) \( P\)-a.s., then \( P(X > 0) > 0 \) implies that for any \( t \in [0, T] \), one has \( P(y^{g,T}_t > y^{g,T}_t) > 0 \). In particular, \( y_0^{g,T} > y_0^{g,T,0} \).

**Proof.** The condition (A3) means that \((y_t^{g,T,0}, z_t^{g,T,0})_{t \in [0, T]} = (0, 0)\). Define \( g_n(t, y, z) \) as in Lemma 2.4. From conditions (a), (b) and the properties of \( g_n \), one has \( g_n(t, y, z) > 0 \) for each \((t, y) \in [0, T] \times \mathbb{R} \) and \( g \geq g_n \). By the standard strict comparison theorem, one has \( y_0^{g,T} > 0 \) and \( y_0^{g,T} \geq y_0^{g_n,T} \), which yields \( y_0^{g,T} > 0 = y_0^{g,T,0} \), as required. \( \square \)
3.2. The representation theorem for the generator from the solutions of the associated BSDE

Besides the comparison theorem, the representation theorem for the generator from the solutions of the associated BSDE is also a very useful tool for studying the properties of BSDEs. The first theorem of this kind was proved by Briand, Coquet, Hu, Memin and Peng [1] for proving the converse comparison theorem. The authors of [1] proved it under a relatively strict condition. Recently Jiang [14] extended it to a more general situation. But all these representation theorems are based on the standard assumptions. For the case when $g$ satisfies (A1) and (A2), we will show it here.

**Theorem 3.4.** Let $g$ satisfy (A1) and (A2), and let $1 \leq p < 2$. Then for each pair $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the following equality:

$$g(t, y, z) = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} [y_{t+\varepsilon}^{g,-,y} + z(W_{t+\varepsilon} - W_t) - y]$$

holds for almost every $t \in [0, T]$.

**Proof.** Let $(t, y, z)$ be given. For simplicity we denote by $(y_s^e, z_s^e)_{s \in [0, t+\varepsilon]}$ the solution of the BSDE

$$y_s^e = y + z(W_{t+\varepsilon} - W_t) + \int_s^{t+\varepsilon} g(r, y_r^e, z_r^e)dr - \int_s^{t+\varepsilon} z_r^e dW_r, \quad s \in [0, t+\varepsilon].$$

For $s \in [t, t+\varepsilon]$, we define $\hat{y}_s^e := y_s^e - (y + z(W_s - W_t))$ and $\hat{z}_s^e := z_s^e - z$. Then, one has

$$\hat{y}_s^e = \int_s^{t+\varepsilon} g(r, \hat{y}_r^e + y + z(W_r - W_t), \hat{z}_r^e + z)dr - \int_s^{t+\varepsilon} \hat{z}_r^e dW_r.$$  

(4)

We first prove that the following statement:

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[ \sup_{s \in [t, t+\varepsilon]} |\hat{y}_s^e|^2 + \int_t^{t+\varepsilon} |\hat{z}_s^e|^2 ds \right] = 0$$

holds. We denote as $(y_{m}^{s,e}, z_{m}^{s,e})$ the solution of $(g_{m}, t + \varepsilon, y + z(W_{t+\varepsilon} - W_t))$, i.e.,

$$y_{m}^{s,e} = y + z(W_{t+\varepsilon} - W_t) + \int_s^{t+\varepsilon} g_{m}(r, y_{m}^{r,e}, z_{m}^{r,e})dr - \int_s^{t+\varepsilon} z_{m}^{r,e} dW_r, \quad s \in [0, t+\varepsilon].$$

(3)

For a given $m > C$ (defined in Lemma 2.4), it follows from Lemma 2.4, and Proposition 3.2 in [14], that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[ \sup_{s \in [t, t+\varepsilon]} |\hat{y}_{m}^{s,e}|^2 + \int_t^{t+\varepsilon} |\hat{z}_{m}^{s,e}|^2 ds \right] = 0,$$

(5)

where $\hat{y}_{m}^{s,e} := y_{m}^{s,e} - (y + z(W_s - W_t))$ and $\hat{z}_{m}^{s,e} := z_{m}^{s,e} - z$. One the other hand, by Lemma 2.6-(ii), one has

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[ \sup_{s \in [t, t+\varepsilon]} |\hat{y}_{m}^{s,e} - \hat{y}_s^e|^2 + \int_t^{t+\varepsilon} |\hat{z}_{m}^{s,e} - \hat{z}_s^e|^2 ds \right] = 0,$$

(6)

for a given $m > C$. Combining (6) with (7) yields (5).
Now define
\[
G^\varepsilon_t := \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} g(r, \tilde{y}^\varepsilon_r + y + z(W_r - W_t), \tilde{z}^\varepsilon_r + z)dr \bigg| \mathcal{F}_t \right],
\]
\[
H^\varepsilon_t := \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} g(r, y + z(W_r - W_t), z)dr \bigg| \mathcal{F}_t \right],
\]
\[
I^\varepsilon_t := \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} g(r, y, z)dr \bigg| \mathcal{F}_t \right].
\]
From Eq. (4), it follows that \(\frac{1}{\varepsilon}(y^\varepsilon_t - y) = \frac{1}{\varepsilon}\tilde{y}^\varepsilon_t = G^\varepsilon_t\). So
\[
\lim_{t \to 0^+} E[|G^\varepsilon_t - H^\varepsilon_t|^2] = 0.
\]
We now want to prove \(\lim_{t \to 0^+} E[|G^\varepsilon_t - H^\varepsilon_t|^2] = 0\). For any given \(m > C\), one has
\[
g(r, \tilde{y}^\varepsilon_r + y + z(W_r - W_t), \tilde{z}^\varepsilon_r + z) - g(r, y + z(W_r - W_t), z)
\]
\[
= g(r, \tilde{y}^\varepsilon_r + y + z(W_r - W_t), \tilde{z}^\varepsilon_r + z) - g_m(r, \tilde{y}^\varepsilon_r + y + z(W_r - W_t), \tilde{z}^\varepsilon_r + z)
\]
\[
+ g_m(r, \tilde{y}^\varepsilon_r + y + z(W_r - W_t), \tilde{z}^\varepsilon_r + z) - g_m(r, y + z(W_r - W_t), z)
\]
\[
+ g_m(r, y + z(W_r - W_t), z) - g(r, y + z(W_r - W_t), z) = \Delta^m_r + \psi^m_r,
\]
where \(\Delta^m_r = g_m(r, \tilde{y}^\varepsilon_r + y + z(W_r - W_t), \tilde{z}^\varepsilon_r + z) - g_m(r, y + z(W_r - W_t), z)\) and \(\psi^m_r\) stands for the remaining terms of the above equality. Clearly, \(|\Delta^m_r| \leq m(|\tilde{y}^\varepsilon_r| + |\tilde{z}^\varepsilon_r|)\) and \(|\psi^m_r| \leq 2\phi\left(\frac{2C}{m-C}\right)\).

In the light of the Lipschitz continuity of \(g_m\), Hölder’s inequality, Jensen’s inequality and (5), we deduce that for any given \(m > C\), one has
\[
E \left[ |G^\varepsilon_t - H^\varepsilon_t|^2 \right] = E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} (\Delta^m_r + \psi^m_r)dr \bigg| \mathcal{F}_t \right]^2
\]
\[
\leq \frac{1}{\varepsilon^2} E \left[ \int_t^{t+\varepsilon} (|\Delta^m_r| + |\psi^m_r|)dr \right]^2
\]
\[
\leq \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} (4m^2 (|\tilde{y}^\varepsilon_r|^2 + |\tilde{z}^\varepsilon_r|^2) + 2|\psi^m_r|^2)dr \right]
\]
\[
\leq \frac{4m^2}{\varepsilon} E \left[ \int_t^{t+\varepsilon} (|\tilde{y}^\varepsilon_r|^2 + |\tilde{z}^\varepsilon_r|^2)dr \right] + 2\phi^2 \left(\frac{2C}{m-C}\right).
\]
Thus \(\lim_{\varepsilon \to 0^+} E[|G^\varepsilon_t - H^\varepsilon_t|^2] \leq \lim_{\varepsilon \to 0^+} E[|G^\varepsilon_t - H^\varepsilon_t|^2] \leq 2\phi^2(\frac{2C}{m-C})\), as required.

By Jensen’s inequality, Hölder’s inequality and the Lipschitz continuity of \(g\) in \(y\), one has
\[
E \left[ |H^\varepsilon_t - I^\varepsilon_t|^2 \right] \leq \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} K^2 |z|^2 |W_r - W_t|^2 dr \right] \to 0, \quad \text{as } \varepsilon \to 0^+.
\]
In view of Lemma 1.3.4 in [15, p.9], we deduce that for \(1 \leq p < 2\),
\[
\lim_{\varepsilon \to 0^+} E \left[ |I^\varepsilon_t - g(t, y, z)|^p \right] = \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} E \left[ \int_t^{t+\varepsilon} g(r, y, z)dr \bigg| \mathcal{F}_t \right] - g(t, y, z) \right]^p = 0
\]
holds for almost every \(t \in [0, T]\). The proof is complete. \(\square\)

If \(g\) is deterministic, then we have:
Corollary 3.5. Let $g$ satisfy (A1) and (A2), and let $g$ be deterministic. Then:

(i) for each pair $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the following equality:

$$g(t, y, z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} [y_{t+\varepsilon, y+z(W_{t+\varepsilon}-W_t) - y}$$

holds for almost every $t \in [0, T]$;

(ii) furthermore, if $g$ also satisfies (A3), then for each pair $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the following equality:

$$g(t, y, z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} [y_{0}^{g(t+\varepsilon, y+z(W_{t+\varepsilon}-W_t)} - y]$$

holds for almost every $t \in [0, T]$.

Proof. (i) It is not hard to check that $y_{t}^{g, t+\varepsilon, y+z(W_{t+\varepsilon}-W_t)}$ is deterministic (see [23, Remark 4.2]), which implies that (i) holds true.

(ii) Using the same arguments as Proposition 3.1 in [1], we deduce

$$y_{t}^{g, t+\varepsilon, y+z(W_{t+\varepsilon}-W_t)} = y_{0}^{g, t+\varepsilon, y+z(W_{t+\varepsilon}-W_t)},$$

as required. $\square$

Remark 3.6. If $g(\cdot, y, z)$ is continuous for each $(y, z)$, we can see that Theorem 3.4 and Corollary 3.5 both hold for every $t \in [0, T]$.

At the end of this subsection, we must point out that the representation theorem for $g$ is very important for us in studying the properties of solutions of BSDEs, particularly for establishing equivalence relations between the properties of solutions and those of the corresponding $g$. Here is an example.

Example 3.7. Let $g$ satisfy (A1) and (A2), and let $g(\cdot, y, z)$ be continuous for each $(y, z)$. Then the following statements are equivalent:

(i) for almost every $t \in [0, T]$, $g(t, \cdot, \cdot, \cdot)$ is subadditive, i.e., for each $(y_i, z_i)$ $(i = 1, 2)$, one has $g(t, y_1 + y_2, z_1 + z_2) \leq g(t, y_1, z_1) + g(t, y_2, z_2)$, $dP \times dP$-a.e.;

(ii) for each $t \in [0, T]$, $y_{t}^{g, T}$, is subadditive, i.e., for each $X_i \in L^2(\mathcal{F}_T)$ $(i = 1, 2)$, one has $y_{t}^{g, T, X_1 + X_2} \leq y_{t}^{g, T, X_1} + y_{t}^{g, T, X_2}$, $P$-a.s.

3.3. The converse comparison theorem

The first converse comparison theorem of BSDEs was initiated by Briand, Coquet, Hu, Memin and Peng [1, 2000]. To be precise, the authors of [1] proved that if $g_i$ $(i = 1, 2)$ is Lipschitz continuous in $(y, z)$ and continuous in $t$, and satisfies (A3), and $(g_i(t, y, z))_{t \in [0, T]} \in L^2_{\mathbb{F}}(0, T)$ for each $(y, z)$, then the following statements are equivalent:

(i) $y_{t}^{g_1, T, X} \leq y_{t}^{g_2, T, X}$ for each $X \in L^2(\mathcal{F}_T)$;

(ii) $g_1(t, y, z) \leq g_2(t, y, z)$ for each $(t, y, z)$.

We now introduce our converse comparison theorem, which is just an application of Theorem 3.4.
Theorem 3.8. Let the assumptions (A1) and (A2) hold for \( g_i \) \((i = 1, 2)\), and let \( g_i(\cdot, y, z) \) be continuous for each \((y, z)\). Then the following statements are equivalent:

(i) \( y_{t}^{g_1, T, X} \leq y_{t}^{g_2, T, X}, \forall X \in L^2(\mathcal{F}_T), \forall t \in [0, T]; \)

(ii) \( g_1(t, y, z) \leq g_2(t, y, z), \) \( P \)-a.s., \( \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d. \)

By virtue of Corollary 3.5, we can obtain the following result.

Theorem 3.9. Let \( g_i \) \((i = 1, 2)\) satisfy (A1) and (A2) and be deterministic. Then the following statements are equivalent:

(i) for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d, g_1(t, y, z) \leq g_2(t, y, z), \) a.e.;

(ii) for each \( X \in L^2(\mathcal{F}_T) \) and \( t \in [0, T], y_{t}^{g_1, T, X} \leq y_{t}^{g_2, T, X}; \)

(iii) for each \( X \in L^2(\mathcal{F}_T), y_{0}^{g_1, T, X} \leq y_{0}^{g_2, T, X}. \)

Remark 3.10. Unlike for the standard BSDEs, we still cannot establish the converse comparison theorem, like Theorem 4.4 in [1], in which \( g \) is independent of \( y \) (but still stochastic). In my opinion, the main obstacle is that we don’t have a strict comparison theorem in our situation.

3.4. The continuous dependence theorem

In the case when \( g \) is a standard generator, i.e., \( g \) is Lipschitz continuous in \((y, z)\), the continuous dependence of solutions to associated BSDEs is an immediate consequence of Proposition 2.2 in [1]. In our situation, we don’t have this kind of inequality. But we still have the following continuous dependence theorem.

We now consider the following BSDEs:

\[
y_t^n = X^n + \int_t^T g_n(s, y_s^n, z_s^n)ds - \int_t^T z_s^n dW_s, \quad t \in [0, T]
\]

where \( n = 0, 1, 2, \ldots \). For each \( n, g_n : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) satisfies the following assumptions:

(H1) \[ \{g_n(t, y, z)\}_{t \in [0, T]} \in L^2_{\mathcal{F}}(0, T), \]
and there exists a constant \( A > 0 \) such that \( |g_n(t, y, z)| \leq A(1 + |y| + |z|), \forall n, t, y, z; \)

(H2) for each \( n, \) we have

\[ |g_n(t, y_1, z_1) - g_n(t, y_2, z_2)| \leq K |y_1 - y_2| + \phi_n(|z_1 - z_2|), \quad \forall (t, y_i, z_i), \quad i = 1, 2 \]

where \( K > 0 \) is a constant and \( \phi_n \) is a function depending on \( n \) and satisfying all properties of \( \phi \) in (A2);

(H3) \( g_n \) converges uniformly to \( g_0 \) with respect to \((t, y, z)\).

Theorem 3.11. Let \( g_n \) \((n = 0, 1, \ldots)\) satisfy (H1)–(H3), and let \( X_n \in L^2(\mathcal{F}_T) \) satisfying \( X^n \to X^0 \) in \( L^2(\mathcal{F}_T) \) as \( n \to \infty. \) Then for each \( t \in [0, T], \) one has

\[
\lim_{n \to \infty} E[ \sup_{t \in [0, T]} |y_{t}^{g_n, T, X^n} - y_{t}^{g_0, T, X^0}|^2 ] = 0.
\]

Proof. It is easy to check by Lemma 2.4 that for each \( m > C \) and \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, \) one has \( \lim_{n \to \infty} \overline{g_n}^m(t, y, z) = \overline{g_0}^m(t, y, z) \) (resp. \( \lim_{n \to \infty} \underline{g_n}^m(t, y, z) = \underline{g_0}^m(t, y, z), \) where
Lemma 2.6. Theorem 3.11. m is just one special case. In fact, we can prove that under more general assumptions, the uniqueness of the solution to \( (g_0, T, X) \) is equivalent to continuous dependence with respect to \( g_0 \) and \( X \) (see Jia and Yu [18]). Clearly, Theorem 3.11 is just one special case.

4. A new class of \( g \)-expectation and its properties

In what follows we first give the notion of our new \( g \)-expectation, and then discuss its properties.

For convenience, we denote Peng’s standard \( g \)-expectation by \( \mathcal{E}^g[\cdot] \). In the following definitions, we always assume that \( g \) satisfies (A1)–(A3).

Definition 4.1. The \( g \)-expectation \( \mathcal{E}^g[\cdot] : L^2(\mathcal{F}_T) \leftrightarrow \mathbb{R} \) is defined by

\[
\mathcal{E}^g[X] = y_0^{g,T,X}.
\]

The conditional \( g \)-expectation of \( X \) with respect to \( \mathcal{F}_t \) is defined by

\[
\mathcal{E}^g[X|\mathcal{F}_t] = y_t^{g,T,X}.
\]
The above definition is the same as Peng’s except that the Lipschitz continuity of \( g \) becomes (A2). But it is worth noting that a pretty important difference between the new \( g \)-expectation and the standard one is: unlike for standard \( g \)-expectation, for a given \( X \in L^2(\mathcal{F}_T) \), the random variable \( \eta \in L^2(\mathcal{F}_t) \) satisfying
\[
E^g[X1_A] = E^g[\eta 1_A], \quad \text{for all } A \in \mathcal{F}_t
\]
may be not unique, because the strict comparison theorem no longer holds. We can obtain this phenomenon from Example 3.2.

4.1. Some properties of \( g \)-expectation

First we illustrate the relationship between these two notions of \( g \)-expectation.

**Theorem 4.2.** Let \( E^g[\cdot|\mathcal{F}_t] \) be defined as before. Then:

(i) there exists a sequence of functions \( g_n : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) converging uniformly to \( g \) with respect to \((t, y, z)\); and for each \( n > C = \max \{L, K\}, g_n \) is a standard generator, i.e., \( g_n \) is Lipschitz continuous in \((y, z)\) and \( g_n(t, y, 0) \equiv 0 \) for each \((t, y)\);

(ii) for each \( X \in L^2(\mathcal{F}_T) \) and \( n > C \), the sequence of standard \( g \)-expectations \( \mathcal{E}^{g_n}[\cdot] \) defined by \( g_n \) satisfies
\[
E[|\mathcal{E}^{g_n}[X|\mathcal{F}_t] - E^g[X|\mathcal{F}_t]|] \leq H\phi \left( \frac{2C}{n - C} \right), \quad \forall t \in [0, T],
\]
where \( H \) is a constant only depending on \( g \) and \( T \);

(iii) for each \( X \in L^2(\mathcal{F}_T) \), one has
\[
\lim_{n \to \infty} E[|\mathcal{E}^{g_n}[X|\mathcal{F}_t] - E^g[X|\mathcal{F}_t]|^2] \to 0, \quad \forall t \in [0, T].
\]

**Proof.** (i) For any given \( n > C \) and \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \), we define
\[
g_n(t, y, z) = \inf_{v \in \mathbb{Q}^d} \{g^+(t, y, v) + n |z - v|\} - \inf_{v \in \mathbb{Q}^d} \{g^-(t, y, v) + n |z - v|\}
\]
where \( g = g^+ - g^- \) and \( g^+ = \max \{g, 0\} \geq 0, g^- = \max \{-g, 0\} \geq 0 \). Clearly, \( g^\pm(t, y, 0) \equiv 0 \) for each \((t, y) \in [0, T] \times \mathbb{R} \), and \( g^\pm \) satisfies (A1), (A2). It follows from Lemma 2.4 that for \((y_i, z_i) \in \mathbb{R} \times \mathbb{R}^d \) (\( i = 1, 2 \)), one has
\[
|g_n(t, y_1, z_1) - g_n(t, y_2, z_2)| \leq 2K |y_1 - y_2| + 2n |z_1 - z_2|, \quad dP \times dt \text{-a.s.}
\]
Again by Lemma 2.4, we deduce that for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d \) and \( n > C \), one has
\[
|g_n(t, y, z) - g(t, y, z)| \leq 2\phi \left( \frac{2C}{n - C} \right), \quad dP \times dt \text{-a.s.},
\]

namely \( g_n \) converges uniformly to \( g \) as \( n \to \infty \). Using the definition of \( g_n \), we can get immediately \( g_n(t, y, 0) \equiv 0 \) for each \((t, y)\).

(ii) By (i), it is clear that for a given \( n > C \), \( \mathcal{E}^{g_n}[\cdot] \) is a standard \( g \)-expectation. And it is not hard to deduce that
\[
\underline{g}_n(t, y, z) \leq g_n(t, y, z) \leq \overline{g}_n(t, y, z), \quad dP \times dt \text{-a.s., } \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d
\]
where \( \underline{g}_n \) and \( \overline{g}_n \) are defined as in Lemma 2.4.
For a given $X \in L^2(\mathcal{F}_T)$, it follows from the comparison theorem for BSDEs that for $t \in [0, T]$, one has

$$y_t^{g_n,T,X} \leq \mathcal{E}^{g_n}[X|\mathcal{F}_t] \leq y_t^{\tilde{g}_n,T,X} \quad \text{and} \quad y_t^{g_n,T,X} \leq \mathcal{E}^{g}[X|\mathcal{F}_t] \leq y_t^{\tilde{g}_n,T,X}. $$

Therefore by Lemma 2.6-(i), one has

$$E \left[|\mathcal{E}^{g_n}[X|\mathcal{F}_t] - \mathcal{E}^{g}[X|\mathcal{F}_t]|\right] \leq 2E[|y_t^{g_n,T,X} - \mathcal{E}^{g}[X|\mathcal{F}_t]|] + 2E[|y_t^{\tilde{g}_n,T,X} - \mathcal{E}^{g}[X|\mathcal{F}_t]|] \leq H\phi \left(\frac{2C}{n-C}\right), \quad \forall t \in [0, T]$$

where $H$ is a constant only depending on $g$ and $T$. 

(iii) It is a consequence of Theorem 3.11. The proof is complete. □

We now prove that as the standard $g$-expectation, our new $g$-expectation is an $\mathcal{F}_t$-consistent nonlinear expectation introduced and systematically studied by Coquet, Hu, Memin and Peng [8, 2002].

**Theorem 4.3.** The $g$-expectation defined by Definition 4.1 is an $\mathcal{F}_t$-consistent nonlinear expectation, i.e., for any $X, Y \in L^2(\mathcal{F}_T)$, one has the following properties:

(i) Monotonicity: $\mathcal{E}^{g}[X|\mathcal{F}_t] \leq \mathcal{E}^{g}[Y|\mathcal{F}_t]$, P-a.s., if $X \leq Y$, P-a.s.

(ii) Constant preservation: $\mathcal{E}^{g}[X|\mathcal{F}_t] = X$, P-a.s., if $X \in L^2(\mathcal{F}_t)$.

(iii) Consistency: $\mathcal{E}^{g}[\mathcal{E}^{g}[X|\mathcal{F}_t]|\mathcal{F}_t] = \mathcal{E}^{g}[X|\mathcal{F}_t]$, P-a.s., for $s \leq t \leq T$.

(iv) Adherence to the “0–1 law”: $\mathcal{E}^{g}[\mathbb{1}_A X|\mathcal{F}_t] = \mathbb{1}_A \mathcal{E}^{g}[X|\mathcal{F}_t]$, P-a.s., $\forall A \in \mathcal{F}_t$.

**Proof.** (i) can be obtained directly from the comparison theorem (Theorem 3.1). Using the uniqueness of the solution to $(g, T, X)$ defined on $[t, T]$, we can see that (ii) holds true.

(iii) In the light of Theorem 4.2, we can deduce that for each $X \in L^2(\mathcal{F}_T)$, one has

$$\mathcal{E}^{g_n}[X|\mathcal{F}_t] \to \mathcal{E}^{g}[X|\mathcal{F}_t], \quad \text{in } L^2 \text{ as } n \to \infty$$

where $g_n$ ($n > C$) is defined as in Theorem 4.2. On the other hand, from Theorem 4.2 and Lemma 36.6 in [23], it follows that for $n > C$, one has

$$\mathcal{E}^{g_n}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}^{g_n}[X|\mathcal{F}_s].$$

The above equation with Theorem 3.11 yields (iii) on putting $n \to \infty$.

(iv) We multiply BSDE $(g, T, X)$ defined on $[t, T]$ by $1_A$, $A \in \mathcal{F}_t$. By virtue of $g(r, y, 0) \equiv 0$ for each $(r, y)$, we have

$$y_s^{g,T,X} 1_A = X 1_A + \int_s^T 1_A g(r, y_r^{g,T,X}, z_r^{g,T,X}) \, dr - \int_s^T 1_A z_r^{g,T,X} \, dW_r$$

where $g_n$ ($n > C$) is defined as in Theorem 4.2. On the other hand, from Theorem 4.2 and Lemma 36.6 in [23], it follows that for $n > C$, one has

$$\mathcal{E}^{g_n}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}^{g_n}[X|\mathcal{F}_s].$$

The above equation with Theorem 3.11 yields (iii) on putting $n \to \infty$.

We will always write in the sequel $\mathcal{E}^\phi$ for $g = \phi(|z|)$ and $\mathcal{E}^{-\phi}$ for $g = -\phi(|z|)$.

**Proposition 4.4.** For $\mathcal{E}^{g}[\cdot]$ with the generator satisfying (A1)–(A3), the following statements are equivalent:

(i) $\mathcal{E}^{-\phi}[X-Y|\mathcal{F}_t] \leq \mathcal{E}^{g}[X|\mathcal{F}_t] - \mathcal{E}^{g}[Y|\mathcal{F}_t] \leq \mathcal{E}^\phi[X-Y|\mathcal{F}_t], \forall X, Y \in L^2(\mathcal{F}_T), \forall t \in [0, T]$;
(ii) \( g \) is independent of \( y \);
(iii) \( \mathbb{E}^g[X + \eta|\mathcal{F}_t] = \mathbb{E}^g[X|\mathcal{F}_t] + \eta, \forall X \in L^2(\mathcal{F}_T) \) and \( \eta \in L^2(\mathcal{F}_t) \).

**Proof.** The readers can refer to [23,1] for the proof of (ii)\( \Leftrightarrow \) (iii). We now only prove (ii)\( \Rightarrow \) (i) and (i)\( \Rightarrow \) (iii).

(ii)\( \Rightarrow \) (i): For given \( X, Y \in L^2(\mathcal{F}_T) \), one has

\[
\mathbb{E}^g[X|\mathcal{F}_t] - \mathbb{E}^g[X|\mathcal{F}_t] = X - Y + \int_t^T (g(s, z^g,T,X) - g(s, z^g,T,Y)) \, ds - \int_t^T (z^g,T,X - z^g,T,Y) \, dW_s.
\]

Note that \( -\phi(|z^g,T,X - z^g,T,Y|) \leq g(s, z^g,T,X) - g(s, z^g,T,Y) \leq \phi(|z^g,T,X - z^g,T,Y|) \). By the comparison theorem, we get (i).

(i)\( \Rightarrow \) (iii): For given \( X \in L^2(\mathcal{F}_T) \) and \( \eta \in L^2(\mathcal{F}_t) \), one has

\[
\mathbb{E}^{-\phi}[\eta|\mathcal{F}_t] \leq \mathbb{E}^g[X + \eta|\mathcal{F}_t] - \mathbb{E}^g[X|\mathcal{F}_t] \leq \mathbb{E}^\phi[\eta|\mathcal{F}_t].
\]

From Theorem 4.3-(ii), it follows that \( \mathbb{E}^{-\phi}[\eta|\mathcal{F}_t] = \mathbb{E}^\phi[\eta|\mathcal{F}_t] = \eta \), as required. The proof is complete. \( \square \)

**Remark 4.5.** For \( g \)-expectation with the generator satisfying (A1)–(A3) and \( X_i \in L^2(\mathcal{F}_T) \) \( (i = 1, 2) \), we always have

\[
\mathbb{E}^{-\phi,-K}[X_1 - X_2|\mathcal{F}_t] \leq \mathbb{E}^g[X_1|\mathcal{F}_t] - \mathbb{E}^g[X_2|\mathcal{F}_t] \leq \mathbb{E}^\phi,K[X_1 - X_2|\mathcal{F}_t], \quad \forall t,
\]

where \( \mathbb{E}^\phi,K[\cdot] \) stands for \( g \)-expectation with the generator \( K|y| + \phi(|z|) \), and \( \mathbb{E}^{-\phi,-K}[\cdot] \) stands for \( g \)-expectation with the generator \( -K|y| - \phi(|z|) \).

**Remark 4.6.** So far we still cannot establish a result like Theorem 7.1 in [8] or Theorem 6.3 in [12].

Using the continuous dependence theorem and classical convergence theorem in \( L^2 \)-space, we can easily deduce the following convergence theorem.

**Theorem 4.7.** Let \( X_n \in L^2(\mathcal{F}_T) \) \( (n = 1, 2, \ldots) \) and \( E[X^2] < \infty, E[Z^2] < \infty \). Then:

(i) if \( X_n \uparrow X \) (or \( X_n \downarrow X \)), \( P \)-a.s., then for \( t \in [0, T] \), one has

\[
\lim_{n \to \infty} \mathbb{E}^g[X_n|\mathcal{F}_t] = \mathbb{E}^g[ \lim_{n \to \infty} X_n|\mathcal{F}_t] = \mathbb{E}^g[X|\mathcal{F}_t], \quad P \text{-a.s.};
\]

(ii) if \( X_n \geq Z, \ P \text{-a.s.}, \) and \( \lim_{n \to \infty} X_n \in L^2(\mathcal{F}_T), \) then for \( t \in [0, T] \), one has

\[
\mathbb{E}^g[ \lim_{n \to \infty} X_n|\mathcal{F}_t] \leq \lim_{n \to \infty} \mathbb{E}^g[X_n|\mathcal{F}_t], \quad P \text{-a.s.};
\]

(iii) if \( X_n \leq Z, \ P \text{-a.s.}, \) and \( \lim_{n \to \infty} X_n \in L^2(\mathcal{F}_T), \) then for \( t \in [0, T] \), one has

\[
\mathbb{E}^g[ \lim_{n \to \infty} X_n|\mathcal{F}_t] \geq \lim_{n \to \infty} \mathbb{E}^g[X_n|\mathcal{F}_t], \quad P \text{-a.s.};
\]

(v) if \( |X_n| \leq Z, \ P \text{-a.s.}, \) and \( \lim_{n \to \infty} X_n = X \), then for \( t \in [0, T] \), one has

\[
\lim_{n \to \infty} \mathbb{E}^g[X_n|\mathcal{F}_t] = \mathbb{E}^g[ \lim_{n \to \infty} X_n|\mathcal{F}_t] = \mathbb{E}^g[X|\mathcal{F}_t], \quad P \text{-a.s.}
\]
In [5], Chen proved that a standard \( g \)-expectation \( \mathbb{E}^g[\cdot] \) can be extended continuously to \( \mathcal{L}^1(\Omega, \mathcal{F}_T, P) \), where
\[
\mathcal{L}^1(\Omega, \mathcal{F}_T, P) = \bigcup_{p>1} L^p(\Omega, \mathcal{F}_T, P).
\]

For our new \( g \)-expectation, we also have the same result.

**Proposition 4.8.** \( \mathbb{E}^g[\cdot] \) as a nonlinear operator defined on \( L^2(\mathcal{F}_T) \) can be extended continuously to \( \mathcal{L}^1(\Omega, \mathcal{F}_T, P) \).

**Proof.** Suppose that \( g_n \) is defined as in Theorem 4.2 for large positive integer \( n > C \). Note that \( g_n \) satisfies all conditions required in [5].

For each \( X \in \mathcal{L}^1(\Omega, \mathcal{F}_T, P) \), there exists \( p > 1 \) such that \( X \in L^p(\Omega, \mathcal{F}_T, P) \). Set \( X_k = X 1_{\{|X| \leq k\}}, k = 1, 2, \ldots \). Then for each \( k \geq 1 \), \( \mathbb{E}^g[X_k] \) and \( \mathbb{E}^{g_n}[X_k] \) are well-defined, and for each \( n \), the limit of \( \lim_{k \to \infty} \mathcal{E}^{g_n}[X_k] \) exists. Define \( \mathcal{E}^{g_n}[X] = \lim_{k \to \infty} \mathcal{E}^{g_n}[X_k] \).

For any large \( n, l > C \), we have
\[
|\mathcal{E}^{g_{n+l}}[X] - \mathcal{E}^{g_n}[X]| = \left| \lim_{k \to \infty} (\mathcal{E}^{g_{n+l}}[X_k] - \mathcal{E}^{g_n}[X_k]) \right| 
\leq \lim_{k \to \infty} |\mathcal{E}^{g_{n+l}}[X_k] - \mathbb{E}^g[X_k]| + \lim_{k \to \infty} |\mathcal{E}^{g_n}[X_k] - \mathbb{E}^g[X_k]| 
\leq H\phi\left(\frac{2C}{n + l - C}\right) + H\phi\left(\frac{2C}{n - C}\right) \to 0, \quad \text{as } n \to \infty.
\]

This implies that the limit of \( \mathbb{E}^{g_n}[X] \) exists; thus we can define \( \mathbb{E}^g[X] = \lim_{n \to \infty} \mathcal{E}^{g_n}[X] \), which is a nonlinear operator defined on \( \mathcal{L}^1(\Omega, \mathcal{F}_T, P) \). The proof is complete. \( \square \)

The corresponding converse comparison theorems for \( \mathbb{E}^g[\cdot] \) can be obtained by Theorems 3.8 and 3.9.

**4.2. A remark on Jensen’s inequality for \( g \)-expectation**

Jensen’s inequality is a fundamental result in probability theory. Because of its extensive applications, the problem of whether Jensen’s inequality for \( g \)-expectation holds or not is brought forward naturally. A counterexample due to Briand, Coquet, Hu, Memin and Peng [1] tells us that Jensen’s inequalities for standard \( g \)-expectations need not hold any longer even if \( g \) is pretty simple. Motivated by this counterexample, Chen, Kulperger and Jiang [7] and Jiang and Chen [16] proved an interesting result: let \( g \) be a standard generator and independent of \( y \); then the following statements are equivalent:

(i) Jensen’s inequality for the corresponding \( g \)-expectation holds, i.e., for any convex function \( f \) and \( X \in L^2(\mathcal{F}_T) \) satisfying \( f(X) \in L^2(\mathcal{F}_T) \), one has
\[
\mathcal{E}^g[f(X)|\mathcal{F}_t] \geq f(\mathcal{E}^g[X|\mathcal{F}_t]), \quad P\text{-a.s.}, \quad \forall t \in [0, T];
\]

(ii) \( g \) is superhomogeneous, i.e., for any \( \lambda \in \mathbb{R} \),
\[
\forall (t, z) \in [0, T] \times \mathbb{R}^d, \quad g(t, \lambda z) \geq \lambda g(t, z), \quad P\text{-a.s.}
\]

Recently, Jiang [15] and Hu [11] independently generalized this result, to the case in which the assumption that \( g \) is independent of \( y \) is removed. In other words, if Jensen’s inequality for \( g \)-expectation holds true, then \( g \) must be independent of \( y \). We naturally want to ask: what happens to our new \( g \)-expectation?
For the sake of simplicity, we only consider the following case: \( g \) satisfies (A1)–(A3) and is continuous in \( t \) for each \( (y, z) \).

For any given \( \lambda, b \in \mathbb{R} \), set \( f(x) = \lambda x + b \), which is a convex function. Thanks to Theorem 3.4, (i) for \( \mathbb{E}^{\phi}[\cdot] \) implies that for any \( (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \), one has

\[
g(t, \lambda y + b, \lambda z) \geq \lambda g(t, y, z).
\]

This yields that \( g \) satisfies (ii).

On the other hand, for any \( \lambda \geq 0 \), one has \( g(t, \lambda z) = \lambda g(t, z), \forall (t, z) \in [0, T] \times \mathbb{R}^d \). Thus it follows from Remark 2.1 that, for \( \lambda > 0 \) and \( x \in \mathbb{R}_+ \), one has

\[
\phi(\lambda x) = \sup \left\{ \left| g(t, z_1) - g(t, z_2) \right| : z_1, z_2 \in \mathbb{R}^d, |z_1 - z_2| \leq \lambda x \right\} \\
= \sup \left\{ \left| g(t, z_1) - g(t, z_2) \right| : z_1, z_2 \in \mathbb{R}^d, \frac{|z_1 - z_2|}{\lambda} \leq x \right\} \\
= \sup \left\{ |g(t, \lambda z_1) - g(t, \lambda z_2)| : z_1, z_2 \in \mathbb{R}^d, |z_1 - z_2| \leq x \right\} \\
= \sup \left\{ \lambda \left| g(t, z_1) - g(t, z_2) \right| : z_1, z_2 \in \mathbb{R}^d, |z_1 - z_2| \leq x \right\} = \lambda \phi(x).
\]

This yields \( \phi(x) = \phi(1)x \) for \( x \in \mathbb{R}_+ \), that is, \( g \) is Lipschitz continuous with respect to \( z \). In other words, if Jensen’s inequality for our new \( g \)-expectation holds true, then \( g \) must be Lipschitz continuous in \( z \).

**Theorem 4.9.** For \( \mathbb{E}^{\phi}[\cdot] \) with the generator satisfying (A1)–(A4), the following statements are equivalent:

(i) Jensen’s inequality for \( \mathbb{E}^{\phi}[\cdot] \) holds true, i.e., for any convex function \( f \) and \( X \in L^2(\mathcal{F}_T) \) satisfying \( f(X) \in L^2(\mathcal{F}_T) \), one has

\[
\mathbb{E}^{\phi}[f(X)|\mathcal{F}_t] \geq f(\mathbb{E}^{\phi}[X|\mathcal{F}_t]), \quad P\text{-a.s., } \forall t \in [0, T];
\]

(ii) \( g \) is independent of \( y \), Lipschitz continuous and superhomogeneous in \( z \).

**Remark 4.10.** As a matter of fact, this phenomenon is also one of the motivations for us to introduce a new concept: \( g \)-convex functions (see [17]).

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**References**


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