THE EILENBERG–MOORE SPECTRAL SEQUENCE AND STRONGLY HOMOTOPY MULTIPLICATIVE MAPS

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0. Introduction

0.1. The main purpose of this paper is to prove the following collapse result for the Eilenberg–Moore (cohomology) spectral sequence for the fibre square

\[
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow & & \downarrow p \\
X & \longrightarrow & B
\end{array}
\]

with $\pi_1(B)$ acting trivially on the cohomology of the fibre of $p$.

Theorem. Assume that $H^* E$, $H^* B$, $H^* X$ are polynomial algebras in at most countably many variables. If the ground ring, which is supposed to be a principal ideal domain, has characteristic 2, assume further that $\text{Sq}_1$ vanishes on $H^* X$ and on $H^* E$. Then:
(1) \( H^* E' \cong \text{Tor}_{H^* B}(H^* X, H^* E) \) as graded modules;
(2) the Eilenberg–Moore spectral sequence collapses;
(3) \( E^0 H^*(E') \cong \text{Tor}_{H^* B}(H^* X, H^* E) \) as bigraded algebras.

In [7], Hirsch conjectured that when \( B, X \) are generalized Eilenberg–MacLane spaces, while \( E \) is contractible, then (with \( \mathbb{Z}_2 \) as ground ring) the spectral sequence collapses. In [12], Schochet gave a counterexample to this conjecture. The theorem above gives a positive answer to the conjecture in case \( \text{Sq}_1 = 0 \) on \( H^* X \), and \( X \) and \( B \) are simply connected with finitely generated homotopy groups (cf. [13]).

Collapse results for the spectral sequence of the fibre square

\[
\begin{array}{ccc}
G/H & \to & BH \\
\downarrow & & \downarrow \\
* & \to & BG,
\end{array}
\]

where \( H, G \) are (Lie-)groups with various other restrictions, have been obtained by various authors. Our theorem contains most of the collapse results in this direction that we are aware of, especially those of Baum [1, Theorem 7.4(i)] and Gugenheim and May [5, Theorem 4.3, Theorems A, B]. For [5, Theorem 4.3], we remark that our theorem applies to \( BT^n \) directly. One must then extend just like Gugenheim and May do in [5], following Baum [1].

0.2. If \( C^* E, C^* B, C^* X \) were commutative, then for the polynomial algebras \( H^* E, H^* B, H^* X \) we would get a commutative diagram

\[
\begin{array}{ccc}
H^* X & \leftarrow & H^* B & \longrightarrow & H^* E \\
\downarrow & & \downarrow & & \downarrow \\
C^* X & \leftarrow & C^* B & \longrightarrow & C^* E
\end{array}
\]

of algebra maps with the vertical maps including identities on homology. It would then follow that \( E^2 = E^\infty \) in the spectral sequence. For very nice spaces, one may choose cochains that are commutative (this was used by Baum and Smith [2]), but in general this is impossible. Now the main idea of the present paper is to extend the category \( \text{DA} \) of differential graded algebras to a bigger category \( \text{DASH} \) having enough morphisms to ensure that there is a morphism \( H^* X \to C^* X \) which induces the identity as homology (whenever \( H^* X \) is a polynomial algebra). This was done by Halperin and Stasheff [15]. However, only in very special cases could they prove that (1) commutes; so they got few actual collapse results from their work. In general, (1) does not commute. For this reason we introduce a homotopy notion in \( \text{DASH} \), and we prove that under the assumptions of the theorem we have:

(A) (1) does homotopy commute.

This is one of the 3 basic steps in the proof of the theorem above. The other two are:
(B) Tor and the Eilenberg–Moore spectral sequence, viewed as functors on the
category of diagrams $C' \rightarrow A \rightarrow B$ in DA extend over the similar category based on
DASH.

(C) If the diagram

$$
\begin{array}{ccc}
C' & \rightarrow & A \\
\downarrow & & \downarrow \\
C' & \rightarrow & A' \rightarrow B'
\end{array}
$$

homotopy commutes in DASH and all the vertical arrows are homology isomorphisms,
then the Eilenberg–Moore spectral sequences for the two rows are isomorphic.

0.3. (B) is essentially contained in [15]. Another proof of (B) together with a proof
of (C), based on the approach to differential homological algebra given by Gugenheim
(see [5]), was developed by Gugenheim and the present author (unpublished). [10],
which should be considered a preliminary and primitive version of the present paper,
also contains proofs of (variants of) (B) and (C). In the present author’s joint paper
with Gugenheim [6], we prove (B) and (C) for the relative theories; the proofs are
essentially due to Gugenheim (though they grew out of our joint efforts to under-
stand the earlier proofs). In this paper we prove (B) and (C) for the absolute theo-
ries. Our proofs (given in 5.3 and 5.4) are based on the fact that DASH (which ex-
tends DA) can also be viewed as a subcategory of DA (see 3.8), so that the exten-
sions of Tor and the spectral sequence are obtained simply as restrictions.

0.4. The proof of (A) is given in 7.4. It is the most difficult part of the paper insofar
as it depends on a rather detailed knowledge of DASH. Especially we have had to
extend $\otimes$ over DASH. As noted in [15], this is possible. However, the extension is
not functorial in both variables simultaneously. This is the main source of troubles.
We get around it by proving that up to homotopy the extended $\otimes$-product is fully
functorial.

0.5. Thus from a technical point of view the main bulk of the paper centers around
the study of DASH with its $\otimes$-product and its homotopy notion.

Section 1 is essentially a standard introduction to the duality of the bar and co-
bar construction and their relation to twisting cochains. It owes much to [8]. We
allow our (co-)algebras to be $\mathbb{Z}$-graded, and we replace the usual simply connected-
ness hypotheses by the cocompleteness of coalgebras introduced in [61]. Also we
study how homotopies of twisting cochains fit with the natural homotopy notions
for (co-)algebras.

Section 2 introduces the category TEX of trivialized algebra extensions (dual to
the E-Z data of [6]). TEX has pullbacks, tensor products and compositions. The
dual of the Eilenberg–Zilber theorem is stated as the existence of a certain functor
Recollections about the bar and cobar constructions and twisting cochains

1.1. Throughout this paper, $R$ is a fixed commutative principal ideal domain (with unit). All modules are unitary $R$-modules. Differential graded modules are $\mathbb{Z}$-graded, where $\mathbb{Z}$ is the integers, and their differentials have degree $-1$. Given two such differential graded modules $M$ and $N$, a map $f : M \to N$ of degree $k$ consists of $R$-homomorphisms $f_n : M_n \to N_{n+k}$ for all $n$. The differential of $f$ is

$$D(f) = df - (-1)^k fd_M.$$ 

The category of differential graded modules is called $\text{DM}$. Its morphisms are the maps $f$ of degree 0 having $Df = 0$. $M$ is the full subcategory of objects with differential 0.

1.2. $\text{DA}$ is the category of augmented, differential graded algebras. If $A \in \text{DA}$, the structure maps $\phi_A : A \otimes A \to A$, $\eta_A : R \to A$, $\varepsilon_A : A \to R$, $d_A : A \to A$ satisfy the...
usual requirements. The morphisms of $\mathcal{D}A$ preserve all the structure. $A$ is the full subcategory of objects with differential 0.

1.3. A coaugmented differential graded coalgebra $C$ is the usual thing, i.e. $C \in \mathcal{D}M$, and we have morphisms

$$
\Delta : C \rightarrow C \otimes C, \quad \eta : R \rightarrow C, \quad \epsilon : C \rightarrow R
$$

in $\mathcal{D}M$ with standard identities. The exact sequence

$$
0 \rightarrow R \xrightarrow{\eta} C \xrightarrow{p} JC \rightarrow 0
$$

in $\mathcal{D}M$ defines the functor $J$ into $\mathcal{D}M$. $\Delta$ induces $J\Delta : JC \rightarrow JC \otimes JC$, with

$$(p \otimes p)\Delta = (J\Delta)p. \quad J\Delta \text{ is coassociative, just like } \Delta.
$$

The iterates $\Delta^{(n)} : C \rightarrow C^{\otimes n}$ are defined by

$$
\Delta^{(0)} = \epsilon, \quad \Delta^{(1)} = C, \quad \Delta^{(2)} = \Delta, \quad \Delta^{(n)} = (\Delta^{(n-1)} \otimes C) \Delta, \quad n > 1.
$$

Similarly one has $(J\Delta)^{(n)}$ and $p^{\otimes n} \Delta^{(n)} = (J\Delta)^{(n)} p$.

There is a filtration

$$
C_{[0]} \subseteq C_{[1]} \subseteq \ldots \subseteq C_{[n]} \subseteq \ldots
$$

with

$$
C_{[n]} = \ker(p^{\otimes n+1} \Delta^{(n+1)} = (J\Delta)^{(n+1)} p : C \rightarrow (JC)^{\otimes n+1}).
$$

We note that $C_{[0]} = \text{Im } \eta$, and that $C_{[n]} \subseteq C_{[n+1]}$ because

$$(J\Delta)^{(n+1)} = (J \Delta \otimes (JC)^{\otimes n-1})(J\Delta)^{(n)}, \quad n > 0,$$

while $C_{[0]} \subseteq C_{[1]}$ is completely trivial. Also notice that in general one does not have

$$
\Delta(C_{[n]}) \subseteq \text{Im}(\Sigma_{i=0}^{n} C_{[i]} \otimes C_{[n-i]} \rightarrow C \otimes C).
$$

In fact, if $C$ is generated by $1, t$ of degree 0, with $4t = 0$ (over the integers) and

$$
\Delta(1) = 1 \otimes 1, \quad \Delta(t) = 1 \otimes t + 2t \otimes t + t \otimes 1,
$$

then $2t \in C_{[1]}$, $t \in C_{[2]}$, but the image of

$$
C_{[0]} \otimes C_{[2]} \oplus C_{[1]} \otimes C_{[1]} \oplus C_{[1]} \oplus C_{[2]} \oplus C_{[0]}
$$

in $C \otimes C$ is generated by $1 \otimes 1, 1 \otimes t, t \otimes 1$, so it does not contain $\Delta t$.

We shall require of all our coalgebras that $C = \bigcup C_{[n]}$, i.e., $C$ is cocomplete in the above filtration. Together with morphisms that preserve all the structure we then have the category $\mathcal{D}C$.

1.4. The exact sequence

$$
0 \rightarrow IA \xrightarrow{i} A \xrightarrow{\epsilon} R \rightarrow 0
$$
in $\text{DM}$ defines $l : DA \to \text{DM}$. $\phi$ induces $l \phi : IA \otimes IA \to IA$, with $i(l \phi) = \phi(i \otimes i)$.
The sequence splits to the right by $\eta$. The corresponding left splitting will be denoted $q : A \to IA$. Similarly we let $j : JC \to C$ be the right splitting of $0 \to R \to C \to JC \to 0$
corresponding to the left splitting $\epsilon : C \to R$.

1.5. The suspension functor $s : \text{DM} \to \text{DM}$ has $(sM)_n = M_{n-1}$ and $d_{sM} = -d_M$. We
have a map $\sigma : M \to sM$ of degree 1 (which is the identity in each degree) and
$Do = 0$. If $f : M \to N$ has degree $k$, then so does $sf : sM \to sN$, and

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
| & \downarrow \sigma & \downarrow \sigma \\
\downarrow s & & \downarrow \downarrow \downarrow s \\
sM & \xrightarrow{sf} & sN
\end{array}
$$

commutes in the graded sense, i.e.,

$$(sf) \sigma = (-1)^k \sigma f.$$  

Thus $d_{sM} = s(d_M)$.

1.6. We recall the definition of the reduced, normalized bar construction as a functor
$B : DA \to DC$. As a graded module,

$$BA = \bigcap_{j \geq 0} (s(IA))^\otimes j,$$

with inclusions and (restrictions of) projections

$$i_j : (s(IA))^\otimes j \to BA,$$

$$p_j : BA \to (s(IA))^\otimes j.$$  

We have

$$(p_j \otimes p_k) \Delta = p_{j+k}, \quad \eta = i_0, \quad \epsilon = (\eta).$$

By this definition, $\Delta$ maps into $\bigcap_{j+k(s(IA))^\otimes j \otimes (s(IA))^\otimes k}$, however, $p_{j+k}(\alpha) = 0$
for $j + k$ big, for any $\alpha \in BA$, so $\Delta(\alpha)$ actually belongs to the submodule $BA \otimes BA$.
The differential on $BA$ is

$$d = d_{\otimes} + d_\sigma,$$

where $d_{\otimes}$ is the coproduct of the differentials on $(s(IA))^\otimes j$, while $d_\sigma$ is the unique
derivation of $(BA, \Delta)$ (cf. [8, 6]) for which

$$p_1 d_\sigma = \sigma(l \phi) (\sigma^{-1} \otimes \sigma^{-1}) p_2.$$
One easily gets the following formulas for $d_\phi$

$$p_j d_\phi = \sum_{\nu=1}^{j} ((s|A)^{\otimes \nu} \otimes \sigma(I\phi) (\sigma^{-1} \otimes \sigma^{-1}) \otimes (s|A)^{(j-\nu)}) p_{j+1},$$

$$d_\phi i_j = \sum_{\nu=1}^{j-1} i_{j-1} ((s|A)^{(j-\nu)} \otimes \sigma(I\phi) (\sigma^{-1} \otimes \sigma^{-1}) \otimes (s|A)^{(j-\nu)}).$$

$d_\otimes$ and $d_\phi$, and hence $d$, are derivations with respect to $\Delta$. Also,

$$d_\otimes^2 = d_\phi^2 = d_\otimes d_\phi + d_\phi d_\otimes = 0.$$

If $a_i \in I|A$, let

$$[a_1 \ldots \ l_n] = i_n(oa_1 \otimes \ldots \otimes o_a_n), \quad n \geq 0.$$  

Then one recovers the usual formulas

$$\Delta [a_1 \ldots \ l_n] = \sum_{\nu=0}^{n} [a_1 \ldots \ l_{\nu}] \otimes [a_{\nu+1} \ldots \ l_n],$$

$$\eta(1) = 1,$$

$$\epsilon [a_1 \ldots \ l_n] = \begin{cases} 0 & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

$$d[a_1 \ldots \ l_n] = - \sum_{\nu=0}^{n} [-1] a_1 \ldots \ l_{\nu-1} da_\nu a_{\nu+1} \ldots \ l_n + \sum_{\nu=0}^{n} [\bar{a}_1 \ldots \ l_{\nu-1} \bar{a}_\nu a_{\nu+1} \ldots \ l_n],$$

where $\bar{a} = (-1)^{\deg(a)+1} a.$

As for co-compactness of $BA$, we note that

$$(BA)_{[n]} = \prod_{j \leq n} (s|A)^{(\otimes j)}.$$  

1.7 Dually the reduced, normalized cobbar construction is the functor $\Omega : DC \to DA$ described as follows: As a graded module,

$$\Omega C = \prod_{k \geq 0} (s^{-1}(JC))^{\otimes k}$$

with inclusions and (restrictions of) projections

$$i_k : (s^{-1}(JC))^{\otimes k} \to \Omega C, \quad q_k : \Omega C \to (s^{-1}(JC))^{\otimes k}.$$
In terms of these one has
\[ \phi(i_i \circ i_j) = i_{i \cdot j}, \quad \eta = j_0, \quad \epsilon = q_1, \]
where \(d_\phi\) is the coproduct of the differentials on \((s^{-1}(JC))^\otimes k\) while \(d_\Delta\) is the unique derivation of \((\Omega C, \phi)\) (cf. \(8, 6\)) with
\[ d_\Delta i_1 = i_2(\sigma^{-1} \otimes \sigma^{-1})(J \Delta) \sigma. \]
Explicit formulas are
\[ d_\Delta i_k = \sum_{\nu=1}^k j_{k+1}((s^{-1}(JC))^{\otimes \nu-1} \otimes (\sigma^{-1} \otimes \sigma^{-1})(J \Delta) \sigma \otimes (s^{-1}(JC))^{\otimes k-\nu}) \]
\[ q_k d_\Delta = \sum_{\nu=1}^k ((s^{-1}(JC))^{\otimes \nu-1} \otimes (\sigma^{-1} \otimes \sigma^{-1})(J \Delta) \sigma \otimes (s^{-1}(JC))^{\otimes k-\nu}) q_{k-1}. \]
\(d_\Delta\) and \(d_\Sigma\) and hence \(d\) are derivations with respect to \(\phi\). Also,
\[ d_\Sigma^2 = d_\Sigma d_\Delta + d_\Delta d_\Sigma = 0. \]
If \(c_i \in JC\), write
\[ \langle c_1 | \ldots | c_n \rangle = i_n(\sigma^{-1} c_1 \otimes \ldots \otimes \sigma^{-1} c_n), \quad n > 0. \]
Then
\[ \phi(\langle c_1 | \ldots | c_n \rangle \otimes \langle c_{n+1} | \ldots | c_{n+m} \rangle) = \langle c_1 | \ldots | c_{n+m} \rangle. \]
\[ \eta(1) = \langle \rangle. \]
\[ \epsilon(c_1 | \ldots | c_n) = \begin{cases} 0 & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases} \]
And if \((J \Delta)(c_i) = \sum c_j' \otimes c_j''\), then
\[ d(\langle c_1 | \ldots | c_n \rangle) = - \sum_i (c_1 | \ldots | c_i-1 | d c_i | c_{i+1} | \ldots | c_n) \]
\[ - \sum_{i,j} \langle c_1 | \ldots | c_i-1 | c'_i | c''_i | c_{i+1} | \ldots | c_n \rangle. \]

1.8. Recall that the functor \(\text{Hom} : \text{DM}^{op} \times \text{DM} \to \text{DM}\) has \(\text{Hom}_n(M, N) = \) the set of maps \(f : M \to N\) of degree \(n\) and differential \(D\) as in 1.1. It lifts to a functor \(\text{Hom} : \text{DC}^{op} \times \text{DA} \to \text{DA}\) when one defines the product \(\cup\) in \(\text{Hom}(C, A)\) by
\[ f \cup g = \phi(f \otimes g) \Delta. \]
For augmentation and unit we have
\[ \epsilon(f) = \epsilon_A f \eta_C(1), \quad \eta(1) = \eta_A \epsilon_C : C \to A. \]

Now the twisting cochain functor \( T : DC^{op} \times DA \to ENS \) is given by
\[ T(C, A) = \{ t \in \operatorname{Hom}_{-1}(C, A) \mid Dt = t \cup t, \ t\eta = 0 = \epsilon t \}. \]

One easily checks that
\[ t^A = t \circ p_1 : BA \to IA \to A, \]
\[ t_C = j_1 \circ p : C \to JC \to s \circ JC \to \Omega C, \]
give elements \( t^A \in T(BA, A), t_C \in T(C, \Omega C) \). They are natural in the sense that for any \( g \in DA(A, B) \) or \( f \in DC(C, D) \) one has
\[ g t^A = t^B B(g), \quad \Omega(f) t_C = t_D f. \]

Furthermore, they are universal in the sense of:

1.9. Proposition. There are natural bijections
\[ \beta : T(C, A) \to DC(C, BA), \quad \omega : T(C, A) \to DA(\Omega C, A), \quad C \in DC, \ A \in DA, \]
with inverses \( t^A \) and \( t_C \), respectively.

Proof. It is obvious that \( t^A \) is a natural transformation \( DC(C, B(A)) \to T(C, A) \). To define an inverse \( \beta \) for \( t^A \), we note that if \( t^A(\beta(t)) = t \in T(C, A) \) with \( \beta(t) \in DC(C, BA) \), then
\[ p_0 \beta(t) = \epsilon, \quad C \to R, \]
\[ p_1 \beta(t) = \sigma q i \sigma^{-1} p_1 \beta(t) = \sigma q i t : C \to A \to IA \to sIA, \]
\[ p_{k+1} \beta(t) = (p_k \& p_1) \Delta \beta(t) \]
\[ = (p_k \beta(t) \& p_1 \beta(t)) \Delta : C \to (s(IA))^\otimes k+1. \]

Thus, if \( \beta(t) \) exists, it is uniquely determined by \( t \). On the other hand, given \( t \), the above inductive formula for \( p_k \beta(t) \) gives a map \( \beta(t) : C \to \Pi_{j \geq 0} (s(IA))^\otimes j \) of degree 0. Since we have
\[ p_k \beta(t) = (p_1 \beta(t))^\otimes k \Delta^{(k)}, \text{ all } k, \]
and since \( p_1 \beta(t) \) vanishes on \( \operatorname{Im} \eta \), we immediately see that \( p_k \beta(t)(x) = 0 \) for \( x \in C[k] \). Thus the cocompleteness of \( C \) guarantees that \( \beta(t) \) actually maps into \( BA \), and it is now easy to prove that \( \beta(t) \in DC(C, BA) \). \( \Box \)

The dual case is slightly simpler.
1.10. **Corollary.** \( B \) and \( \Omega \) are adjoint functors with the required isomorphism being

\[ \beta \omega^{-1} : DA(\Omega C, A) \to DC(C, BA) . \]

1.11. In \( DA(A, B), DC(C, D), T(C, A) \) one has homotopy notions which fit nicely with \( \beta \) and \( \omega \). Let \( f_0, f_1 \in DC(C, D) \), \( g_0, g_1 \in DA(A, B) \), \( t_0, t_1 \in T(C, A) \). A homotopy \( h \) from \( f_0 \) to \( f_1 \) is a map \( h : C \to D \) such that

\[ Dh = f_1 - f_0, \quad \Delta h = (f_0 \otimes h + h \otimes f_1) \Delta ; \]

we write \( h : f_0 \simeq f_1 \) (in \( DC \)). A homotopy \( k \) from \( g_0 \) to \( g_1 \) is a map \( k : A \to B \) such that

\[ Dk = g_0 - g_1, \quad k\phi = \phi(g_0 \otimes k + k \otimes g_1) ; \]

we write \( k : g_0 \simeq g_1 \) (in \( DA \)). Finally, a homotopy from \( t_0 \) to \( t_1 \) is a map \( l : C \to A \) such that

\[ Dl = t_0 \cup l \cup l \cup t_1, \quad \Delta \eta = \eta, \quad \epsilon l = \epsilon ; \]

we write \( l : t_0 \simeq t_1 \) (in \( T \)).

Now, if \( h : f_0 \simeq f_1 \) in \( DC(C, BA) \), one easily sees that \( t^A h + \eta e : t^A f_0 \simeq t^A f_1 \) in \( T(C, A) \). Thus \( h \to t^A h + \eta e \) defines a map

\[ \{ h \mid h : f_0 \simeq f_1 \text{ in } DC(C, BA) \} \to \{ l \mid l : t^A f_0 \simeq t^A f_1 \text{ in } T(C, A) \} . \]

Proceeding exactly like in the proof of 1.9, one easily constructs an inverse \( \beta \) for this mapping. Also, if \( k : g_0 \simeq g_1 \) in \( DA(\Omega C, A) \), then \( k t_C + \eta e : g_0 t_C \simeq g_1 t_C \) in \( T(C, A) \), and this sets up a bijection (whose inverse we call \( \omega \))

\[ \{ k \mid k : g_0 \simeq g_1 \text{ in } DA(\Omega C, A) \} \leftrightarrow \{ l \mid l : g_0 t_C \simeq g_1 t_C \text{ in } T(C, A) \} . \]

1.12. In general \( \simeq \) is not an equivalence relation on \( DA(A, B) \) or \( DC(C, D) \). However, on \( T(C, A) \), and hence on \( DA(\Omega C, A) \) and on \( DC(C, BA) \) \( \simeq \) is an equivalence relation.

In fact, if \( l : t_0 \simeq t_1 \) in \( T(C, A) \), then \( x = l - \eta e \) vanishes on \( \text{Im} \eta \), so

\[ x_{\Delta(n)} = \phi(n) x^{\Delta(n)} \]

vanishes on \( C_{[n - 1]} \). Therefore

\[ v = \eta e - x + x \cup 2 - x \cup 3 + x \cup 4 - ... \]

is a well-defined map \( v : C \to A \). Moreover,

\[ l \cup v = (\eta e + x) \cup (\eta e - x + x \cup 2 - x \cup 3 + ... ) = \eta e = y \cup l , \]

so we denote \( v \) by \( l^{-1} \). Then

\[ 0 = D(l \cup l^{-1}) = t_0 \cup l \cup l^{-1} - l \cup t_1 \cup l^{-1} + l \cup D(l^{-1}) , \]

so

\[ 0 = l^{-1} \cup 0 = l^{-1} \cup t_0 - t_1 \cup l^{-1} + D(l^{-1}) , \]
i.e.

\[ t^1 : t_1 \simeq t_0 \text{ in } T(C, A). \]

This shows that \( \simeq \) is symmetric on \( T(C, A) \). For reflexivity and transitivity, we note that in \( T(C, A) \) we have \( \eta e : t \simeq t \) and

\[ t : t_0 \simeq t_1, \quad t' : t_1 \simeq t_2 \Rightarrow t \cup t' : t_0 \simeq t_2. \]

2. Trivialized extensions in \( DA \)

2.1. Let \( A \in DA \). In this section we study the set of all those \( X \in DA \) which are homotopy equivalent to \( A \) in a very strong (and explicit way). The topological analogue (which should not be pressed too hard, though) is a space \( A \) and its strong deformation retracts with given retractions.

Let \( \alpha \in DA(X, A) \). A trivialization of \( \alpha \) is a pair \( (\rho, h) \) with \( \rho \in DM(A, X) \), \( h \in \text{Hom}(X, X) \) and

\[ \begin{align*}
\alpha \rho &= A, \\
\rho \alpha &= X - Dh, \\
\rho \eta_A &= \eta_X, \\
\epsilon_X \rho &= \epsilon_A, \\
\alpha h &= 0, \\
h \rho &= 0, \\
h^2 &= 0.
\end{align*} \]

The triple \( (\alpha, \rho, h) \) is called a trivialized extension of \( A \) (with total algebra \( X \), projection \( \alpha \), section \( \rho \) and homotopy \( h \)). \( TEX \) denotes the class of all such triples, \( TEX_A \) the subclass with fixed base algebra \( A \).

A strict morphism of trivialized extensions is a pair \( (f_1, f_2) \) of morphisms in \( DA \) such that

\[
\begin{array}{c}
X \xrightarrow{f_1} X' \\
\downarrow \alpha \downarrow \alpha' \\
A \xrightarrow{f_2} A'
\end{array}
\]

commutes in \( DA \) and \( f_1 \rho = \rho' f_2, f_1 h = h' f_1 \).

We need more general morphisms than the strict ones. To describe them we need the following proposition, which is also the basis for the description of shm maps given in 3.3.

2.2. Proposition. Let \( E = (\alpha, \rho, h) \in TEX \) with \( \alpha : X \to A \). There is \( t^E \in T(BA, X) \) with \( \alpha t^E = t^A \) and a homotopy \( h^E : t^X \simeq t^E B(\alpha) \) in \( T(BX, X) \). Moreover, \( h^E \beta(t^E) = \eta e, \alpha h^E = \eta e \).
Proof. We define $t^E$ and $h^E$ by the identities

\begin{align}
(1) \quad t^E &= h(t^E \cup t^E) + \rho t^A, \\
(2) \quad h^E &= h(t^X \cup h^E - h^E \cup t^E B(\alpha)) + \eta_X \epsilon_{BX}.
\end{align}

These definitions must be interpreted as follows: Since $t^E i_0 = t^E \eta$ must be 0 and $h^E i_0 = h^E \eta$ must be $\eta$, we have started the inductive definition of $t^E i_j$ and $h^E i_j$. But then (1) and (2) determine $t^E i_{j+1}$ and $h^E i_{j+1}$ in terms of $t^E i_k, h^E i_k$ for $k < j$.

The identity $\alpha t^E = t^A$ is now obvious, and once this is established, an easy induction shows that indeed $t^E \in T(BA, X)$ and that

$$Dh^E = t^X \cup h^E - h^E \cup t^E B(\alpha).$$

Also, $h^E \eta = \eta$ because $t^E \eta = 0$, and $\epsilon h^E = \epsilon$ because $\epsilon h = \epsilon \alpha h = 0$. Thus $h^E : t^X \sim t^E B(\alpha)$.

Finally, $\alpha h^E = \eta_X \epsilon_{BX}$ since $\alpha h = 0$ and

$$h^E \beta(t^E) = h(t^X \cup h^E - h^E \cup t^E B(\alpha)) \beta(t^E) + \eta \epsilon.$$

Since $\beta(t^E) \in DC$ and

$$t^X \beta(t^E) = t^E, \quad B(\alpha) \beta(t^E) = \beta(\alpha t^E) = \beta(t^A) = BA,$$

we get, with $h' = h^E \beta(t^E)$,

$$h' = h(t^E \cup h' - h' \cup t^E) + \eta \epsilon.$$

Now an easy induction, starting from

$$h' i_0 = h^E \beta(t^E) \eta = h^E \eta = \eta,$$

shows that

$$h' i_j = 0 \quad \text{for } j > 0,$$

so

$$h^E \beta(t^E) = h' = \eta \epsilon,$$

as claimed. \(\square\)

2.3. We now define a morphism $(f_1, f_2) : E = (\alpha, \rho, h) \to E' = (\alpha', \rho', h')$, where $\alpha = X \to A, \alpha' : X' \to A'$, to be a pair such that the diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{f_1} & X' \\
\downarrow \alpha & & \downarrow \alpha' \\
A & \xrightarrow{f_2} & A'
\end{array} \quad \text{and} \quad 
\begin{array}{ccc}
X & \xrightarrow{f_1} & X' \\
\uparrow t^E & & \uparrow t^{E'} \\
BA & \xrightarrow{B(f_2)} & BA'
\end{array}
$$
commute in $\text{DA}$, and as maps of modules, respectively. With the obvious composition this defines the category $\text{TEX}$ of trivialized extensions. $\text{TEX}_A$ is the subcategory consisting of all $(f_1, f_2)$ with $f_2 = A \cdot A \to A'$. It is clear that any strict morphism is a morphism in $\text{TEX}$.

2.4. If $f \in \text{DA}(A, B)$, then $(A, A, 0), (B, B, 0) \in \text{TEX}$, and $(f, f) \in \text{TEX}((A, A, 0), (B, B, 0))$. In this way $\text{DA}$ is embedded as a subcategory of $\text{TEX}$. Thus we shall identify $A \in \text{DA}$ and $(A, A, 0) \in \text{TEX}$. This identification is consistent with the notations $t^A$ and $t^E$, i.e.,

\[ t^A(A, A, 0) = t^E (BA \to A). \]

2.5. There are two main reasons for our study of $\text{TEX}$. The first one is the following reformulation of the classical Eilenberg–Zilber theorem. The second one will be presented when we study tensor products of shn maps (see 3.3).

We let $\text{SS}_*$ denote the category of based semisimplicial sets with $C : \text{SS}_* \to \text{DM}$ and $C^* : \text{SS}_* \to \text{DA}$ the normalized chain and cochain functors, respectively. In general, $C X$ is not cocomplete, so we cannot assert that $C$ takes values in $\text{DC}$. In fact, $C X \in \text{DC}$ if and only if $X$ is reduced, i.e., $X_0$ consists of the base point alone. However, $C X$ is of course a coaugmented, differential, graded coalgebra. Now the classical Eilenberg–Zilber theorem (see e.g. [14]) gives natural maps

\[ V : C X \otimes CY \to C(X \times Y), \quad f : C(X \times Y) \to C X \otimes CY, \quad \phi : C(X \times Y) \to C(X \times Y), \]

with

\[ f V = C X \otimes CY, \quad V f = C(X \times Y) - D \phi, \quad f \phi - 0, \quad \phi V - 0, \quad \phi^2 - 0. \]

Moreover, $V, f \in \text{DM}$, and $(\text{see } [4]) V$ is a map of coaugmented, differential graded coalgebras. And it is easily seen that $f$ preserves counit as well as coaugmentation.

By dualizing we get $\text{EZ}_{X, Y} = (V^*, f^*, \phi^*) \in \text{TEX}((C X \otimes CY)^*)$ with total algebra $C^*(X \times Y)$. Moreover, naturality of $V, f$ and $\phi$ guarantees functoriality of $\text{EZ}$, i.e. we have the following reformulation of the dual of the classical Eilenberg–Zilber theorem:

2.6. Eilenberg–Zilber theorem. There is a functor $\text{EZ} : \text{SS}_* \times \text{SS}_* \to \text{TEX}$ with

\[ \text{EZ}(X, Y) = \text{EZ}_{X, Y} = (V^*, f^*, \Phi^*) , \]

base algebra $(C X \otimes CY)^*$ and total algebra $C^*(X \times Y)$.

This form of the Eilenberg–Zilber theorem is convenient for our later discussion of $C^* X$ as strongly homotopy commutative algebra.

2.7. There is a partially defined composition in $\text{TEX}$ defined as follows: Let $E_i = (\alpha_i, \rho_i, h_i) \in \text{TEX}$ with $\alpha_i : X_i \to A_i$. If $X_2 = A_1$, then
\[ E_1 \circ E_2 = (\alpha_2 \alpha_1, \rho_1 \rho_2, h_1 + \rho_1 h_2 \alpha_1) \in \text{TEX}. \]

This composition is functorial, i.e., if \((f_1, g_1) : E_i \to E_i\) and \(g_1 = f_2\), then \((f_1, g_2) : E_1 \circ E_2 \to E_1 \circ E_2\). This follows immediately from

\[ t^{E_1 \circ E_2} = t^{E_1} \beta(t^{E_2}) . \]

One proves (1) by showing that \(t^1 \beta(t^2)\) satisfies the defining equation for \(t^{E_1 \circ E_2}\). Here \(t^i\) is short for \(t^E_i\). One has

\[
(h_1 + \rho_1 h_2 \alpha_1) (t^1 \beta(t^2) \cup t^1 \beta(t^2)) + \rho_1 \rho_2 t^{A_2} = \\
= h_1 (t^1 \cup t^1) \beta(t^2) + \rho_1 h_2 (\alpha_1 t^1 \beta(t^2) \cup \alpha_1 t^1 \beta(t^2)) + \rho_1 \rho_2 t^{A_2} = \\
= t^1 \beta(t^2) - \rho_1 t^2 + \rho_1 h_2 (t^2 \cup t^2) + \rho_1 \rho_2 t^{A_2} = t^1 \beta(t^2),
\]

as desired.

Composition is also associative in the obvious way.

2.8. Proposition. If \(F \in \text{TEX}_A\), while \(f \in \text{DA}(B, A)\), then there is \(f^* F \in \text{TEX}_B\) and a morphism \((f', f) : f^* F \to F\) such that any morphism \((g, f) : F \to F\) factors uniquely through \((f', f), i.e., there is a unique \((l, B) \in \text{TEX}_R(F, f^* F)\) with \((f', f)(l, B) = (g, f)\).

Proof. We have the pullback diagram

\[
\begin{array}{ccc}
  f^* X & \xrightarrow{f'} & X \\
  \downarrow{f^*(\alpha)} & & \downarrow{\alpha} \\
  B & \xrightarrow{f} & A
\end{array}
\]

in \(\text{DA}\) with \(f^* X \subseteq B \otimes X\) consisting of all \((b, x)\) with

\[ fb = \alpha x, \quad f^*(\alpha) (b, x) = b, \quad f'(b, x) = x . \]

The product in \(f^* X\) has \((b, x)(b', x') = (bb', xx')\). ⊗

If we require \((f', f)\) to be a strict morphism in \(\text{TEX}\), then we get a unique trivialization \((f^*(\rho), f^*(\beta))\) of \(f^*(\alpha)\). The universal property is then easy to check. As usual, the universal property ensures that \(f^*\) is a functor \(\text{TEX}_A \to \text{TEX}_B\), and if \(g \in \text{DA}(C, B)\), then we get a natural equivalence from \(g^* f^*\) to \((fg)^*\).
2.9. The relationship between composition and pullback is nice. If \( E_i = (\alpha_i, \rho_i, h_i) \in \mathbf{TEX} \) with \( \alpha_i : X_i \to A_i \) and \( A_1 = X_2 \), and \( g \in \mathbf{DA}(B, A_2) \), then we have

\[
(g', g) : g^* E_2 \to E_2 \text{ and } (g'', g') : g''^* E_1 \to E_1.
\]

Then

\[
(g'', g) : g''^* E_1 \circ g^* E_2 \to E_1 \circ E_2,
\]

and the induced morphism

\[
g''^* E_1 \circ g^* E_2 \to g^*(E_1 \circ E_2)
\]

is an isomorphism which is natural in all ingredients.

2.10. Proposition. \( \otimes : \mathbf{DA} \times \mathbf{DA} \to \mathbf{DA} \) extends to a functor \( \otimes : \mathbf{TEX} \times \mathbf{TEX} \to \mathbf{TEX} \) such that

(i) \( \otimes \) is associative and has \( R \) as a two-sided unit;

(ii) \( (E_1 \circ E_2) \otimes (F_1 \circ F_2) = (E_1 \otimes F_1) \circ (E_2 \otimes F_2) \), provided \( E_2 \circ F_1 \in \mathbf{DA} \) (and the compositions exist).

Proof. If \( E_i = (\alpha_i, \rho_i, h_i) \) with \( \alpha_i : X_i \to A_i \), then we let

\[
E_1 \otimes E_2 = (\alpha_1 \otimes \alpha_2, \rho_1 \otimes \rho_2, h_1 \otimes X_2 + \rho_1 \alpha_1 \otimes h_2).
\]

And if \( (f_i, g_i) : E_i \to E_i' = (\alpha_i', \rho_i', h_i') \), then we let

\[
(f_1, g_1) \otimes (f_2, g_2) = (f_1 \otimes f_2, g_1 \otimes g_2).
\]

The results then follow by trivial computations except for the fact that

\( (f_1, g_1) \otimes (f_2, g_2) \)

really is a morphism of trivialized extensions. For the proof of that we need the following formulas:

\[
(1) \quad t^F \otimes B[a_1 \otimes h_1] = t^F [qa_1] \otimes h_1 + n \varepsilon(a_1) \otimes h_1,
\]

\[
(2) \quad t^F \otimes B[a_1 \otimes h_1 \ldots \otimes a_n \otimes h_n] =
\]

\[
(z - 1)^r t^F [qa_1 \ldots \otimes qa_n] \otimes b_1 b_2 \ldots b_n, \quad n > 1.
\]

Here \( F = (\alpha, \rho, h) \), with \( \alpha : X \to A, B \in \mathbf{DA} \), \( a_i \otimes b_i \in I(A \otimes B) \) and

\( r = \Sigma_v \deg(b_1 \ldots b_v) \deg([a_{v+1}]) \). (1) is true by simple inspection, and (2) follows by induction on \( n \), using the defining equations for \( t^F \) and \( t^B \).

Using (1) and (2), one now easily proves that the restriction of \( \otimes \) to \( \mathbf{TEX} \times \mathbf{DA} \) is functorial with the definitions given above. Similarly, \( \otimes \mid \mathbf{DA} \times \mathbf{TEX} \) is functorial.

Returning now to the above \( (f_i, g_i) : E_i \to E_i' \) we note that by (ii)

\[
E_1 \otimes E_2 = E_1 \circ A_1 \otimes X_2 \circ E_2 = (E_1 \otimes X_2) \circ (A_1 \otimes E_2),
\]

\[
E_1' \otimes E_2' = (E_1' \otimes X_2') \circ (A_1' \otimes E_2').
\]

Also, by what we have already shown,

\[
(f_1, g_1) \otimes (f_2, g_2) = (f_1 \otimes f_2, g_1 \otimes g_2) : E_1 \otimes X_2 \to E_1' \otimes X_2',
\]

\[
(g_1, g_1) \otimes (f_2, g_2) = (g_1 \otimes f_2, g_1 \otimes g_2) : A_1 \otimes E_2 \to A_1' \otimes E_2'.
\]
But then the functoriality of composition ensures that

\[(f_1, g_1) \otimes (f_2, g_2) = (f_1 \otimes f_2, g_1 \otimes g_2) \cdot E_1 \otimes E_2 \rightarrow E_1' \otimes E_2'
\]

is indeed a morphism of trivialized extensions. □

2.11. \(\otimes\) is definitely not commutative (up to natural isomorphism). In fact, the pair

\[(\tau_X, \tau_A), \text{ where } \tau_X : X_1 \otimes X_2 \rightarrow X_2 \otimes X_1, \tau_A : A_1 \otimes A_2 \rightarrow A_2 \otimes A_1 \text{ are twisting maps,}
\]

define a natural isomorphism

\[(\tau_X, \tau_A) : E_1 \otimes E_2 \rightarrow E_2 \otimes E_1,
\]

where \(\otimes' : \mathbf{TEX} \times \mathbf{TEX} \rightarrow \mathbf{TEX}\) is a different extension of \(\otimes\) to \(\mathbf{TEX}\) given by

\[(\alpha_1, \rho_1, h_1) \otimes' (\alpha_2, \rho_2, h_2) = (\alpha_1 \otimes \alpha_2, \rho_1 \otimes \rho_2, X_1 \otimes h_2 + h_1 \otimes \rho_2 \alpha_2).
\]

Instead of 2.10(ii), we have for \(\otimes'\),

\[\bigl( (E_1 \otimes E_2) \otimes' (E_1' \otimes E_2') \bigr) \otimes' \bigl( (E_1 \otimes E_2) \otimes' (E_1' \otimes E_2') \bigr),
\]

provided \(E_1\) or \(F_2 \in \mathbf{DA}\) (and the compositions exist).

For later use we record the following:

2.12. Proposition. If \(E = (\alpha, \rho, h), E' = (\alpha', \rho', h')\) have \(\alpha = \alpha' : X \rightarrow A\), then \(t^E \simeq t^{E'}\) in \(T(BA, X)\).

**Proof.** We have by 2.2,

\[h^{E'} : t^X \simeq t^{E'} B(\alpha)
\]

in \(T(BX, X)\). Thus

\[\beta h^{E'} : BX = \beta(t^X) \simeq \beta(t^{E'} B(\alpha)) = \beta(t^{E'}) B(\alpha)
\]

in \(\mathbf{DC}(RX, RX)\). Consequently,

\[\beta(h^{E'}) \beta(t^{E'}) : \beta(t^{E'}) \simeq \beta(t^{E'}) B(\alpha) \beta(t^{E'}) = \beta(t^{E'})
\]

in \(\mathbf{DC}\), so \(h^{E'} \beta(t^{E'}) : t^E \simeq t^{E'}\) in \(T(BA, X)\). □

We shall especially need:

2.13. Corollary. If \(E_i \in \mathbf{TEX}_{A_i}\), then there is a homotopy

\[l : t^{E_1 \otimes E_2} \simeq t^{E_1 \otimes E_2},
\]

in \(T\) with \(l B(A_1 \otimes \eta) = \eta \epsilon\).

**Proof.** By the above we may take

\[l = h^{E_1 \otimes E_2} \beta(t^{E_1 \otimes E_2}).\]
Now we have
\[(X_1 \otimes \eta, A_2 \otimes \eta) : E_1 \otimes R \to E_1 \otimes E_2\]
in \(\text{TEX}\), so
\[t^{E_1 \otimes E_2} : B(A_1 \otimes \eta) = (X_1 \otimes \eta) t^{E_1} R = t^{E_1} \otimes \eta,\]
so
\[\beta(t^{E_1 \otimes E_2}) B(A_1 \otimes \eta) = \beta(t^{E_1} \otimes \eta) = \beta(t^{E_1}) \otimes \eta\]
and
\[\beta^{E_1 \otimes E_2} B(A_1 \otimes \eta) = h^{E_1 \otimes E_2} (\beta(t^{E_1}) \otimes \eta)\]
\[= h^{E_1 \otimes E_2} (X_1 \otimes \eta) \beta(t^{E_1})\]
\[= (X_1 \otimes \eta) h^{E_1 \otimes E_2} \beta(t^{E_1})\]
\[= (h^{E_1} \otimes \eta) \beta(t^{E_1})\]
\[= h^{E_1} \beta(t^{E_1}) \otimes \eta\]
\[= \eta X_1 \in X_1 \otimes \eta = \eta \in : A_1 \otimes R \to X_1 \otimes X_2. \square\]

2.14. Proposition. There is a functor \(P : D \times \to \text{TEX}\) such that \(P(A)\) is an initial object in \(\text{TEX}_A\). One has

\[P(A) = (\alpha_A, \rho_A, h_A)\]

Here \(\alpha_A = \omega(t_A) = \Omega BA \to A; \rho_A\) and \(h_A\) are defined below. One has

\[t^{P(A)} = t_{BA} : BA \to \Omega BA.\]

Proof. First we warn that \(h^A = h^{(A, A, 0)} : BA \to A\), while \(h_A : B \Omega BA \to \Omega BA\), i.e., \(h_A \neq h^A\).

We define \(\rho_A\) by

\[\rho_A = t_{BA} \ i_1 \sigma q + \eta \in : A \to \Omega BA.\]

Since \(\alpha_A t_{BA} = t^A\), one has \(\alpha_A \rho_A = \iota_A \psi_A + \eta \eta_A = \iota_A\). Also \(\rho_A \eta_A = \eta_{\Omega BA}\) and \(\epsilon_{\Omega BA} \rho_A = \epsilon_A\). To define \(h_A : \Omega BA \to \Omega BA\), we consider a typical generator

\[\gamma = \langle a_{11}, \ldots, a_{1k(1)} \rangle, \ldots, \langle a_{n1}, \ldots, a_{nk(n)} \rangle, \quad a_{ij} \in IA, \quad k(i) > 0.\]

Now define graded submodules \(S_i \subset \Omega BA\) as follows:

- \(S_0\) is generated by all \(\gamma\) with \(n = 0\) or \(n = 1\) and \(k(1) > 1\);
- \(S_1\) is generated by all \(\gamma\) with \(n > 1, k(1) = 1, k(2) > 1\);
- \(S_i\) is generated by all \(\gamma\) with \(k(1) = k(2) = 1\) and \(n = i\), for \(i \geq 2\).

Then \(\Omega BA = \bigcup_{i \geq 0} S_i\) as graded module, and we define \(h_A\) by specifying \(h_i = h_A | S_i\) inductively on \(i\). We start by taking \(h_0 = 0\). To get the induction going, we introduce maps \(\chi_i : S_i \to S_0, i > 0\), given by
The identity \( a_A h_A = 0 \) is shown by induction. and \( a_A p_A = 0, h_A = 0 \) both are immediate. Thus we have constructed \( P(A) \). The functoriality of \( p \) just amounts to the obvious naturality of \( a_A, \rho_A \) and \( h_A \).

An easy computation shows that

\[ d \chi_i(\gamma) \equiv \gamma \pmod{\Pi_{j=0}^{i-1} S_j} \]  
for all \( \gamma \in S_i \).

Suppose now that \( h_0, h_1, \ldots, h_{i-1} \) are defined and \( h_j(S_j) \subset S_0 \) for \( j = 0, 1, \ldots, i-1 \). Also let \( \gamma \in S_i \). Then we must have

\[ h_A(\gamma) = h_A(d \chi_i(\gamma)) + h_A(\gamma - d \chi_i(\gamma)) \]

so

\[ D h_A(\chi_i(\gamma)) = (\Omega BA - \rho_A \alpha_A)(\chi_i(\gamma)) \]  
holds if and only if

\[ d h_A(\gamma) = \chi_i(\gamma) + h_A(\gamma - d \chi_i(\gamma)) \]

Here we have used that \( \alpha_A \chi_i = 0 \). Since \( \gamma - d \chi_i(\gamma) \in \Pi_{j=0}^{i-1} S_j \), (2) defines \( h_i \). In the induction we have used \( D h_A = \Omega BA - \rho_A \alpha_A \) only for elements of the form \( \chi_i(\gamma) \), so we still have to prove

\[ d h_A(\xi) + h_A d(\xi) = \xi - \rho_A \alpha_A(\xi), \quad \xi \in S_i. \]

This is done by induction on \( i \).

If \( i = 0 \), we note that \( S_0 \) is generated by \( \langle \cdot \rangle, \langle [a] \rangle \) for all \( a \in \mathcal{A} \). and \( \text{Im}(\chi_i) \) for all \( i \geq 1 \). For \( \xi \in \text{Im}(\chi_i) \), (3) is known to hold, and for \( \xi = \langle \cdot \rangle \) or \( \xi = \langle [a] \rangle \), it is easy to verify (3).

Now let us assume that (3) is true for \( i < n \), and let \( \xi \in S_n \). If \( \eta = d \chi_n(\xi) - \xi \), then (3) holds for \( \eta \), and one has

\[ d h_A(\xi) = d \chi_n(\xi) - d h_A(\eta) \]

\[ = d \chi_n(\xi) - h_A(d \xi) - \eta + \rho_A \alpha_A(\eta) \]

\[ = \xi - h_A d(\xi) + \rho_A \alpha_A(d \chi_n(\xi) - \xi) \]

\[ = \xi - h_A d(\xi) - \rho_A \alpha_A(\xi). \]

Here we have used that \( \alpha_A d \chi_n(\xi) = d \alpha_A \chi_n(\xi) = 0 \), and that \( d \chi_n(\xi) - \eta = \xi \). This finishes the proof of (3).

The identity \( \alpha_A h_A = 0 \) is easy to show by induction, and \( h_A \rho_A = 0, h_A^2 = 0 \) both are immediate. Thus we have constructed \( P(A) \). The functoriality of \( P \) just amounts to the obvious naturality of \( \alpha_A, \rho_A \) and \( h_A \).
We next check that \( t^{P(A)} = t_{BA} : BA \to \Omega BA \). By definition,

\[
t^{P(A)}([a]) = \rho_A t_A([a]) = \rho_A(a) = (\langle [a] \rangle) = t_{BA}([a]), \quad a \in IA.
\]

Now assume that

\[
t^{P(A)}([a_1 \mid \ldots \mid a_i]) = t_{BA}([a_1 \mid \ldots \mid a_i]) = (\langle [a_1 \mid \ldots \mid a_i] \rangle)
\]

whenever \( i < n \). Then

\[
t^{P(A)}([a_1 \mid \ldots \mid a_n]) = \sum_i (-1)^{\text{deg}([a_1 \mid \ldots \mid a_i])} h^A(\langle [a_1 \mid \ldots \mid a_i] \mid [a_{i+1} \mid \ldots \mid a_n] \rangle)
\]

\[= h^A(\langle [a_1 \mid a_2 \mid \ldots \mid a_n] \rangle).
\]

For \( n > 2 \), one has

\[
h^A(\langle [a_1 \mid a_2 \mid \ldots \mid a_n] \rangle) = \chi_1(\langle [a_1 \mid a_2 \mid \ldots \mid a_n] \rangle) = \langle [a_1 \mid \ldots \mid a_n] \rangle,
\]

so this case is alright. For \( n = 2 \) one has

\[
d \chi_2(\langle [\bar{a}_1 \mid a_2] \rangle) = \chi_2(\langle [a_1 \mid a_2] \rangle) \equiv \langle [\bar{a}_1 \mid a_2] \rangle \pmod{S_0},
\]

so

\[
h^A(\langle [\bar{a}_1 \mid a_2] \rangle) = \chi_2(\langle [\bar{a}_1 \mid a_2] \rangle) = \langle [a_1 \mid a_2] \rangle,
\]

and

\[
t^{P(A)}([a_1 \mid a_2]) = ([a_1 \mid a_2]),
\]

as desired. This finishes the proof of \( t^{P(A)} = t_{BA} \).

Finally, it is now easy to see that \( P(A) \) is initial in \( \text{TEX}_A \). Indeed let \( E = (\alpha, \rho, h) \in \text{TEX}_A \), with \( \alpha : X \to A \). Any morphism \( f : P(A) \to E \) in \( \text{TEX}_A \) must have \( f t^{P(A)} = t^E \), i.e., \( f = \omega(t^E) \), where \( \omega : T(BA, X) \to DA(\Omega BA, X) \). On the other hand, \( \omega(t^E) \) is a morphism from \( P(A) \) to \( E \); in fact, \( \alpha \omega(t^E) \) has

\[
\alpha \omega(t^E) t_{BA} = \alpha \cdot t^E = t^A = \omega(\alpha_A) t_{BA}^A,
\]

so \( \alpha \omega(t^E) = \alpha_A \). This concludes the proof of the proposition. \( \Box \)

2.15. Corollary. \( \alpha_A : \Omega BA \to A \) is a homotopy equivalence for all \( A \in DA \).

Apparently, it was formerly only known that \( \alpha_A \) is a homology isomorphism for all flat \( A \) (see e.g. [8]).

(The author thanks the referee for pointing out that the trivialization of \( \alpha : \Omega BA \to A \) has also been obtained by Byron Drachman; see [2a].)
3. The category $\mathbf{DASH}$ of differential graded algebras and strongly homotopy multiplicative maps

3.1. $\mathbf{DASH}$ has the same objects as does $\mathbf{DA}$. $\mathbf{DASH}(A, B) = T(BA, B)$. We write $f : A \Rightarrow B$ for $f \in \mathbf{DASH}(A, B)$. If $f : A \Rightarrow B, g : B \Rightarrow C$, then the composition $g \circ f$ is defined to be $g \beta(f) : A \Rightarrow C$. Since $\beta(g \beta(f)) = \beta(g) \beta(f)$, $\mathbf{DASH}$ is indeed a category. Moreover, $A \mapsto BA, f \mapsto \beta(f)$ define a functor $\beta : \mathbf{DASH} \to \mathbf{DC}$. This is an embedding, and it represents $\mathbf{DASH}$ as the full subcategory of $\mathbf{DC}$ determined by all $BA, A \in \mathbf{DA}$. $\mathbf{DA}$ embeds into $\mathbf{DASH}$ by the identity on objects and by sending $f \in \mathbf{DA}(A, B)$ to $f \circ \iota^A \in \mathbf{DASH}(A, B)$. We identify $\mathbf{DA}$ with its image in $\mathbf{DASH}$. This identifies $f \in \mathbf{DA}(A, B)$ with $f \circ \iota^A \in T(BA, B)$, so we must be careful. With this identification,

$$\beta : \mathbf{DA} = B : \mathbf{DA} \to \mathbf{DC}.$$

Any $f : A \Rightarrow B$ is called an shm (strongly homotopy multiplicative) map. If $f \in \mathbf{DA}$, we call $f$ multiplicative. For a justification of the name shm, we refer to the Appendix. If $f : A \Rightarrow B, g : B \Rightarrow C$, then we note that

$g \circ f = gB(f)$ if $f$ is multiplicative,

$g \circ f = gf$ if $g$ is multiplicative.

3.2. In $\mathbf{DASH}(A, B)$, the homotopy relation is an equivalence relation according to 1.12. Moreover, if $l : t_0 \sim t_1 : A \Rightarrow B$, while $f : A' \Rightarrow A, g : B \Rightarrow B'$, then

$$l \beta(l) : t_0 \circ f \sim t_1 \circ f : A' \Rightarrow B.$$

$$g \beta(l) + \eta \epsilon : g \circ t_0 \sim g \circ t_1 : A \Rightarrow B'.$$

Thus we have a homotopy category $\mathbf{DASH}_h$ with the same objects as $\mathbf{DASH}$ and with $\mathbf{DASH}_h(A, B) = \{A, B\}$ is the set of homotopy classes of shm maps from $A$ to $B$. By abuse of language, we shall write $l \circ f = l \beta(l)$ and $g \circ l = g \beta(l) + \eta \epsilon$. Then

$$g \circ (l \circ f) = (g \circ l) \circ f, \quad (g \circ g) \circ l = g' \circ (g \circ l), \quad (l \circ f) \circ f' = l \circ (f \circ f').$$

3.3. Proposition. $\otimes : \mathbf{DA} \times \mathbf{DA} \to \mathbf{DA}$ extends to a functor in each variable separately,

$$\otimes : \mathbf{DASH} \times \mathbf{DASH} \to \mathbf{DASH},$$

with the following properties:

(i) $\otimes$ is associative and has $R$ as a unit;

(ii) if $g_1 : A_1 \Rightarrow B_1, f_1 : B_1 \Rightarrow C_1$, and either $f_2$ or $g_1$ is multiplicative, then

$$(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = f_1 \circ g_1 \otimes f_2 \circ g_2.$$

Especially, $\otimes$ is $\mathbf{DA} \times \mathbf{DASH}$ and $\otimes$ is $\mathbf{DASH} \times \mathbf{DA}$ are functorial in both variables.
Proof. On objects we must take $A_1 \times A_2 = A_1 \otimes A_2$. Now, if $f_i : A_i \Rightarrow B_i$, we have $\omega(f_i) \in DA(\Omega BA, B_i)$. Also, in $DA(\Omega BA_1 \times A_2), \Omega BA_1 \otimes \Omega BA_2$, there is a nice $\kappa_{A_1, A_2}$, namely the unique morphism in $\text{TEX} \times A_1 \otimes A_2$ from $P(A_1 \times A_2)$ to $P(A_1) \otimes P(A_2)$.

We now define $f_1 \otimes f_2$ by

$$\omega(f_1 \otimes f_2) = (\omega(f_1) \otimes \omega(f_2)) \kappa_{A_1, A_2}.$$ 

The associativity of $\otimes$ follows immediately from the observation that

$(\kappa_{A_1, A_2} \otimes \Omega BA_1) \kappa_{A_1 \otimes A_2, A_3}$ and $(\Omega BA_1 \otimes \kappa_{A_2, A_3} \otimes A_2)$ both represent morphisms from $P(A_1 \times A_2 \times A_3)$ to $PA_1 \otimes PA_2 \otimes PA_3$, so that they coincide.

Also, $R$ is a left unit because

$$k_{R, A} = \text{id} : \Omega BA = \Omega BA(A \otimes R) \Rightarrow \Omega BA \otimes \Omega BR = \Omega BA.$$

Similarly, $R$ is a right unit.

If $f_i = g_i t_{A_i}$ is multiplicative, then

$$\omega(f_i) = g_i \alpha_{A_i}$$

(because $\omega(f_i)t_{B_{A_i}} = f_i = g_i t_{A_i} = g_i \alpha_{A_i} t_{B_{A_i}}$), so

$$\omega(f_1 \otimes f_2) = (g_1 \otimes g_2) (\alpha_{A_1} \otimes \alpha_{A_2}) \kappa_{A_1, A_2}$$

$$= (g_1 \otimes g_2) \alpha_{A_1} \otimes A_2$$

$$= \omega((g_1 \otimes g_2) t_{A_1} \otimes A_2),$$

so $\otimes$ does extend $\times$ on DA.

The functoriality of $\otimes$ in each variable separately follows from (ii) for the proof of which we now set the stage.

DASH$(A, B)$ admits a description in terms of $\text{TEX}_A$ and DA as follows: Any $E = (\alpha, \rho, h) \in \text{TEX}_A$, with $\alpha : X \Rightarrow A$ gives $t^E : A \Rightarrow X$. Thus, if $f \in DA(X, B)$, then we have

$$f \circ t^E = f t^E : A \Rightarrow B.$$

Since for any $g : A \Rightarrow B$ one has

$$\omega(g) t_{PA} = \omega(g) t_{BA} = g,$$

every morphism in DASH can be represented in the form $f t^E$, i.e. DASH is generated by DA and all $t^E, E \in \text{TEX}$.

What are the relations? We claim that $f_1 t^E_1 = f_2 t^E_2 : A \Rightarrow B$ if and only if there exist $E \in \text{TEX}_A$ and $k_i \in \text{TEX}_A(E, E_i)$ such that $f_1 k_1 = f_2 k_2$. Indeed, if such $k_i$ are given, then $k_i t^E = t^E_i$. so

$$f_1 t^E_1 = f_1 k_1 t^E = f_2 k_2 t^E = f_2 t^E_2.$$
If conversely \( f_1 t^{E_1} = f_2 t^{E_2} : A \to B \), then we take \( k_i \) to be the unique morphism \( P(A) \to E_i \). Then

\[
j_1 k_1 t^{BA} = f_1 t^{E_1} = f_2 t^{E_2} = f_2 k_2 t^{BA}.
\]

so \( f_1 k_1 = f_2 k_2 \).

How is composition in terms of this representation? We let \( f t^E : A \to B \) and \( g t^F : B \to C \), where \( f, g \in \text{DA}, E \in \text{TEX}_A, F \in \text{TEX}_B \). Now, if \( (f, f) : F' \to F \) for some \( F' \), then we have

\[
f_1 t^{F'} = t^F \delta(f) = t^F \circ f.
\]

\[
g t^F \circ f t^{E'} = g \circ t^F \circ f \circ t^E = g f_1 t^{E'} \circ t^F = g f_1 t^{E'} t^F.
\]

Note that for \( (f', f) : F' \to F \) one may always take \( (f', f) : f^* F \to F \), thus getting the special case

\[
g t^F \circ f t^{E'} = g f_1 t^{E'} f^* F \circ E.
\]

We introduced this description in order to study \( \circ \). So how does \( \circ \) look in this description? We claim that

\[
f_1 t^{E_1} \circ f_2 t^{E_2} = (f_1 \circ f_2) t^{E_1 \circ E_2}.
\]

Indeed, if \( k_i : P(A_i) \to E_i \) in \( \text{TEX}_{A_i} \), then

\[
f_i t^{E_i} = f_i k_i t^{P(A_i)} = f_i k_i t^{BA_i},
\]

so \( \omega(f_i t^{E_i}) = f_i k_i \), and

\[
\omega(f_1 t^{E_1} \circ f_2 t^{E_2}) = (f_1 k_1 \circ f_2 k_2) k_{A_1, A_2} = (f_1 \circ f_2)(k_1 \circ k_2) k_{A_1, A_2}.
\]

Similarly,

\[
\omega((f_1 \circ f_2) t^{E_1 \circ E_2}) = (f_1 \circ f_2) k_{12},
\]

where \( k_{12} \in \text{TEX}_{A_1 \circ A_2} (P(A_1 \circ A_2), E_1 \circ E_2) \). The initiality of \( P(A_1 \circ A_2) \) guarantees that \( (k_1 \circ k_2) k_{A_1, A_2} = k_{12} \), so the claim is proved.

### 3.4

With the above description of \( \circ \), we return to the proof of 3.3(ii). We let \( E_i \in \text{TEX}_{A_i} \) with total algebra \( X_i, F_i \in \text{TEX}_{B_i} \) with total algebra \( Y_i, C_i \). Finally, let \( (f_i, f_i) : G_i = f_i^* F_i \to F_i \) be the canonical morphism. Then

\[
g_i t^{F_i} \circ f_i t^{F_i} = g_i f_i t^{G_i \circ F_i}.
\]

so

\[
(g_1 t^{F_1} \circ f_1 t^{F_1}) \circ (g_2 t^{F_2} \circ f_2 t^{F_2}) = (g_1 f_1 \circ g_2 f_2) t^{G_1 \circ F_1 \circ G_2 \circ F_2}.
\]

Also, \( (f_1 \circ f_2, f_1 \circ f_2) \in \text{TEX}(G_1 \circ G_2, F_1 \circ F_2) \), so

\[
(g_1 t^{F_1} \circ g_2 t^{F_2}) \circ (f_1 t^{F_1} \circ f_2 t^{F_2}) = (g_1 \circ g_2)(f_1 \circ f_2) t^{(G_1 \circ G_2) \circ (F_1 \circ F_2)}.
\]
By 2.10(ii), these two shm maps coincide, provided $G_2 \in DA$ or $E_1 \in DA$. Now, if $f_1 \circ F_1$ is multiplicative, then we may take $F_1 \in DA$, and if $g_2$ is multiplicative, then we may take $F_2 \in DA$. Then also $G_2 \in DA$, and the proof is finished. □

3.5. Proposition. ~ is a congruence relation with respect to $\otimes$. Thus there is an induced $\otimes$ on $DASH_h$. On $DASH_h$, tensor is functorial (in both variables simultaneously) associative, commutative and has $R$ as two-sided unit.

Proof. Let

$$l_i : f_i \sim g_i : A_i \Rightarrow B_i$$

in $DASH$. Then

$$\omega(l_i) : \omega(f_i) \sim \omega(g_i) : \Omega B A_i \Rightarrow B_i$$

in $DA$, and it is easily seen that

$$h' = (\omega(l_1) \otimes \omega(g_1)) k_{A_1, A_2} : \omega(f_1 \otimes g_1) \Rightarrow \omega(f_2 \otimes g_1),$$

$$h'' = (\omega(l_2) \otimes \omega(g_2)) k_{A_1, A_2} : \omega(f_2 \otimes g_1) \Rightarrow \omega(f_2 \otimes g_2).$$

Now the required homotopy from $f_1 \otimes g_1$ to $f_2 \otimes g_2$ is $\omega^{-1}(h') \cup \omega^{-1}(h'')$. From the above it follows that $\otimes$ induces a functor in each variable separately (still called $\otimes$) on $DASH_h$.

To prove that this is actually functorial in both variables simultaneously, we only have to show that

$$(B_1 \otimes f_2) \circ (f_1 \otimes A_2) \sim (f_1 \otimes B_2) \circ (A_1 \otimes f_2)$$

for all $f_i : A_i \Rightarrow B_i$. We let $f_i = g_i \circ t^{E_i}$, where $E_i \in TEX_{A_i}$ has total algebra $X_i$, and $g_i \in DA(X_i, B_i)$. By 3.3(ii), we then have

$$(B_1 \otimes f_2) \circ (f_1 \otimes A_2) = (g_1 \otimes g_2) \circ (t^{X_1} \otimes t^{E_2}) \circ (t^{E_1} \otimes t^{A_2}),$$

$$(f_1 \otimes B_2) \circ (A_1 \otimes f_2) = (g_1 \otimes g_2) \circ (t^{E_1} \otimes t^{E_2}).$$

Also,

$$(t^{X_1} \otimes t^{E_2}) \circ (t^{E_1} \otimes t^{A_2}) = t^{X_1 \otimes E_2} \circ t^{E_1 \otimes A_2}$$

$$= t^{E_1 \otimes E_2} \circ (E_1 \otimes A_2)$$

$$= t^{E_1 \otimes E_2} \circ (E_1 \otimes A_2).$$

so Corollary 2.13 gives the required homotopy.

The associativity of $\otimes$ and the naturality for $\otimes$ on $DASH_h$ is inherited from $DASH$. Thus we only have to show that on $DASH_h$, $\otimes$ is commutative, i.e.,

$$\tau : A_1 \otimes A_2 \Rightarrow A_2 \otimes A_1$$

is a natural with respect to shm maps $f_i : A_i \Rightarrow B_i$. If $f_i$ is multiplicative, this is obvious, so we may take $f_i = t^{E_i}$, where $E_i \in TEX_{A_i}$ has total algebra $B_i$. We then have
3.4. For later use, we note a few auxiliary results about the above homotopies.

First, let \( f_i : A_i \Rightarrow B_i \), \( i = 1, 2 \). Then we get a homotopy

\[
h : (B_1 \otimes f_2) \circ (f_1 \otimes A_2) \sim f_1 \otimes f_2 : A_1 \otimes A_2 \Rightarrow B_1 \otimes B_2.
\]

This \( h \) came up as \( (g_1 \otimes g_2) \circ l \), where \( g_i \) is multiplicative, and \( l : F_1 \otimes F_2 \sim F_1 \otimes F_2 \).

Here 2.13 says that \( l \circ (A_1 \otimes \eta) = \eta \varepsilon \). It follows that \( h \circ (A_1 \otimes \eta) = \eta \varepsilon \). It is easy to extend this result to:

(i) If \( f_i : A_i \Rightarrow B_i \), \( g_i : B_i \Rightarrow C_i \), then

\[
(g_1 \otimes g_2) \circ (f_1 \otimes f_2) \sim (g_1 \circ f_1) \otimes (g_2 \circ f_2)
\]

by a homotopy \( k \) with \( k \circ (A_1 \otimes \eta) = \eta \varepsilon \).

In fact,

\[
(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \otimes C_2) \circ (B_1 \otimes g_2) \circ (f_1 \otimes B_2) \circ (A_1 \otimes f_2).
\]

so the required homotopy is

\[
k = (g_1 \otimes C_2) \circ h \circ (A_1 \otimes f_2),
\]

where \( h : (B_1 \otimes g_2) \circ (f_1 \otimes B_2) \sim f_1 \otimes g_2 \) has \( h \circ (A_1 \otimes \eta) = \eta \varepsilon \).

Similarly one gets:

(ii) If \( f_i : A_i \Rightarrow B_i \), then \( (f_1 \otimes f_2) \circ \tau \sim \tau \circ (f_2 \otimes f_1) \) by a homotopy \( h \) with \( h \circ (A_1 \otimes \eta) = \eta \varepsilon \).

Finally we need:

(iii) For any \( f : A \Rightarrow B \), the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\otimes f} & A' \\
\downarrow \tau & & \downarrow \tau \\
A' \otimes A & \xrightarrow{A' \otimes f} & A' \otimes B
\end{array}
\]

commutes.
For the proof, let \( f = g t^E \) with \( E \in \text{TEX}_A \), and use the fact that \((\tau, \tau)\) is a strict morphism in \( \text{TEX} \) from \( E \cong A' \) to \( A' \cong E \) when \( A' \in DA \).

3.7. Proposition. The forgetful functor \( DA \to DM \) and the homology functor \( H : DA \to A \) extend to functors \( \# : \text{DASH} \to DM, H : \text{DASH} \to A \). For \( f : A \to B \), one has

\[
f_{\#} = \omega(f) \rho_A : A \to \Omega BA \to B
\]

\[
= f i_1 \circ q_A + \eta \epsilon : A \to B.
\]

\( H(f) = H(f_{\#}) : HA \to HB. \)

Moreover, \( \# \) preserves tensor products and homotopies.

**Proof.** Take \( f_{\#} = \omega(f) \rho_A \). If \( f \in DA \), then \( f = g t^A \) for some \( g \in DA(A, B) \), and

\[
f_{\#} = (g t^A)_\# = \omega(g t^A) \rho_A = g \omega(t^A) \rho_A = g \alpha_A \rho_A = g.
\]

so \( \# \) does extend the forgetful \( DA \to DM. \)

To check the functoriality of \( \# \), we note that

\[
\rho_A = i_{BA} i_1 \circ q_A + \eta \epsilon : A \to \Omega BA
\]

so

\[
f_{\#} = f i_1 \circ q_A + \eta \epsilon : A \to B.
\]

Thus, if also \( f' : B \to C \), then we have to show that

\[
f' \circ (f) i_1 \circ q_A = f' i_1 \circ q_B f i_1 \circ q_A.
\]

But

\[
BA = \sum i_\nu p_\nu,
\]

so

\[
f' \beta(f) i_1 \circ q_A = \sum f' i_\nu p_\nu \beta(f) i_1 \circ q_A.
\]

The definition of \( \beta(f) \) given in 1.9 immediately gives

\[
p_\nu \beta(f) i_1 = \delta_1 i_\nu \circ q_{Bf}.
\]

so

\[
f' \beta(f) i_1 \circ q_A = f' i_1 \circ q_B f i_1 \circ q_A,
\]

as desired.

Now the part of the proposition relating to \( H \) reduces to showing that \( H(f) \in A(HA, HB) \) when \( f \in \text{DASH}(A, B) \). And that is clear since \( \rho_A, \alpha_A \) are homotopy inverses, so that

\[
H(f) = H(f_{\#}) = H(\omega(f)) H(\rho_A) = H(\omega(f)) H(\alpha_A)^{-1}.
\]
As for the identity \((f \otimes g)_{\#} = f_{\#} \otimes g_{\#}\), we note that

\[
(f \otimes g)_{\#} = (\omega(f) \otimes \omega(g)) k_{A,B} \rho_A \otimes \rho_B,
\]
so it suffices to prove that

\[
k_{A,B} \rho_A \otimes \rho_B = \rho_A \otimes \rho_B.
\]

Now, if \(E = (\alpha, \rho, \eta) \in \text{TEX}\), then

\[
t^E i_1 \circ q = \rho t^A i_1 \circ q = \rho - \eta \epsilon.
\]

so we have (since \(k_{A,B} = d^P(A \otimes B) - d^P(A \otimes B)\))

\[
\rho_A \otimes \rho_B = d^P \otimes PB i_1 \circ q_A \otimes B + \eta \epsilon = k_{A,B} \rho_A \otimes B.
\]

as desired.

Finally, if \(l : t_0 \sim t_1\) in \(\text{DASH}(A, B)\), then

\[
\omega(l) : \omega(t_0) \sim \omega(t_1)
\]
in \(\text{DA}\) and

\[
\omega(l) \rho_A = l_# : t_0 # \sim t_1 #
\]
in \(\text{DM}\).

3.8. We have the functor \(\Omega \beta : \text{DASH} \to \text{DA}\). It is clearly an embedding of \(\text{DASH}\) into \(\text{DA}\), and its restriction to \(\text{DA} \subset \text{DASH}\) is \(\Omega B : \text{DA} \to \text{DA}\). We note that for \(f : A \Rightarrow B\) we have a commutative diagram

\[
\begin{array}{ccc}
H(\Omega BA) & \to & H(\Omega BB) \\
\downarrow H(\alpha_A) & & \downarrow H(\alpha_B) \\
H(A) & \to & H(B)
\end{array}
\]

Indeed,

\[
\alpha_B \Omega \beta(f) t_{BA} = \alpha_B t^B_{BB} \beta(f) = t^B \beta(f) = f,
\]
so

\[
f_{\#} = \alpha_B \Omega \beta(f) \rho_A,
\]

\[
H(f) = H(\alpha_B) H(\Omega \beta(f)) H(\alpha_A)^{-1}.
\]

3.9. Proposition. (i) If \(f_i : A_i \Rightarrow B_i, i = 1, 2, \ldots\) then there is a unique shm map

\[
f_{\#} = \otimes_{i=1}^\infty f_i
\]
such that the diagrams
(where the vertical arrows are the multiplicative inclusions) commute for all $n$.

(ii) If further we have $g_i : B_i \Rightarrow C_i$, $i = 1, 2, \ldots$, then

\[
\bigotimes_{i=1}^\infty g_i \circ \bigotimes_{i=1}^\infty f_i \sim \bigotimes_{i=1}^\infty (g_i \circ f_i).
\]

(iii) If all $g_i$ or all $f_i$ are multiplicative, then equality holds in (2).

(iv) Finally, if $h_i : f_i \sim g_i : A_i \Rightarrow B_i$, then

\[
\bigotimes_{i=1}^\infty f_i \sim \bigotimes_{i=1}^\infty g_i.
\]

**Proof.** (i) Since $B(\bigotimes_{i=1}^\infty A_i) = \bigcup_n B(\bigotimes_{i=1}^n A_i)$, with inclusions $\bigotimes_{i=1}^n A_i \otimes \eta_{A_i}$, we get $f^{[\infty]}$ by noticing that

\[
(f_1 \otimes \ldots \otimes f_{n+1}) B(\bigotimes_{i=1}^n A_i \otimes \eta) =
\]

\[
= (f_1 \otimes \ldots \otimes f_{n+1}) \circ (A_1 \otimes \ldots \otimes A_n \otimes \eta) =
\]

\[
= f_1 \otimes \ldots \otimes f_n \otimes \eta = (B_1 \otimes \ldots \otimes B_n \otimes \eta) \circ (f_1 \otimes \ldots \otimes f_n \otimes R).
\]

(ii) To get the homotopy (2), we let

\[
A[n] = \bigotimes_{i=1}^n A_i, \quad f[n] = \bigotimes_{i=1}^n f_i.
\]

It then suffices to construct homotopies

\[
h[n] : g[n] \circ f[n] \sim (g \circ f)[n],
\]

with

\[
h^{[n+1]} A[n] \otimes \eta = (B[n] \otimes \eta) \circ h[n].
\]

We take $h[1] = \eta \varepsilon$, and assume $h[n]$ known. Then
4. Strongly homotopy commutative algebras

4.1. Definition. A **strongly homotopy commutative algebra (shc algebra)** is a pair $(A, \Phi)$ with $A \in \text{DA}$ and:

(i) $\Phi : A \otimes A \to A$;

(ii) $\Phi_\eta = \Phi : A \otimes A \to A$;

(iii) $\Phi \circ (A \otimes \eta) = \Phi \circ (\eta \otimes A) = A$, i.e., $\eta$ is a unit for $\Phi$;

(iv) $\Phi \circ (\Phi \otimes A) \sim \Phi \circ (A \otimes \Phi)$, i.e., $\Phi$ is homotopy associative;

(v) $\Phi \circ \tau \sim \Phi : A \otimes A \to A$, i.e., $\Phi$ is homotopy commutative.

A morphism $f$ from $(A, \Phi)$ to $(B, \Phi)$ is an $f : A \to B$ such that:
From the properties of \( g \) versus \( \sim \), it is easily seen that this defines a category. We call it \( \text{shcA} \). We can embed the full subcategory \( \text{DA}^c \) of \( \text{DA} \) determined by the commutative objects of \( \text{DA} \) into \( \text{shcA} \) by sending \( A \) to \((A, \phi)\), with \( \phi \) considered in \( \text{DASH} \).

Condition (vi) follows from the fact that \( f \eta = 0 \), so it could be left out.

For later use, we note that homotopies

\[
h : \Phi \circ (\Phi \otimes A) \sim \Phi \circ (A \otimes \Phi), \quad k : \Phi \circ \tau \sim \Phi
\]

may be chosen such that

\[
h \circ (A \otimes A \otimes \eta) = \eta \epsilon, \quad k \circ (A \otimes \eta) = \eta \epsilon.
\]

In fact, starting from arbitrary homotopies \( h' \), \( k' \) showing homotopy associativity and commutativity for \( \Phi \), we note that

\[
h'' = h' \circ (A \otimes A \otimes \eta) \cdot (A \otimes \Phi) : \Phi \circ (A \otimes \Phi) \sim \Phi \circ (A \otimes \Phi),
\]

\[
k'' = k' \circ (A \otimes \eta) \cdot \Phi : \Phi \sim \Phi.
\]

Hence

\[
h = h' \cup (h'')^{-1}, \quad k = k' \cup (k'')^{-1}.
\]

4.2. Proposition. \( \otimes : \text{DA}^c \times \text{DA}^c \to \text{DA}^c \) extends to a functor in each variable separately,

\[\otimes : \text{shcA} \times \text{shcA} \to \text{shcA},\]

which is associative, has \( R \) as a unit, and is commutative up to natural equivalence.

Proof. Given \( A, B \in \text{shcA} \), we define \( \Phi : (A \otimes B) \otimes (A \otimes B) \to A \otimes B \) to be the composition

\[\Phi = (\Phi \otimes \Phi) \circ (A \otimes \tau \otimes B),\]

with \( \tau : B \otimes A \to A \otimes B \) the twist map. Since \( \otimes \) has all the standard properties, at least up to homotopy, it is a standard exercise to verify that this makes \( A \otimes B \in \text{shcA} \). Also, for the same reason, we easily show that \( f \otimes g \) belongs to \( \text{shcA} \) whenever \( f \) and \( g \) do.

The associativity of \( \otimes \) on \( \text{shcA} \) follows easily from 3.3(ii) since the twist maps
involved are multiplicative. It is trivial to see that $R$ is a unit for $\otimes$. Finally, the commutativity of $\otimes$ on $\mathbf{shcA}$ means that $\tau : B \otimes A \to A \otimes B$ is an isomorphism in $\mathbf{shcA}$, and that is true by inspection. $\square$

4.3. Given $(A_i, \Phi_i) \in \mathbf{shcA}$, $i = 1, 2, 3, \ldots$, we may form $A = \bigotimes_{i=1}^{\infty} A_i$ and (see 3.9)

$$\Phi = \left( \bigotimes_{i=1}^{\infty} \Phi_i \right) \circ \tau : A \otimes A \Rightarrow A,$$

with $\tau$ the obvious twist map. To see that this $A \in \mathbf{shcA}$, we let $\Phi^\infty = \bigotimes_{i=1}^{\infty} \Phi_i$, and look at the diagram

$$
\begin{array}{ccccccc}
\bigotimes A_i \times^3 & \xrightarrow{\tau \otimes A} & \bigotimes (A_i \otimes A_i) \otimes (\bigotimes A_i) & \xrightarrow{\Phi^\infty \otimes A} & (\bigotimes A_i) \otimes (\bigotimes A_i) \\
\downarrow A \otimes \tau & & (1) \downarrow \tau & & (2) \downarrow \tau \\
\bigotimes (A_i \otimes A_i) & \xrightarrow{\tau} & \bigotimes (A_i \otimes A_i) & \xrightarrow{\bigotimes (\Phi_i \otimes A_i)} & \bigotimes (A_i \otimes A_i) \\
\downarrow A \otimes \Phi^\infty & & (3) \downarrow \bigotimes (\Phi_i \otimes \Phi_i) & & (4) \downarrow \Phi^\infty \\
\bigotimes A_i & \xrightarrow{\tau} & \bigotimes A_i & \xrightarrow{\Phi^\infty} & \bigotimes A_i \\
\end{array}
$$

Here (1) commutes, (4) homotopy commutes by 3.9(iii), (iv). Also it is easy enough to see that (2) and (3) commute. Therefore the product on $A = \bigotimes_{i=1}^{\infty} A_i$ is homotopy associative. Similarly we can show that it is homotopy commutative. And it is trivial to see that $\eta$ is a true unit.

Altogether we have seen that $\mathbf{shcA}$ is closed under countable $\otimes$-products as far as objects go. If $f_i \in \mathbf{shcA}(A_i, B_i)$, then we may easily verify that $\bigotimes_{i=1}^{\infty} f_i \in \mathbf{shcA}$.

4.4. **Proposition.** If $(A, \Phi) \in \mathbf{shcA}$, then $\Phi$ is a map of $\mathbf{shc}$-algebras.

**Proof.** This follows from the homotopy commutativity of $\Phi$ by standard manipulations (using also the homotopy associativity of $\Phi$). $\square$

It follows from this that the iterates $\Phi^{[n]} : A^{\otimes n} \Rightarrow A$ defined by

$$\Phi^{[0]} = \eta, \quad \Phi^{[1]} = A, \quad \Phi^{[n]} = \Phi \circ (\Phi^{[n-1]} \otimes A), \quad n \geq 1,$$

are maps of $\mathbf{shc}$-algebras. Since

$$\Phi^{[n]} \circ (A^{\otimes n-1} \otimes \eta) = \Phi^{[n-1]},$$

we get $\Phi^{[n]} : A^{\otimes \infty} \Rightarrow A$. To see that also $\Phi^\infty$ is a morphism in $\mathbf{shcA}$, we must show that the diagram
Here we let $h_i$ be the homotopy numbered $i$. Then $h_1, h_2, h_4, h_5$ are manufactured from the associativity homotopy for $\Phi$, while $h_3$ comes from the commutativity homotopy for $\Phi$. Thus, by the remarks at the end of 4.1, we can control the restrictions of all the $h_i$, and we easily see that

$$h^{[2]} = h_1 \cup h_2 \cup h_3 \cup h_4 \cup h_5$$

has the correct restriction. Then suppose that $h^{[n]}$ is constructed, and consider the string of homotopies

$$\Phi \circ (\Phi^{[n+1]} \otimes \Phi^{[n+1]}) = \Phi \circ (\Phi \circ (\Phi^{[n]} \otimes A) \otimes \Phi \circ (\Phi^{[n]} \otimes A))$$
$\phi \circ (\phi \otimes \phi) \circ (\phi[n] \otimes A \otimes \phi[n] \otimes A)$

$\phi = (\phi \otimes \phi) \circ (A \otimes A \otimes A) \circ (\phi[n] \otimes A \otimes A)$

$\phi = (\phi \otimes (\phi[n] \otimes \phi[n])) \circ (A \otimes A \otimes A) \circ (A[n] \otimes \phi[n])$

$\phi = (\phi \otimes (\phi[n] \otimes \phi[n])) \circ (A \otimes A \otimes A) \circ (A[n] \otimes \phi[n])$

$\phi \circ (\phi[n] \otimes A) \circ (\phi[n] \otimes \phi[n]) \circ (A[n] \otimes \phi[n])$

$\phi \circ (\phi[n] \otimes A) \circ (\phi[n] \otimes \phi[n]) \circ (A[n] \otimes \phi[n])$

Again let $h_4$ be homotopy number $i$. Then $h_1$ and $h_4$ are constructed from the homotopy in 3.6(i). $h_2$ is

$h_2[2] \circ (\phi[n] \otimes A \otimes \phi[n] \otimes A)$

$h_3$ and $h_5$ are constructed from the homotopy in 3.6(ii) and $h[n]$, respectively. Thus we have full control of the restrictions of each $h_i$ and we easily get

$h_1[n+1] \circ (A \otimes A \otimes A \otimes A) = h[n]$

as desired.

We record this as:

4.5. Proposition. If $A \in \text{shcA}$, then the iterated product $\phi[n] : A[n] \to A$ is a morphism of she algebras, $n \leq \infty$. □

For the next proposition we let

$\left[ \cdot, \cdot \right] = \text{DASH}_h(\cdot, \cdot)$

and if $f : A \Rightarrow B$, we let $[f]$ denote its homotopy class.

4.6. Proposition. For $A_i \in \text{DASH}$, $i = 1, 2, \ldots, n$, $n \leq \infty$, there is a natural transformation

$\chi : \Pi_{i=1}^n \left[ A_i, \cdot \right] \to \left[ \otimes_{i=1}^n A_i, \cdot \right]$

of functors $\text{shcA} \to \text{ENS}$. If $C \in \text{shcA}$ and $f_i : A_i \Rightarrow C$, then

$\chi(\left[ f_i \right]) = \phi[n] \circ (\otimes_{i=1}^n f_i)$

Proof. Obvious. □
4.7. Proposition. The normalized cochain functor into \( \text{DM} \) lifts to a functor \( \text{SS} \rightarrow \text{shcA} \) with \( \Phi : C^* X \otimes C^* X \rightarrow C^* X \) the usual cup product.

Proof. Let \( X \in \text{SS} \). Then from 2.6 we have \( EZ = EZ_X, X \) and

\[
\iota^{EZ} : (CX \otimes CX)^* \rightarrow C^*(X \times X).
\]

Composition with the multiplicative maps

\[
\iota : C^* X \otimes C^* X \rightarrow (CX \otimes CX)^*, \quad d^* : C^*(X \times X) \rightarrow C^* X
\]

\( (d : X \rightarrow X \times X \text{ the diagonal}) \) gives us

\[
\Phi = d^* \iota^{EZ} \circ \iota : C^* X \otimes C^* X \rightarrow C^* X.
\]

Now it is easy to see that \( \Phi \) is the usual cup product (based on the diagonal approximation \( f^* \)). Also, the naturality of \( \iota, E_Z, X \) and \( d \) vis-à-vis maps \( f : X \rightarrow Y \) ensures that \( f^* : C^* Y \rightarrow C^* X \), viewed as shm map, commutes (in fact precisely, not just up to homotopy as required) properly with \( \Phi \) and preserves the unit. Thus we just have to show that \( \eta : R \rightarrow C^* X \) is a true unit, and that \( \Phi \) is associative and commutative up to homotopy.

If \( \pi : X \rightarrow * \) is the projection then

\[
(C^*(\pi \times X), (C\pi \otimes CX)^*) : EZ_{*,X} \rightarrow EZ_{*,X}
\]

in \( \text{TEX} \) and \( (\pi \times X)d : X \rightarrow * \times X \) is the natural homeomorphism. Therefore

\[
\Phi \circ (C^* \pi \otimes C^* X) = d^* \iota^{EZ}_{X,X} \circ \iota \circ (C^\pi \otimes C^X)^* \circ \iota
\]

\[
= d^* \iota^{EZ}_{X,X} \circ (C\pi \otimes CX)^* \circ \iota
\]

\[
= d^* \circ C^*(\pi \times X) \circ t^{EZ_{*,X}} \circ \iota.
\]

Now, identifying \( t^{EZ_{*,X}} \) with \( C^* X \in \text{DA} \subseteq \text{TEX} \), we have

\[
(\pi \times X)d = X,
\]

\[
t^{EZ}_{X,X} = C^* X,
\]

\( \iota = \text{identity} \).

Thus \( \eta = C^* \pi \) is a true left unit for \( \Phi \). Similarly, it is a right unit.

The homotopy associativity of \( \Phi \) amounts to showing that the big outer square in diagram (1) homotopy commutes. We write \( X \) instead of \( CX \) and \( t^X.Y \) for \( t^{EZ_{X,Y}} \). In (1), \( t', t'' \) are the twisting cochains associated with the following two objects in \( \text{TEX} \):

\[
E' = ((X \otimes V)^*, (X \otimes f)^*, (X \otimes \Phi)^*), \quad E'' = ((V \otimes X)^*, (f \otimes X)^*, (\Phi \otimes X)^*).
\]

In diagram (1), the four squares in the corners commute for trivial reasons. The second square in the left-hand column commutes because \( (\iota, \iota) \) is a morphism in \( \text{TEX} \) from \( \pi^* \otimes t^{EZ_{X,X}} \) to \( E' \). The remaining three squares along the edges commute for similar reasons. Thus we just have to show that the centre square homotopy commutes. But this is clear since in \( \text{DASH} \) we have that \( t^X.X.X \circ t'' \) is a homotopy inverse of \( (V_X.X \otimes X)^* V_{X \times X,X,X} \), while \( t^X.X.X \circ t' \) is a homotopy inverse of \( (X \otimes V_X.X)^* V_{X,X \times X}^* \).
and it is a classical result that

\[ X \circ X \circ X \xrightarrow{\iota \circ \iota} (X \circ X) \circ X \xrightarrow{\iota \circ X} (X \times X) \circ X \xrightarrow{d \circ X} X \circ X \circ X \]

(1) \[ X \circ (X \times X) \xrightarrow{\iota} (X \circ (X \times X)) \xrightarrow{\iota \times X} (X \times (X \times X)) \xrightarrow{(d \times X)} (X \times X) \circ X \]

\[ X \circ d^* \xrightarrow{\iota} (X \circ d)^* \xrightarrow{\iota \times X} (X \times d)^* \xrightarrow{d^*} X \]

and it is a classical result that

\[ X \circ X \circ X \xrightarrow{\nabla_{X \times X} \circ X} (X \times X) \circ X \]

\[ X \circ \nabla_{X \times X} \xrightarrow{\nabla_{X \times X \times X}} (X \times X) \times X \]

\[ X \circ (X \times X) \xrightarrow{\nabla_{X \times X \times X}} X \times X \times X \]

commutes.

Finally, for the homotopy commutativity we look at the diagram

\[ X \circ X \circ X \xrightarrow{\iota \circ \iota} (X \circ X) \circ X \xrightarrow{\iota \circ X} (X \times X) \circ X \xrightarrow{d \circ X} X \circ X \ circ \]

\[ \xrightarrow{\tau \circ \tau} \]

\[ X \circ d \circ X \xrightarrow{\tau \circ d \circ \tau} (X \circ d) \circ X \xrightarrow{d \circ \tau} X \circ X \]

where \( \tau, \tau \) are twist maps. The two outer squares commute. Also, \( \iota \times X \times X \) is a homotopy inverse of \( \nabla^* : (X \times X)^* \to (X \circ X)^* \), so the middle square homotopy commutes because of the classically known commutativity of the diagram

\[ X \circ X \xrightarrow{\nabla} X \times X \]

\[ \xrightarrow{\tau} \]

\[ X \circ X \xrightarrow{\nabla} X \times X \]

\[ \square \]
4.8. Proposition. If $A \in \text{shcA}$, then

$$a \cup_1 b = \begin{cases} \Phi([1 \otimes a \otimes 1] + [a \otimes 1 \otimes b]) & \text{if } a, b \in IA, \\ 0 & \text{if } a \text{ or } b \in \text{Im } \eta, \end{cases}$$

defines a $\cup_1$ product in $A$ which satisfies

$$d(a \cup_1 b) + da \cup_1 b + (-1)^{\deg(a)} a \cup_1 db = ab - (-1)^{\deg(a) \deg(b)} ha.$$  

Proof. A straightforward computation. \qed

4.9. If $\text{char } R = 2$, then $\cup_1$ gives the usual operation $\text{Sq}_1 : HA \to HA$, $A \in \text{shcA}$. We have $\text{Sq}_1(\text{cls } a) = \text{cls } (a \cup_1 a)$, so

$$\deg \text{Sq}_1(a) = 2 \deg a + 1.$$ 

If $A = C^* X$, $X \in SS_*$, then $\text{Sq}_1 x = \text{Sq}^{n-1} x$ for $x \in H^n X$.

5. The Eilenberg–Moore spectral sequence

5.1. If $A \in \text{DA}$, while $M$, $N$ are (right and left, respectively) differential graded modules over $A$, then one has the Eilenberg–Moore spectral sequence. It is constructed by choosing a proper projective resolution

$$M \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P_n \leftarrow \ldots$$

of $M$ as an $A$-module. One then forms the bicomplex $P_i \otimes_A N$ and assembles it (by means of the coproduct $\ast$) to a single complex

$$\text{Tot}(P_i \otimes_A N) = \bigoplus P_i \otimes_A N.$$ 

This is filtered by the resolution degree. As such it is complete and cocomplete (see [3]). The resulting spectral sequence will here be called $\text{clEM}(M, A, N)$. We have

$$\text{clEM}^2(M, A, N) \simeq \text{clTor}^{HA}(HM, HN), \quad \text{clEM}^*(M, A, N) \simeq E^0(\text{clTor}^A(M, N)).$$ 

Here the subscript $\text{cl}$ means classical (as opposed to the generalized version we are about to present). If $f \in \text{DA}(A, A')$, while $g : M \to M'$, $h : N \to N'$ are $f$-linear, then one has the map of spectral sequences

$$\text{clEM}(g, f, h) : \text{clEM}(M, A, N) \to \text{clEM}(M', A', N'),$$

which on the $E^2$ and $E^\infty$ levels correspond to $\text{clTor}^H(f, Hg, Hh)$ and $\text{clTor}(g, h)$, respectively. In view of the bicompleteness of the filtrations, we see from [3] that $\text{clTor}(g, h)$ is an isomorphism provided $Hf$, $Hg$ and $Hh$ are isomorphisms.

The references that we know of state the above results only for the case where $A, M, N$ are all positively (or negatively) graded. However, the reader will have no trouble generalizing the results to $\mathbb{Z}$-graded algebras and modules.
5.2. We shall consider \( c_{\text{EM}} \) and \( c_{\text{Tor}} \) as functions on the functor category \( \text{DA}^{-} \). Here \( \rightarrow \) is the category with 3 objects, 0, \( \pm 1 \), and two non-identity morphisms. Thus an object \( (\mu, \nu) \) is a diagram \( B \xleftarrow{\mu} A \xrightarrow{\nu} C \) in \( \text{DA} \), and a morphism \( (g, f, h) : (\mu, \nu) \to (\mu', \nu') \) is a commutative diagram

\[
\begin{array}{ccc}
B & \xleftarrow{\mu} & A & \xrightarrow{\nu} & C \\
\downarrow g & & \downarrow f & & \downarrow h \\
B' & \xleftarrow{\mu'} & A' & \xrightarrow{\nu'} & C'
\end{array}
\]

We let \( c_{\text{EM}}(\mu, \nu) = c_{\text{EM}}(B, A, C) \), where the module structures come via \( \mu \) and \( \nu \). Also use the similar notation for \( c_{\text{Tor}} \).

5.3. **Proposition.** \( c_{\text{EM}} \) and \( c_{\text{Tor}} \) extend (up to natural equivalence) to functors

\[
\text{EM} : \text{DASH} \to \text{SS}, \quad \text{Tor} : \text{DASH} \to \text{M}.
\]

Moreover, there are natural isomorphisms

(1) \( \text{EM}^{2}(\mu, \nu) \simeq \text{Tor}(H\mu, H\nu) \),

(2) \( \text{EM}^{\ast}(\mu, \nu) \simeq E^{0} \text{Tor}(\mu, \nu) \).

Finally, if we have \( (g, f, h) : (\mu, \nu) \to (\mu', \nu') \), with \( Hg, Hf, Hh \) isomorphisms, then \( \text{Tor}(g, f, h) \) and \( \text{EM}(g, f, h) \) are isomorphisms.

**Proof.** By 3.8 we have a functor \( \Omega \beta : \text{DASH} \to \text{DA} \). We take \( \text{EM} = c_{\text{EM}} \circ (\Omega \beta)^{-} \) and \( \text{Tor} = c_{\text{Tor}} \circ (\Omega \beta)^{-} \), i.e., for an object \( B \xleftarrow{\mu} A \xrightarrow{\nu} C \) we have

\[
\text{EM}(\mu, \nu) = c_{\text{EM}}(\Omega \beta \mu, \Omega \beta \nu), \quad \text{Tor}(\mu, \nu) = c_{\text{Tor}}(\Omega \beta \mu, \Omega \beta \nu).
\]

Now, if \( (\mu, \nu) \in \text{DA}^{-} \), then we have a commutative diagram

\[
\begin{array}{ccc}
\Omega \beta B & \xleftarrow{\Omega \beta \mu} & \Omega \beta A & \xrightarrow{\Omega \beta \nu} & \Omega \beta C \\
\downarrow \alpha_{B} & & \downarrow \alpha_{A} & & \downarrow \alpha_{C} \\
B & \xleftarrow{\mu} & A & \xrightarrow{\nu} & C
\end{array}
\]

This gives

\[
c_{\text{EM}}(\alpha_{B}, \alpha_{A}, \alpha_{C}) : \text{EM}(\mu, \nu) = c_{\text{EM}}(\Omega \beta \mu, \Omega \beta \nu) \to c_{\text{EM}}(\mu, \nu).
\]

On the \( E^{2} \) level this map is an isomorphism. Hence it is an isomorphism of spectral sequences (from \( E^{2} \) on). It is also natural with respect to morphisms in \( \text{DA}^{-} \). Thus we have shown that \( \text{EM} \) extends \( c_{\text{EM}} \) up to natural equivalence. Using the comparison theorem mentioned in 5.1, we also see that
$\cl\Tor(\alpha_B, \alpha_A, \alpha_C) : \Tor(\mu, \nu) \to \cl\Tor(\mu, \nu)$
is an isomorphism. It is clearly natural, so $\Tor$ extends $\cl\Tor$ up to natural equivalence.

For general $(\mu, \nu)$ in $\text{DASH}^*$, we have

$$EM^2(\mu, \nu) = \cl EM^2(\Omega\beta\mu, \Omega\beta\nu) \sim \cl\Tor(H\Omega\beta\mu, H\Omega\beta\nu) \sim \cl\Tor(H\mu, H\nu),$$

where, in order to get the last isomorphism, we use the diagram $3.8(1)$ for $f = \mu$ and $f' = \nu$. It is straightforward to verify that the above isomorphism is natural. Similarly,

$$EM^\infty(\mu, \nu) = \cl EM^\infty(\Omega\beta\mu, \Omega\beta\nu) \simeq E^0(\cl\Tor(\Omega\beta\mu, \Omega\beta\nu)) \simeq E^0\Tor(\mu, \nu).$$

Finally, if $Hg, Hf, Hh$ are isomorphisms, then by diagram $3.11(1)$ $o$ are $H\Omega\beta g, H\Omega\beta f$ and $H\Omega\beta h$, so an appeal to the classical result settles the last part of the proposition. \(\square\)

This comparison result is not quite strong enough for the application we have in mind. We need:

**5.4. Theorem.** Suppose that

$$\begin{array}{ccc}
B & \xleftarrow{\mu} & A \\
\downarrow{g} & \swarrow{f} & \downarrow{h} \\
B' & \xleftarrow{\mu'} & A'
\end{array} \xrightarrow{\nu} \begin{array}{ccc}
C \\
\downarrow{h} \\
C'
\end{array}$$

homotopy commutes in $\text{DASH}$, and that $Hg, Hf, Hh$ are isomorphisms. Then

$$EM(\mu, \nu) \simeq EM(\mu', \nu'), \quad \Tor(\mu, \nu) \simeq \Tor(\mu', \nu').$$

**Proof.** We know that $\beta : \text{DASH} \to \text{DC}$ preserves homotopies. Also it is easily seen that homotopies are preserved under $\Omega : \text{DC} \to \text{DA}$. Thus it suffices to prove that for any homotopy commutative diagram

$$\begin{array}{ccc}
B & \xleftarrow{\mu} & A \\
\downarrow{g} & \swarrow{f} & \downarrow{h} \\
B' & \xleftarrow{\mu'} & A'
\end{array} \xrightarrow{\nu} \begin{array}{ccc}
C \\
\downarrow{\nu'} \\
C'
\end{array}$$
in $\text{DA}$ one has

$$\cl EM(\mu, \nu) \simeq \cl EM(\mu', \nu'), \quad \cl\Tor(\mu, \nu) \simeq \cl\Tor(\mu', \nu').$$
Let
\[ K : h \nu \sim \nu' f, \quad H : g \mu \sim \mu' f \]
in $\text{DA}$. We denote by $I^*$ the normalized cochains on the standard 1-simplex. $I^*$ has generators $x_0, x_1, x$ of degrees $0, 0, -1$, respectively. Also,
\[
dx_0 = -x, \quad dx_1 = x, \quad dx = 0, \\
x_0^2 = x_0, \quad x_1^2 = x_1, \quad x_0 x_1 = x_1 x_0 = 0, \quad x_0 x = x, \\
x_1 x = 0, \quad xx_1 = x, \quad x^2 = 0.
\]
If we forget about augmentations (which are anyway not of any importance for $\tau_{\text{EM}}$), then we can easily see that $K$ gives rise to a morphism of differential graded algebras $K' : A \rightarrow I^* \otimes C$ with
\[
K'(u) = -x \otimes K(u) + x_0 \otimes h \nu(u) + x_1 \otimes \nu' f(u).
\]
Similarly one has $H' : A \rightarrow I^* \otimes B$. There are then the commutative diagrams
\[
\begin{array}{cccccc}
B & \xleftarrow{\mu} & A & \xrightarrow{\nu} & C \\
\downarrow{g} & & \downarrow{h} & & \\
B' & \xrightarrow{A} & A' & \xrightarrow{C'} & \\
\downarrow{\pi_0 \otimes B'} & & \downarrow{\pi_0 \otimes C'} & & \\
I^* \otimes B' & \xleftarrow{H'} & A & \xrightarrow{K'} & I^* \otimes C' \\
\downarrow{\pi_1 \otimes R'} & & \downarrow{\pi_1 \otimes C'} & & \\
B' & \xleftarrow{\mu'} & A' & \xrightarrow{\nu'} & C'
\end{array}
\]
(in $\text{DA}$ except for augmentations), where $\pi_i : I^* \rightarrow R$ is projection onto $R x_i = R$. Here all vertical arrows induce isomorphisms in homology, so we get the desired results. □

6. Homotopy classes of shm maps $R [x_1, x_2] \Rightarrow A$

6.1. Let $P_n = R [x_1, x_2, \ldots, x_n]$ be the usual polynomial algebra with vanishing differential. Let $A \in \text{DA}$ come with a $\text{U}_1$ product. If $f : P_n \Rightarrow A$, then $f ([x_i]) \in A$ is a cycle of degree $d_i = \deg x_i$. Also for $i \neq j$ we have cycles
\( c_g(f) = f([x_i \mid x_j]) - f([x_j \mid x_i]) + f([x_i]) \cup_1 f([x_j]) \)
of degree \( d_i + d_j + 1 \). Here, and in the following, we utilize the fact that either \( d_i \) is even or \( \text{char } R = 2 \).

Also, if \( h : f \circ g : P_n \to A \), then
\[
dh([x_i]) = f([x_i]) - g([x_i]) ,
\[
d(h([x_i \mid x_j]) - h([x_j \mid x_i]) - f([x_i]) \cup_1 h([x_j]) - h([x_i]) \cup_1 g([x_j])
\]
\[= h([x_i])h([x_j])) = c_{ij}(f) - c_{ij}(g) .
\]

Thus there are well-defined maps
1. \( c : \{P[x, A] \to H_d(A) \}
2. \( c : \{P[x_1, x_2, A] \to H_{d_1}(A) \oplus H_{d_2}(A) \oplus H_{d_1+d_2+1}(A) \}
\]
given by
\[
c([f]) = \text{cls } f([x]), \quad f : P_1 \to A ,
\]
\[
c([f]) = (\text{cls } f([x_1]), \text{cls } f([x_2]), \text{cls } c_{i_2}(f)), \quad f : P_2 \to A .
\]

6.2. Proposition. The maps 6.1 (1), (2) are bijections. Consequently, any \( f : P_1 \to A \) has a multiplicative representative in its homotopy class.

Proof. The proof occupies the rest of Section 6. It proceeds by replacing \( BP_n \) by the exterior coalgebra
\[
\Lambda_n = \Lambda(y_1, y_2, ..., y_n)
\]
with \( \deg y_i = 1 + d_i \). The Koszul complex \( \Lambda_n \otimes P_n \) with \( d(y_i \otimes 1) = 1 \otimes x_i \) and \( BP_n \otimes P_n \) with twisted differential using \( eP_n : BP_n \to P_n \) (see [6]) both give proper projective resolutions of \( R \) over \( P_n \). Then the standard comparison theorem of differential homological algebra gives \( P_n \)-linear maps
\[
\chi : \Lambda_n \otimes P_n \to BP_n \otimes P_n , \quad \chi' : BP_n \otimes P_n \to \Lambda_n \otimes P_n ,
\]
and \( P_n \)-linear homotopies from \( \chi \chi' \) and \( \chi' \chi \) to the respective identities. Factoring out \( P_n \) and remembering that \( d = 0 \) on \( \Lambda_n \), we get
\[
\lambda : \Lambda_n \to BP_n , \quad \lambda' : BP_n \to \Lambda_n ,
\]
with \( \lambda \lambda' = BP_n \) and \( \lambda' \lambda = \Lambda_n \). Here \( \lambda \) may be taken in \( DC \), e.g. take
\[
\lambda(y_{i_1} \ldots y_{i_k}) = \sum \text{sign } \{x_{i_{m(1)}} \mid \ldots \mid x_{i_{m(k)}}\} ,
\]
with \( \pi \) ranging over the symmetric group on \( \{1, 2, \ldots, k\} \), and obtain \( \chi \) by \( P_n \)-linear extension. Also, \( \lambda' \) commutes with coaugmentations and counits. And if one filters \( BP_n \) by letting

\[
\text{filt } [x_1^{m_1} \ldots x_n^{m_n} | \ldots | x_1^{m_1} \ldots x_n^{m_n}] = \sum v_{ij},
\]

while \( \Lambda_n \) has filt \((y_1, \ldots, y_k) = k\), then \( \lambda, \lambda' \) and the homotopy \( h : BP_n \rightarrow \lambda \lambda' \) all preserve filtration. Finally, \( h \) vanishes on \([\ ]\), i.e., on all elements of filtration 0.

We construct \( \bar{\rho} \in DA(\Omega BP_n, \Omega \Lambda_n) \) as follows: First replace \( h \) by \( h - h \lambda - \lambda \lambda' + \lambda \lambda' h\lambda \). Then

\[
h \lambda = 0, \quad \lambda' h = 0, \quad Dh = BP_n - \lambda \lambda'.
\]

An induction on the above filtration reveals that there is a unique map \( \bar{\rho} : BP_n \rightarrow \Omega \Lambda_n \) with

\[
\bar{\rho} = (\rho \cup \bar{\rho}) h + t_{\lambda} \lambda'.
\]

Moreover, \( \rho \lambda = t_{\lambda n} \), so

\[
D \bar{\rho} = (D \bar{\rho} \cup \bar{\rho} - \bar{\rho} \cup D \bar{\rho}) h + (\bar{\rho} \cup \bar{\rho})(BP^n - \lambda \lambda') + (t_{\lambda n} \cup t_{\lambda n}) \lambda'
\]

\[
= \bar{\rho} \cup \bar{\rho} + (D \bar{\rho} \cup \bar{\rho} - \bar{\rho} \cup D \bar{\rho}) h.
\]

If we evaluate this on \( x \in BP_n \), then on the right-hand side, \( D \bar{\rho} \) will be applied only to elements of filtration lower than the filtration of \( x \) (because \( \bar{\rho} \) vanishes on elements of filtration \( 0 \)). Hence on the right-hand side we may, inductively, assume that \( D \bar{\rho} = \bar{\rho} \cup \bar{\rho} \). Then we get

\[
D \bar{\rho} = \rho \cup \rho,
\]

i.e., \( \rho : P_n \rightarrow \Omega \Lambda_n \). We take \( \rho = \omega(\bar{\rho}) \).

Since \( \rho \Omega(\lambda) t_{\lambda n} = \rho t_{BP_n} \lambda = \bar{\rho} \lambda = t_{\lambda n} \), we have

(1) \( \rho \Omega(\lambda) = \Omega \Lambda_n \).

A similar argument shows that there is a unique map \( \bar{k} : BP_n \rightarrow \Omega BP_n \) with

\[
k = \omega(\bar{k}) h + \eta e,
\]

where \( t = t_{BP_n} \), \( t' = \Omega(\lambda) \). Since \( k \lambda = \eta e \lambda = \eta e \), we see that

\[
(t \cup k - k \cup t') \lambda \lambda' = (t \lambda \cup \eta e - \eta e \cup t') \lambda' = t \lambda \lambda' - t' \lambda \lambda'
\]

\[
= t \lambda \lambda' \Omega(\lambda) \bar{k} \lambda \lambda' = t \lambda \lambda' - \Omega(\lambda) t_{\lambda n} \lambda'
\]

\[
= t \lambda \lambda' - t \lambda \lambda' = 0.
\]

Hence

\[
D \bar{k} = (t \cup t \cup D \bar{k} - t \cup D \bar{k} - D \bar{k} \cup t' - \bar{k} \cup t' \cup t') h + (t \cup \bar{k} - \bar{k} \cup t'),
\]

\[
D \bar{k} = (t \cup t \cup D \bar{k} - t \cup D \bar{k} - D \bar{k} \cup t' - \bar{k} \cup t' \cup t') h + (t \cup \bar{k} - \bar{k} \cup t'),
\]
and an easy induction shows that \( Dk = t \cup k \cup k \cup t' \). Then

\[
(2) \quad k = \omega(k) \cdot \Omega BP_n \sim \Omega(\lambda) \rho.
\]

From (1) and (2) we see that the top row in the commutative diagram (subscript \( h \)’s denote homotopy classes)

\[
\begin{array}{c}
DA_h(\Omega A_n, A) \to (\Omega A) \\
\downarrow \omega \\
T_h(\Lambda_n, A) \to \lambda \\
\end{array}
\]

is an isomorphism. Hence so is \( \lambda^* \).

We then return to \( c \). For \( n = 1 \) or \( 2 \), it factors over \( \lambda^* \) to give \( c' \) defined on \( T_h(\Lambda_n, A) \) by

\[
c'(\{f\}) = \text{cls}(f(y_1)), \quad f \in T(\Lambda_n, A), \]

\[
c'(\{f\}) = (\text{cls}(f(y_1)), \text{cls}(f(y_2)), \text{cls}(f(y_1y_2) + f(y_1) \cup f(y_2)), \quad f \in T(\Lambda_n, A).
\]

It is now sufficient to check that \( c' \) is a bijection. For \( n = 1 \), we leave it to the reader. Let \( n = 2 \), and let \( \omega_1, \omega_2, \omega_12 \) by cycles in \( A \) of relevant degrees. Define \( f : A_n \to A \) by

\[
f(1) = 0, \quad f(y_i) = \omega_i, \quad f(y_1y_2) = \omega_12 - \omega_1 \cup \omega_2.
\]

A straightforward computation shows that \( f \in T(\Lambda_n, A) \). Since clearly

\[
c'(f) = (\text{cls} \omega_1, \text{cls} \omega_2, \text{cls} \omega_12),
\]

\( c' \) is onto. To see that \( c' \) is injective, let \( f_i : A_2 \to A \) be given along with \( \xi_i \in A \) such that

\[
d\xi_i = f_0(y_i) - f_1(y_i), \quad i = 1, 2,
\]

\[
d\xi_12 = f_0(y_1y_2) - f_1(y_1y_2) + f_0(y_1) \cup f_0(y_2) - f_1(y_1) \cup f_1(y_2).
\]

Then one easily verifies that \( h : f_0 \simeq f_1 \), where

\[
h(1) = 1, \quad h(y_i) = \xi_i, \quad i = 1, 2
\]

\[
h(y_1y_2) = \xi_12 + f_1(y_1) \cup \xi_2 + f_1(y_2) + \xi_1 \xi_2. \quad \Box
\]
7. The algebraic collapse theorem

7.1. Theorem. Let \( B \leftarrow A \rightarrow C \) be a diagram in \( \text{shcA} \). Assume that:

1. \( HA, HB, HC \) are polynomial algebras with at most countably many generators;
2. if \( \text{char } R = 2 \), then \( \text{Sq}_1 \) vanishes on the polynomial generators of \( HB \) and \( HC \).

Then \( \text{Tor}(H\mu, H\nu) \approx \text{Tor}(\mu, \nu) \), \( \text{EM}(\mu, \nu) \) collapses, and there is no additive extension problem.

**Proof.** By 5.4, it suffices to construct a homotopy commutative diagram

\[
\begin{array}{ccc}
HB & \overset{H\mu}{\longrightarrow} & HA \\
\downarrow{\beta_B} & & \downarrow{\beta_A} \\
B & \overset{\mu}{\longrightarrow} & A \\
& \downarrow{\nu} & \downarrow{\nu} \\
& C & HC
\end{array}
\]

in \( \text{DASH} \), with \( H\beta_X = HX : HX \to HX \), \( X = A, B, C \). \( \square \)

7.2. We first construct \( \beta_X \), following [15].

Let

\[
HX = R[x_1, \ldots, x_n] = \bigotimes_{i=1}^{n} R[x_i], \quad n \leq \infty.
\]

We then have

\[
\beta_X = \Phi[n] \circ \bigotimes_{i=1}^{n} \beta_i,
\]

where \( \beta_i \) is the multiplicative map \( R[X_i] \to X \) sending \( x_i \) to some fixed representative \( \xi_i \) for the class \( x_i \). It is then obvious that

\[
H\beta_X = HX : HX \to HX.
\]

Thus we just have to show that (1) homotopy commutes. This is the difficult part of the present paper, and it was to this end we put up most of the preceding results on shm maps.

7.3. Lemma. \( \beta_X : HX \to X \) is a morphism of \( \text{shc} \) algebras if and only if \( \text{char } R \neq 2 \) or \( \text{char } R = 2 \) and \( \text{Sq}_1(x_i) = 0 \) for all \( i \).

**Proof.** It is easily seen that \( \beta_X \in \text{shcA}(HX, X) \) if and only if \( \beta_i \in \text{shcA}(R[X_i], X) \) for all \( i \), i.e., if and only if
and we have proved the lemma. EI

If \( X = C'(U) \) for some \( 1 \in SS_{\pi} \), then \( Sq_i(x) = 0 \) for all \( i \), since \( Sq_i \) vanishes on all of \( \mathbb{M} \times Y \), because of the Cartan formula. For general \( X \in schA \), we doubt that there is any Cartan formula for \( Sq_i \) (cf. [9]), so we stick to the condition \( Sq_i = 0 \) on the polynomial generators.

7.4. We can now prove Theorem 7.1. We let

\[
 HA = R[x_1, \ldots, x_n], \quad n < \infty, \quad HB = R[y_1, \ldots, y_m], \quad m < \infty.
\]

We then have the diagram
with $\chi$ the natural transformation from 4.6. This diagram commutes since both $\beta_B$ and $\nu$ are in $\mathfrak{seA}$.

We let $[\nu_i] \in [R[x_i], HB]$ be represented by the restriction of $H\nu$ to $R[x_i]$. Then

$$\chi(\{[\nu_i]\}) = H\nu, \quad (\beta_B)_* \chi(\{[\nu_i]\}) = [\beta_B \circ H\nu].$$

Also, let $\beta_i : R[x_i] \to A$ be the multiplicative map sending $x_i$ to $\xi_i$ representing $x_i$. Then

$$\chi(\{[\beta_i]\}) = [\beta_A], \quad \nu_* \chi(\{[\beta_i]\}) = [\nu \circ \beta_A].$$

Thus we see that

$$\beta_B \circ H\nu \sim \nu \circ \beta_A,$$

provided $\nu_* [\beta_i] = (\beta_B)_* [\nu_i]$. These homotopy classes are represented by the $\text{shm}$ maps

$$R[x_i] \xrightarrow{\nu_i} HB \xrightarrow{\beta_B} B,$$

$$R[x_i] \xrightarrow{\beta_i} A \xrightarrow{\nu} B,$$

so by 6.2 we can check whether they are equal by comparing $\beta_B \circ \nu_i([x_i])$ and $\nu \circ \beta_i([x_i])$ in $HB$. But both of these cycles represent $H(\nu)(x_i)$, so we are through. $\Box$

7.5. Although at present we cannot construct morphisms of she algebras $HX \to X$ unless $iHX$ is very nice ($\cong$ polynomial algebra), we note that in the proof of homotopy commutativity of 7.1(1) we did not use any information on how $\beta_B$ and $\beta_C$ were constructed, but only the fact that they are in $\mathfrak{seA}$. Thus we have
7.6. Proposition. Suppose that for the diagram
\[ C \xleftarrow{\mu} A \xrightarrow{\nu} B \]
in \text{shcA} we have
(i) \(HA\) is a polynomial algebra with at most countably many generators;
(ii) there is \( \beta_X \in \text{shcA}(HX, X) \) such that \( H\beta_X \) is an isomorphism for \( X = B, C \).
Then \( \text{EM}(\mu, \nu) \) collapses.

\textbf{Proof.} Construct \( \beta_A \) as above. Also replace \( \beta_X \) by \( \beta_X \circ H(\beta_X)^{-1} \), \( X = B, C \). Then the proof given earlier immediately generalizes to show that

\[ H^C \xleftarrow{H\mu} H^A \xrightarrow{H\nu} H^B \]
\[ C \xleftarrow{\mu} A \xrightarrow{\nu} B \]

homotopy commutes in \textbf{DASH}. \( \square \)

8. Geometric applications

8.1. The main application is to the cohomology of fibre squares. Let
\[ E' \xrightarrow{f} E \]
\[ X \xrightarrow{p} B \]
be a fibre square. Then we have
\[ p^* : C^* B \to C^* E \]
\[ f^* : C^* B \to C^* X \]
in \text{shcA}. We assume that \( \pi_1 B \) acts trivially on \( H^*(\text{fibre of } p) \); then
\[ \text{Tor}(p^*, f^*) \simeq H^*(E') \]
as is well known. As an immediate corollary we then get:

8.2. Theorem. In the above fibre square, assume that:
(i) \( \pi_1(R) \) acts trivially on the cohomology of the fibre of \( p \);
(ii) \( H^* B, H^* E, H^* X \) are polynomial algebras in at most countably many variables;
(iii) if \( \text{char } R = 2 \), then \( \text{Sq}_1 \) vanishes on \( H^* E \) and \( H^* X \).
Then the Eilenberg-Moore spectral sequence collapses.
As mentioned in the Introduction, this covers a variety of the hitherto known collapse results for the Eilenberg–Moore spectral sequence applied to homogeneous spaces.

8.3. The functoriality of Tor on sheaves also gives rise to the product in the geometric Eilenberg–Moore spectral sequence. If \( p : E \to B \) and \( f : X \to B \) are maps of spaces, then we have the commutative diagram

\[
\begin{array}{c}
C^* X \otimes C^* X \leftarrow f^* \otimes f^* \quad C^* B \otimes C^* B \leftarrow \mu \circ \mu \quad C^* E \otimes C^* E
\end{array}
\]

\[
\begin{array}{c}
C^* X \leftarrow f^* \quad C^* B \leftarrow \mu \quad C^* E
\end{array}
\]

giving

(1) \( \text{Tor}(\Phi, \Phi, \Phi) : \text{Tor}(p^* \otimes p^*, f^* \otimes f^*) \to \text{Tor}(p^*, f^*) \).

This can be composed with the classically defined Kunneth type map

(2) \( \text{Tor}(p^*, f^*) \otimes \text{Tor}(p^*, f^*) \to \text{Tor}(p^* \otimes p^*, f^* \otimes f^*) \)

to give the product. It is easy to see that this product coincides with the one defined classically by using instead of (1).

(3) \( \text{cl} \text{Tor}(d^*, d^*, d^*) \quad \text{cl} \text{Tor}(V^*, V^*, V^*) : \text{cl} \text{Tor}(\iota, \iota, \iota) \).

with \( d \) diagonals, \( V \) from the Eilenberg–Zilber theorems, and \( \iota : C^* X \otimes C^* Y \to (CX \otimes CY)^* \). In fact,

\[
\Phi = d \circ t^{FZ} \circ \iota, \quad V^* \circ t^{FZ} = \text{identity},
\]

so

\[
\text{Tor}(V^*, V^*, V^*)^{-1} = \text{Tor}(t^{FZ}, t^{FZ}, t^{FZ}).
\]

Therefore the product is associative. But this can also be seen directly from our definition by noticing that \( \Phi \) is naturally homotopy associative, and that we have the following:
8.4. Lemma. If the diagrams

\[
\begin{array}{ccc}
C & \xleftarrow{\mu} & A \xrightarrow{\nu} B \\
\downarrow g_i & & \downarrow h_i \\
C' & \xleftarrow{\mu'} & A' \xrightarrow{\nu'} B'
\end{array}
\]

\((i = 0, 1)\) in DASH are commutative and homotopic in the sense that we have

\[
G : g_0 \simeq g_1, \quad H : h_0 \simeq h_1, \quad F : f_0 \simeq f_1,
\]

with \(G \circ \mu = \mu' \circ F, \quad \nu' \circ F = H \circ \nu,\) then

\[
\text{Tor}(g_0, f_0, h_0) = \text{Tor}(g_1, f_1, h_1) : \text{Tor}(\mu, \nu) \rightarrow \text{Tor}(\mu', \nu').
\]

\textbf{Proof.} As in the proof of 5.4, we get maps \(G', F', H'\) (in DA except for augmentations)

\[
G' : \Omega BC \rightarrow I^* \otimes \Omega BC', \quad F' : \Omega BA \rightarrow I^* \otimes \Omega BA', \quad H' : \Omega BB \rightarrow I^* \otimes \Omega BB',
\]

with

\[
(\pi_i \otimes \Omega BC')G' = \Omega \beta g_i, \quad (\pi_i \otimes \Omega BA')F' = \Omega \beta f_i, \quad (\pi_i \otimes \Omega BB')H' = \Omega \beta h_i.
\]

Since also

\[
G' \Omega \beta(\mu) = \Omega \beta(\mu') F', \quad \Omega \beta(\nu') F' = H' \Omega \beta(\nu),
\]

as one easily verifies from \(G \circ \mu = \mu' \circ F\) and \(\nu' \circ F = H \circ \nu,\) it suffices to prove that

\[
(1) \quad \text{Tor}(\pi_i \otimes \Omega BC', \pi_i \otimes \Omega BA', \pi_i \otimes \Omega BB')
\]

is independent of \(i.\) But these maps are isomorphisms, due to the comparison theorem mentioned in 5.1, and both have \(\text{Tor}(\eta \otimes \Omega BC', \eta \otimes \Omega BA', \eta \otimes \Omega BB')\) as right inverses, so they do coincide.

9. Remarks about product structures in a purely algebraic set up

9.1. If we have a diagram

\[
\begin{array}{ccc}
C & \xleftarrow{\mu} & A \xrightarrow{\nu} B \\
\downarrow \otimes & & \downarrow \otimes \\
C & \xleftarrow{\mu} & A \xrightarrow{\nu} B
\end{array}
\]

in sheA, then

\[
C \otimes C \xleftarrow{\mu \otimes \mu} A \otimes A \xrightarrow{\nu \otimes \nu} B \otimes B
\]

\[
\downarrow \otimes \downarrow \otimes \\
C & \xleftarrow{\mu} & A \xrightarrow{\nu} B
\]
homotopy commutes in DASH. Hence it still homotopy commutes after applying $\Omega \beta$ to it. Choosing homotopies

\[ H : \Omega \beta(\Phi) \Omega \beta(\nu \otimes \nu) \simeq \Omega \beta(\nu) \Omega \beta(\Phi), \]

\[ K : \Omega \beta(\Phi) \Omega \beta(\mu \otimes \mu) \simeq \Omega \beta(\mu) \Omega \beta(\Phi), \]

in DA, we can proceed like in the proof of 5.4 to get an induced map

\[ \text{Tor}(\mu \otimes \mu, \nu \otimes \nu) \to \text{Tor}(\mu, \nu). \]

Presumably it depends on the choice of $H$ and $K$, though we have not worked out any example to prove the actual dependence.

Also the classical Künneth type map gives us (since $\text{Tor}(\mu, \nu) = \text{clTor}(\Omega \beta(\mu), \Omega \beta(\nu))$)

\[ \text{Tor}(\mu, \nu) \simeq \text{Tor}(\mu, \nu) \to \text{clTor}(\Omega \beta(\mu) \otimes \Omega \beta(\nu), \Omega \beta(\mu) \otimes \Omega \beta(\nu)). \]

Thus to manufacture a product on $\text{Tor}(\mu, \nu)$, we just have to produce a suitable map

\[ \text{clTor}(\Omega \beta(\mu) \otimes \Omega \beta(\mu), \Omega \beta(\nu) \otimes \Omega \beta(\nu)) \to \text{clTor}(\Omega \beta(\mu \otimes \mu), \Omega \beta(\nu \otimes \nu)). \]

For this purpose we consider the diagram

\[ C^\sim \otimes C^\sim \leftarrow A^\sim \otimes A^\sim \leftarrow B^\sim \otimes B^\sim \]

\[ \begin{array}{ccc}
C \otimes C & \leftarrow & A \otimes A \\
\uparrow \kappa_{C,C} & & \uparrow \kappa_{A,A} \\
(C \otimes C) & \leftarrow & (A \otimes A) \\
\downarrow & & \downarrow \\
(B \otimes B) & \rightarrow & (B \otimes B) \\
\end{array} \]

\[ \begin{array}{ccc}
C \otimes C & \leftarrow & A \otimes A \\
\uparrow \omega(\mu) & & \uparrow \omega(\nu) \\
C \otimes C & \leftarrow & (A \otimes A) \\
\downarrow & & \downarrow \\
(B \otimes B) & \rightarrow & (B \otimes B) \\
\end{array} \]

where for abbreviation we write $A^\sim = \Omega BA$ and $\mu^\sim = \Omega \beta(\mu)$. Now $k_{X,Y}$ is always an isomorphism on homology, so, if the diagram commutes, we may take the inverse of $\text{clTor}(k, k, k)$ as our map (3). (4) however, does not commute, so we must be a little bit more careful. We consider then

\[ \begin{array}{ccc}
C \otimes C & \leftarrow & A \otimes A \\
\uparrow \omega(\mu) \otimes \omega(\mu) & & \uparrow \omega(\nu) \otimes \omega(\nu) \\
C \otimes C & \leftarrow & (A \otimes A) \\
\downarrow & & \downarrow \\
(B \otimes B) & \rightarrow & (B \otimes B) \\
\end{array} \]

\[ \begin{array}{ccc}
C \otimes C & \leftarrow & A \otimes A \\
\uparrow \mu \otimes \mu & & \uparrow \nu \otimes \nu \\
C \otimes C & \leftarrow & (A \otimes A) \\
\downarrow & & \downarrow \\
(B \otimes B) & \rightarrow & (B \otimes B) \\
\end{array} \]

It is easily seen that the top rectangles commute, i.e., $\alpha_C \mu^\sim = \omega(\mu)$ etc. Also,

\[ (\alpha_C \otimes \alpha_C) k_{C,C} = \alpha_{C \otimes C}. \]
so\n\[ (\alpha_C \otimes \alpha_C) k_{C,C}(\mu \otimes \mu) = \alpha_C \otimes \alpha_C(\mu \otimes \mu) = \omega(\mu \otimes \mu) = (\omega(\mu) \otimes \omega(\mu)) k_{A,A}, \]
i.e., the outer two squares commute. Finally, $\alpha_C \otimes \alpha_C$ induces an isomorphism in homology. Thus for our map (3) we may take the composition
\[ \text{cl} \text{Tor}(\alpha_C \otimes \alpha_C) k_{C,C} \cdot k_{A,A} \cdot (\alpha_B \otimes \alpha_B) k_{B,B} \cdot \text{cl} \text{Tor}(\alpha_C \otimes \alpha_C, A^\wedge \otimes A^\wedge, \alpha_B \otimes \alpha_B). \]
The composition of (1), (2) and (3) now gives some sort of a product. We have no specific applications of this in mind, so we have not tried to investigate the properties of this product. Presumably they are relatively bad, because of the dependence of (1) on our choice of the homotopies. We conjecture that one could get a decent theory by including in the structure of an she algebra also a fixed associativity homotopy (and commutativity homotopy). Presumably a morphism of she algebra would then have to include an explicit homotopy $f \circ \Phi \sim \Phi \circ (f \otimes f)$. However, for the present these ideas will not be pursued.

Appendix. shm maps as maps with higher homotopies

Any shm map $\tau : A \Rightarrow B$ determines and is determined by its “components”
\[ t_j = t_{ij} \alpha \otimes_j : (IA)^{\otimes_j} \to B. \]
In terms of components, the three defining properties $t \eta = 0, t \eta = 0$ and $D t = t \cup t$ take the form
\[ e t_j = 0, \quad t_0 = 0, \]
\[ d_B f_j + (-1)^j f_j d_{IA^{\otimes_j}} = \]
\[ = \sum_{\nu=1}^{j} (-1)^\nu [\phi_B(f_{\nu} \otimes f_{j-\nu}) - f_{j-1}(IA^{\otimes j-1} \otimes I \phi_A \otimes IA^{\otimes j-\nu-1})]. \]
For $j = 1, 2$, these identities read
\[ d_B f_1 - f_1 d_{IA} = 0, \]
\[ d_B f_2 + f_2 d_{IA^{\otimes 2}} = f_1(l \phi_A) - \phi_B(f_1 \otimes f_2). \]
Thus $f_1 : IA \Rightarrow B$ is in DM, and $f_2$ is a homotopy measuring the deviation of $f_1$ from multiplicativity. For $j \geq 3$, $f_j$ have interpretations as higher coherence homotopies, but they are not very illuminating.

One may also define homotopies in terms of “components”, and $\otimes$ may be defined in such terms. However, it seems that nothing is really gained hereby, so we spare the reader and ourselves these added troubles.
References