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Enumeration of unlabelled graphs with specified degree parities

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Abstract

This paper gives a generating function for unlabelled graphs of order n. The coefficient of each monomial in this function shows the number of unlabelled graphs with given size and the number of odd vertices. Furthermore, the numerical examples are given for $1 \leqslant n \leqslant 9$.

1. Introduction

In this paper we consider enumeration problems of finite undirected graphs without multiple edges or loops. In a graph, a vertex of even degree is called an even vertex and a vertex of odd degree is called an odd vertex. A graph whose vertices are all even is said to be even. We refer to a graph with order n and size (the number of edges) N as an (n,N)-graph. If an (n,N)-graph is rooted at a specified vertex of degree d, it is referred to as a rooted $(d)(n,N)$-graph. We shall first consider the enumeration of unlabelled even rooted $(d)(n,N)$-graphs and then from this enumeration we shall derive a formula for the number of unlabelled (n,N)-graphs with d odd vertices.

Tazawa [5] got a generating function which tells us the number of graphs of order n with d odd vertices. This was derived from Theorem 2 ([1, p. 858]). Read and Robinson [4] gave a generating function which tells us the number of labelled (n,N)-graphs with d odd vertices and, Tazawa and Shirakura [6] gave an alternative counting formula of finding the number. If we tried to resolve the problem treated in this paper, using a modified theorem obtained by adding any information on edges to Theorem 2 in [1], it seems to be very difficult. So we will resolve this problem along Liskovec method [3]. The last section shows the numerical examples for $1 \leqslant n \leqslant 9$.

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2. Main theorem

Let p be a positive integer. Then a p-tuple of nonnegative integers, $(j) = (j_1, j_2, \ldots, j_p)$, satisfying $\sum_{r=1}^{p} r j_r = p$ is called a partition of p. A partition $(j) = (j_1, j_2, \ldots, j_p)$ of p may sometimes be written as $(j) = (1^{j_1} 2^{j_2} \cdots p^{j_p})$. In this paper the following notations are used: For a partition (j) of p,

$$|j| = \sum_{r=1}^{p} r j_r (= p), \qquad j! = \prod_{r=1}^{p} j_r!,$$

$$\pi(j) = \prod_{r=1}^{p} r^{j_r}, \qquad s(j) = \sum_{r=1}^{p} j_r.$$

Let $(j) = (j_1, j_2, \ldots, j_p)$ and $(k) = (k_1, k_2, \ldots, k_q)$ be partitions of p and q, respectively, and consider the following three functions:

$$\beta^+((j);x) = \prod_{1 \leqslant \ell < m \leqslant p} (1 + x^{[\ell,m]})^{(\ell,m) j_\ell j_m}$$

$$\times \prod_{\ell=1}^{p} (1 + x^\ell)^{\ell(j_\ell(j_\ell - 1)/2) + [(\ell-1)/2] j_\ell} \prod_{2|\ell} (1 + x^{\ell/2})^{j_\ell}, \tag{2.1}$$

$$\beta^\pm((j),(k);x) = \prod_{\ell=1}^{p} \prod_{m=1}^{q} (1 + (-1)^{[\ell,m]/m} x^{[\ell,m]})^{(\ell,m) j_\ell k_m}, \tag{2.2}$$

$$\beta^-((k);x) = \prod_{1 \leqslant \ell < m \leqslant q} (1 + (-1)^{(([\ell,m]/\ell) + [\ell,m]/m)} x^{[\ell,m]})^{(\ell,m) k_\ell k_m}$$

$$\times \prod_{m=1}^{q} (1 + x^m)^{m(k_m(k_m - 1)/2) + [(m-1)/2] k_m} \prod_{2|m} (1 - x^{m/2})^{k_m}, \tag{2.3}$$

where $[\ell, m]$ and (ℓ, m) denote the l.c.m. and g.c.d., respectively, and $[r]$ is the greatest integer not exceeding r. Furthermore, in the case of $p = 0$ or $q = 0$, the corresponding expressions are defined as 1. Then we have:

Theorem 1. *Let n be a positive integer. Then the generating function having the number of unlabellled (nonisomorphic) even rooted $(d)(n+1, N)$-graphs as the coefficient of $x^N y^d$ for a nonnegative even integer d is given by*

$$E(n+1, x, y) = \frac{1}{2} \sum_{p+q=n} \sum_{\substack{(j) \\ |j|=p}} \sum_{\substack{(k) \\ |k|=q}} \frac{2^{-s(j)}}{j! \pi(j)} \frac{2^{-s(k)}}{k! \pi(k)} Q(L+R), \tag{2.4}$$

where

$$
\begin{aligned}
Q &= \beta^{+}((j));x)\beta^{\pm}((j),(k);x)\beta^{-}((k);x),\\
L &= \prod_{\ell=1}^{p}(1+(xy)^{\ell})^{j_\ell}\prod_{m=1}^{q}(1-(xy)^{m})^{k_m}, \qquad (2.5)\\
R &= \prod_{\ell=1}^{p}(1+(-1)^{\ell}(xy)^{\ell})^{j_\ell}\prod_{m=1}^{q}(1-(-1)^{m}(xy)^{m})^{k_m}.
\end{aligned}
$$

For example, we have

$$E(4,x,y) = 1 + x^3 + x^3y^2 + x^4y^2. \qquad (2.6)$$

The proof of this theorem will be given in the next section. Now, for a nonnegative even integer d let $\mathcal{M}^{(d)}_{n,N}$ be the set of labelled (n,N)-graphs with d odd vertices and let $\mathcal{L}^{(d)}_{n+1,N+d}$ be the set of labelled even rooted $(d)(n+1,N+d)$-graphs, where the root in $\mathcal{L}^{(d)}_{n+1,N+d}$ is v_0. We shall establish a 1–1 correspondence between $\mathcal{M}^{(d)}_{n,N}$ and $\mathcal{L}^{(d)}_{n+1,N+d}$. Consider any graph G of $\mathcal{M}^{(d)}_{n,N}$. Next we add to G a new vertex v_0. Finally, we construct a graph G' from G and v_0 by specifying that v_0 is adjacent to each of odd vertices of G. Then G' is one which belongs to $\mathcal{L}^{(d)}_{n+1,N+d}$. It is easily seen that this correspondence is 1–1 and that every labelled (n,N)-graphs with d odd vertices can be obtained in this way from some graph in $\mathcal{L}^{(d)}_{n+1,N+d}$. If two labelled graphs of $\mathcal{M}^{(d)}_{n,N}$ are isomorphic, then the corresponding two labelled graphs of $\mathcal{L}^{(d)}_{n+1,N+d}$ are also isomorphic, and vice versa. Hence we have:

Theorem 2. *Let n be a positive integer. Then the number $a^{(d)}_{n,N}$ of unlabelled (n,N)-graphs with d odd vertices is equal to the coefficient of $x^{N+d}y^d$ in the polynomial $E(n+1,x,y)$.*

Let $N_n(x,y)$ be the polynomial having $a^{(d)}_{n,N}$ as the coefficient of x^Ny^d. Then $N_n(x,y)$ is a generating function for prescribed unlabelled graphs, as seen in the just above theorem. If we consider $n=3$ as an example, the coefficient of a term x^4y^2 in (2.6) gives the number of (3,2)-graphs with 2 odd vertices. That is, the generating function $N_3(x,y)$ is $1+x^3+xy^2+x^2y^2$. Note that if we set $y=0$ in $N_n(x,y)$, then Theorem 2 is reduced to the result given by Liskovec [3].

3. The proof of Theorem 1

We take the group A consisting of all permutations acting on a set $V=\{v_0,v_1,\ldots,v_n\}$ which fix v_0. For $g\in A$ we denote by $e(g,N,d)$ the number of even rooted $(d)(n+1,N)$-graphs which are fixed by g, where v_0 is the root vertex. Then Burnside's Lemma shows that the number, $E(n+1,N,d)$, of even rooted $(d)(n+1,N)$-graphs is given by

$$E(n+1,N,d)=\frac{1}{n!}\sum_{g\in A}e(g,N,d). \tag{3.1}$$

If we consider the generating function

$$e(g,x,y)=\sum_{d=0}\sum_{N=0}e(g,N,d)x^N y^d \tag{3.2}$$

for even rooted graphs of order $n+1$ which are fixed by g, $E(n+1,x,y)$ can be written as

$$E(n+1,x,y)=\frac{1}{n!}\sum_{g\in A}e(g,x,y). \tag{3.3}$$

Therefore, our goal is to find a formula for $e(g,x,y)$.

As seen in [3], we consider a graph in which the number $+1$ or -1 is assigned to each vertex. We call such a graph a vertex-signed graph. Each vertex can be referred to as 'positive' or 'negative' depending on the sign assigned to the vertex. For $g\in A$ let Γ be a vertex-signed graph fixed by g. Note that in Γ the allocation of the numbers on the vertices is preserved under g. This implies that all vertices in each cycle of g have the same sign.

Let c_1 and c_2 be cycles of g and let Γ_{c_1,c_2} be the maximal bipartite subgraph of Γ such that its partite sets are c_1 and c_2. Then in Γ_{c_1,c_2} we denote the degree of each vertex in c_1 and the degree of each vertex in c_2 by $d_{c_2}(c_1)$ and $d_{c_1}(c_2)$, respectively. We assign the number

$$\varepsilon_\Gamma(c_1,c_2)=\begin{cases}\varepsilon(c_1)^{d_{c_2}(c_1)}\varepsilon(c_2)^{d_{c_1}(c_2)} & \text{for } c_1\neq c_2,\\[2ex] \varepsilon(c_1)^{d_{c_1}(c_1)} & \text{for } c_1=c_2\end{cases} \tag{3.4}$$

to the edge-set of Γ_{c_1,c_2}, where $\varepsilon(c_i)$ is the sign of a vertex in $c_i (i=1,2)$. Put

$$\varepsilon(\Gamma)=\prod_{\{c_1,c_2\}}\varepsilon_\Gamma(c_1,c_2) \tag{3.5}$$

which is called the sign of Γ, where the product Π is over all unordered pairs of cycles c_1,c_2 in g (c_1 and c_2 are not necessarily distinct). Then it is easily observed that the sign of Γ can be written as

$$\varepsilon(\Gamma)=\prod_{c_i}\varepsilon(c_i)^{d(c_i)}, \tag{3.6}$$

where $d(c_i)$ is the degree of each vertex of c_i in Γ, since $d(c_i)=\sum_{c_j}d_{c_j}(c_i)$. Liskovec [3] gave the following lemma.

Lemma 3.

$$\sum_{\{\varepsilon(c_i)=\pm 1\}} \varepsilon(\Gamma) = \begin{cases} 2^{s(g)} & \text{if } \Gamma \text{ is an even graph,} \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

where the summation is over all possible allocations of +1 or −1 on the vertices which are preserved by g.

Let $H(g,N,d)$ be the set of rooted $(d)(n+1,N)$-graphs which are fixed by g. It follows from Lemma 3 that the following lemma holds. This is also a slight modification of the corresponding result in [3].

Lemma 4.

$$e(g,N,d) = 2^{-s(g)} \sum_{\{\varepsilon(c_i)=\pm 1\}} \sum_{\Gamma \in H(g,N,d)} \varepsilon(\Gamma). \tag{3.8}$$

Now consider a vertex-signed rooted $(d)(n+1,N)$-graph Γ which is fixed by $g \in A$. Let W be the set of positive vertices of Γ. Of course, $g(W) = W$. Put $W_1 = W$ and $W_2 = V - W$. We denote by $f_i = g|W_i$ the permutation on W_i obtained by restricting g to $W_i (i=1,2)$. Then with respect to the subgraphs $\Gamma[W_i]$ induced by $W_i, f_i(\Gamma[W_i]) = \Gamma[W_i]$ holds for $i = 1,2$. Moreover, for the maximal bipartite subgraph Γ_{W_1,W_2} of Γ such that its partite sets are W_1 and W_2, we have

$$\varepsilon(\Gamma) = \varepsilon(\Gamma_{W_1,W_2})\varepsilon(\Gamma[W_2]). \tag{3.9}$$

Therefore, summing $\varepsilon(\Gamma)$ over all Γ's in $H(g,N,d)$ with the positive vertex set W, we obtain

$$\sum_{\Gamma \in H(g,N,d)} \varepsilon(\Gamma) = \sum_{\substack{K+L+M=N \\ a+b=d}} \gamma^+(f_1,K,a)\gamma_1^{\pm}(f_1,f_2,L,b)\gamma^-(f_2,M) + \sum_{\substack{K+L+M=N \\ a+b=d}} \gamma^+(f_1,K)\gamma_2^{\pm}(f_1,f_2,L,b)\gamma^-(f_2,M,a), \tag{3.10}$$

where the first sum of the right-hand side is for the case that the root vertex v_0 is in W_1 and the second one is for the case that it is in W_2, and where

- $\gamma^+(f_1,K,a)$: The number of rooted $(a)(|W_1|,K)$-graphs fixed by f_1. Here $|W_1|$ means the cardinality of W_1.
- $\gamma^+(f_1,K)$: The number of $(|W_1|,K)$-graphs fixed by f_1.
- $\gamma_i^{\pm}(f_1,f_2,L,b)$: The sum of signs of rooted (b) bipartite graphs fixed by g in which the partite sets are W_1 and W_2 and in which the root vertex v_0 is in W_i for $i = 1,2$.
- $\gamma^-(f_2,M,a)$: The sum of signs of rooted $(a)(|W_2|,M)$-graphs fixed by f_2.
- $\gamma^-(f_2,M)$: The sum of signs of $(|W_2|,M)$-graphs fixed by f_2.

By applying (3.10) to Lemma 4, the equality

$$e(g,N,d) = 2^{-s(g)}(Q_1 + Q_2) \tag{3.11}$$

is obtained, where

$$Q_1 = \sum\nolimits^{(1)} \sum_{\substack{K+L+M=N \\ a+b=d}} \gamma^+(f_1,K,a)\gamma_1^{\pm}(f_1,f_2,L,b)\gamma^-(f_2,M),$$

$$Q_2 = \sum\nolimits^{(2)} \sum_{\substack{K+L+M=N \\ a+b=d}} \gamma^+(f_1,K)\gamma_2^{\pm}(f_1,f_2,L,b)\gamma^-(f_2,M,a) \tag{3.12}$$

and the summations $\sum^{(1)}$ and $\sum^{(2)}$ are taken over all subsets W (sets of positive vertices) of V satisfying $g(W) = W$ provided that the former is for $v_0 \in W$ and the latter is for $v_0 \notin W$.

Now we introduce the generating functions $\gamma^+(f_1,x,y), \gamma^+(f_1,x), \gamma_i^{\pm}(f_1,f_2,x,y)$ $(i = 1,2), \gamma^-(f_2,x,y)$ and $\gamma^-(f_2,x)$ for $\gamma^+(f_1,K,a), \gamma^+(f_1,K), \gamma_i^{\pm}(f_1,f_2,L,b)$, $\gamma^-(f_2,M,a)$ and $\gamma^-(f_2,M)$, respectively. Then $e(g,x,y)$ in (3.3) becomes

$$e(g,x,y) = 2^{-s(g)}\{ \sum\nolimits^{(1)} \gamma^+(f_1,x,y)\gamma_1^{\pm}(f_1,f_2,x,y)\gamma^-(f_2,x) + \sum\nolimits^{(2)} \gamma^+(f_1,x)\gamma_2^{\pm}(f_1,f_2,x,y)\gamma^-(f_2,x,y)\}. \tag{3.13}$$

Let c_1 and c_2 be cycles of g whose lengths are ℓ and m, respectively. We consider the 2-subsets which have one vertex in each of those cycles and we define

$$K(c_1,\ c_2)=\{\text{size of } \Gamma_{c_1,c_2} | \Gamma \text{ is a graph fixed by } g\}.$$

Then we have the following remark (see [2, p.116]).

Remark: (1) The case $c_1 \neq c_2$. Then c_1 and c_2 induce (ℓ,m) cycles of length $[\ell,m]$ on those 2-subsets. Thus $K(c_1,c_2) = \{k[\ell,m] | k = 0,1,...,(\ell,m)\}$. Each vertex of c_1 is in $[\ell,m]/\ell$ of the 2-subsets of one such cycle of length $[\ell,m]$. So if the size of Γ_{c_1,c_2} is $k[\ell,m]$, clearly $d_{c_2}(c_1) = k[\ell,m]/\ell$ and $d_{c_1}(c_2) = k[\ell,m]/m$.

(2) The case $c_1 = c_2$. Suppose ℓ is even. This cycle c_1 induces $(\ell-2)/2$ cycles of length ℓ and one cycle of length $\ell/2$ on those 2-subsets. Thus $K(c_1,c_2) = \{k\ell | k = 0,1,...,(\ell-2)/2\} \cup \{\ell/2\}$. Each vertex of c_1 is in two of the 2-subsets of one such cycle of length ℓ and also it is in exactly one of the 2-subsets of the cycle of length $\ell/2$. So $d_{c_1}(c_1) = 2k$ or $2k+1$, depending on whether the size of Γ_{c_1,c_2} is $k\ell$ or $k\ell + (\ell/2)$. When ℓ is odd, the cycle c_1 induces $(\ell-1)/2$ cycles of length ℓ on those 2-subsets. Thus $K(c_1,c_2) = \{k\ell | k = 0,1,...,(\ell-1)/2\}$. Each vertex of c_1 is in two of the 2-subsets of one such cycle of length ℓ. So if the size of Γ_{c_1,c_2} is $k\ell$, clearly $d_{c_1}(c_1) = 2k$.

Let us come back to (3.13). For example, let c_1 and c_2 be cycles of f_1 and f_2, respectively, whose lengths are ℓ and m, where we assume that neither c_1 nor c_2 contain the root vertex v_0. Note here that each vertex of c_1 is positive while each vertex of c_2 is negative. Then it is easy to see from remark that if there are $k[\ell,m]$ edges that join vertices of c_1 to vertices of c_2, there are $\binom{(\ell,m)}{k}$ different ways. Thus, it follows that

$$\sum_{k=0}^{(\ell,m)}(+1)^{k[\ell,m]/\ell}(-1)^{k[\ell,m]/m}\binom{(\ell,m)}{k}x^{k[\ell,m]}=(1+(-1)^{[\ell,m]/m}x^{[\ell,m]})^{(\ell,m)}$$

contributes to $\gamma_1^{\pm}(f_1,f_2,x,y)$ and $\gamma_2^{\pm}(f_1,f_2,x,y)$, as seen in (3.5). Considering similarly the other cycles of f_1 and f_2, we have

Lemma 5. *Let p and q be the numbers of vertices in f_1 and f_2, respectively, excepting the root vertex v_0, and let $(j)=(j_1,j_2,\ldots,j_p)$ and $(k)=(k_1,k_2,\ldots,k_q)$ be the cycle structures of f_1 and f_2, respectively, where j_ℓ is the number of cycles of length ℓ in f_1 not containing v_0 and k_m is the number of cycles of length m in f_2 not containing v_0. Then the followings hold:*

$$\gamma^+(f_1,x,y)=\beta^+((j),x)\prod_{\ell=1}^{p}(1+(xy)^\ell)^{j_\ell},$$

$$\gamma_1^{\pm}(f_1,f_2,x,y)=\beta^{\pm}((j),(k),x)\prod_{m=1}^{q}(1-(xy)^m)^{k_m},$$

$$\gamma^+(f_1,x)=\beta^+((j),x),$$

$$\gamma^-(f_2,x)=\beta^-((k),x),$$

$$\gamma_2^{\pm}(f_1,f_2,x,y)=\beta^{\pm}((j),(k),x)\prod_{\ell=1}^{p}(1+(-1)^\ell(xy)^\ell)^{j_\ell},$$

$$\gamma^-(f_2,x,y)=\beta^-((k),x)\prod_{m=1}^{q}(1-(-1)^m(xy)^m)^{k_m}.$$

By applying Lemma 5 to (3.13) it follows from (3.3) that Theorem 1 holds. This completes the proof of Theorem 1.

4. Numerical examples

The numerical examples of $a_{n,N}^{(d)}$, the number of unlabelled (n,N)-graphs with d odd vertices, will be shown in this section for $1\leqslant n\leqslant 9$. Each of nine numerical tables corresponds to each $n=1,2,\ldots,9$. The row headings of such a table are the sizes of graphs and the column headings are the numbers of odd vertices. The entry $a_{n,N}^{(d)}$ is made at the (N,d)th position of the table corresponding to the size N and the number d of odd vertices for each n.

$n = 1$

	0
0	1

$n = 2$

	0	2
0	1	
1		1

$n = 3$

	0	2
0	1	
1		1
2		1
3	1	

$n = 4$

	0	2	4
0	1		
1		1	
2		1	1
3	1	1	1
4	1	1	
5		1	
6			1

$n = 5$

	0	2	4
0	1		
1		1	
2		1	1
3	1	1	2
4	1	3	2
5	1	4	1
6	1	3	2
7	1	1	2
8		1	1
9		1	
10	1		

$n = 6$

	0	2	4	6
0	1			
1		1		
2		1	1	
3	1	1	2	1
4	1	3	4	1
5	1	7	5	2
6	3	8	9	1
7	2	11	9	2
8	2	9	11	2
9	1	9	8	3
10	2	5	7	1
11	1	4	3	1
12	1	2	1	1
13		1	1	
14			1	
15				1

$n = 7$

	0	2	4	6
0	1			
1		1		
2		1	1	
3	1	1	2	1
4	1	3	4	2
5	1	7	8	5
6	3	12	19	7
7	4	21	32	8
8	4	35	44	14
9	6	44	61	20
10	6	47	74	21
11	6	47	74	21
12	6	44	61	20
13	4	35	44	14
14	4	21	32	8
15	3	12	19	7
16	1	7	8	5
17	1	3	4	2
18	1	1	2	1
19		1	1	
20		1		
21	1			

$n = 8$

	0	2	4	6	8
0	1				
1		1			
2		1	1		
3	1	1	2	1	
4	1	3	4	2	1
5	1	7	8	7	1
6	3	12	24	14	3
7	4	27	51	29	4
8	7	52	105	52	5
9	9	102	187	96	8
10	16	156	328	148	15
11	18	234	480	232	16
12	25	299	660	303	25
13	24	365	774	368	26
14	29	376	836	376	29
15	26	368	774	365	24
16	25	303	660	299	25
17	16	232	480	234	18
18	15	148	328	156	16
19	8	96	187	102	9
20	5	52	105	52	7
21	4	29	51	27	4
22	3	14	24	12	3
23	1	7	8	7	1
24	1	2	4	3	1
25		1	2	1	1
26			1	1	
27				1	
28					1

$n = 9$

	0	2	4	6	8
0	1				
1		1			
2		1	1		
3	1	1	2	1	
4	1	3	4	2	1
5	1	7	8	7	2
6	3	12	24	18	6
7	4	27	58	45	14
8	7	60	140	113	25
9	13	130	329	253	46
10	21	272	729	521	94
11	36	533	1474	1038	171
12	58	969	2740	1937	291
13	83	1590	4697	3277	473
14	118	2398	7310	5074	715
15	156	3338	10 296	7165	978
16	189	4250	13 183	9141	1224
17	213	4913	15 319	10 558	1400
18	228	5150	16 120	11 076	1466
19	213	4913	15 319	10 558	1400
20	189	4250	13 183	9141	1224
21	156	3338	10 296	7165	978
22	118	2398	7310	5074	715
23	83	1590	4697	3277	473
24	58	969	2740	1937	291
25	36	533	1474	1038	171
26	21	272	729	521	94
27	13	130	329	253	46
28	7	60	140	113	25
29	4	27	58	45	14
30	3	12	24	18	6
31	1	7	8	7	2
32	1	3	4	2	1
33	1	1	2	1	
34		1	1		
35		1			
36	1				

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