JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 45, 357–374 (1974)

# On the Reconstruction of a Function on a Circular Domain from a Sampling of Its Line Integrals*

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## INTRODUCTION

We consider here the problem of approximating a function $\rho(x, y)$ on the unit disk $(x^2 + y^2 \leqslant 1)$, given an empirical estimate of its line integral along each of the $N(N-1)/2$ chords connecting pairs of elements in a set of $N$ points uniformly spaced along the circumference of the disk. This problem arises in a practical way in connection with the analysis of data from a positron scanning device [1] being used for clinical studies of the brain, and questions of convergence and numerical stability in the presence of relatively large experimental errors are relevant in judging the adequacy of any proposed solution. These questions were met in what was thought to be a satisfactory manner by a method developed and implemented as a computer code several years ago [2]. However, when attempts were made to analyze more recently acquired data, serious difficulties appeared, prompting the present re-examination.

In the earlier solution, it was recognized that the $N(N-1)/2$ line integrals comprise a sampling, at a discrete set of points, of a certain function $\hat{\rho}(p, \theta)$, known as the Radon transform of $\rho$. Since analytical expressions for the inverse Radon transform in the general case were available in the literature [3], these were adapted to the problem at hand in a way which, it was hoped, would ensure the necessary numerical stability.

In the following, we shall show, first, that the Radon transformation of functions on a closed circular domain possesses a relatively simple structure which is not immediately apparent from the more general theory. Specifically, we shall show that there is an essentially unique system of orthogonal functions, $\{\varphi_\alpha\}$, on the disk, whose Radon transforms are also orthogonal with respect to an appropriate, and rather natural, weight function which will be

* Worked performed under the auspices of the U.S. Atomic Energy Commission.

defined. The $\varphi_\alpha$, as it turns out, are *polynomials* in $x$ and $y$, and their Radon transforms $\hat{\varphi}_\alpha(p, \theta)$ are closely related to the classical Chebyshev polynomials of the second kind. Moreover, for any $N$, the points of the sampling mesh fall precisely at the zeroes of one of the polynomials in this latter system. From these facts, a procedure which is both stable and computationally efficient follows quite naturally. The reconstructed function $\rho(x, y)$ this procedure finally yields can be described as the unique polynomial in $x$ and $y$ of some specified total degree, $M \leqslant N - 2$, such that its Radon transform fits the data in a least-squares sense.

The procedure for obtaining this polynomial from any real data set is summarized below in Eq. (4.15)–(4.16), which are based on the result given in Theorem 5. Strictly speaking, the latter can be derived directly from Theorem 1 of Section 3, together with the definitions, (2.4)–(2.6), of the functions $\{F_{n,k}\}$. The remaining material, especially Theorems 2, 3, and 4 in Section 3, while not logically necessary in the development of this final result, nevertheless appears to shed some light on its significance, and may suggest generalizations to other, related types of problems.

## 1. Preliminary Definitions and Notation

We shall throughout represent a point in the plane either by Cartesian coordinates, $x$ and $y$, or by polar coordinates, $r$ and $\psi$, related to $x$ and $y$ in the usual fashion, so that

$$x + iy = r \cdot e^{i\psi}. \tag{1.1}$$

Every straight line in the plane will be characterized as the locus of points satisfying

$$x \cos \theta + y \sin \theta = r \cos(\theta - \psi) = p, \tag{1.2}$$

for some real distance parameter $p$ and angle $\theta$.

For convenience, we allow $p$ to assume negative values in (1.2), so that every straight line has exactly two distinct parametric representations, $\langle p, \theta \rangle$ and $\langle p', \theta' \rangle$, with $p' = -p$ and $\theta' = \theta + \pi \pmod{2\pi}$.

By $\mathbb{D}$ we shall mean the closed unit disk,

$$\mathbb{D} = \{\langle x, y \rangle \mid x^2 + y^2 \leqslant 1\}, \tag{1.3}$$

and by $\mathbb{C}$, the set of parameters,

$$\mathbb{C} = \{\langle p, \theta \rangle \mid -1 \leqslant p \leqslant 1, 0 \leqslant \theta < 2\pi\},$$

which topologically form the surface of a (truncated) cylinder. Then every chord, or straight line intersecting $\mathbb{D}$, is represented by exactly two points on $\mathbb{C}$, related by reflection through the center point of the cylinder.

We shall find it convenient to consider arbitrary complex-valued functions $f$ on $\mathbb{D}$ satisfying the square-integrability condition,

$$\|f\|_{\mathbb{D}}^2 = \iint_{\mathbb{D}} |f(x, y)|^2 \, dx \, dy < \infty, \tag{1.4}$$

and we adopt the standard notation, $L^2(\mathbb{D})$, for the Hilbert space associated with these functions, with inner product,

$$(f_2, f_2)_{\mathbb{D}} = \iint_{\mathbb{D}} f_1(x, y) \overline{f_2(x, y)} \, dx \, dy, \tag{1.5}$$

in which the bar denotes complex conjugation. We shall also refer to the Hilbert space, $L^2(\mathbb{C}, W)$, of complex-valued functions on $\mathbb{C}$, square-integrable with respect to a specified weight function, $W(p, \theta)$, which may in principle be any nonnegative real-valued function, integrable on $\mathbb{C}$. The corresponding inner product is given by

$$(g_1, g_2)_{\mathbb{C},W} = \int_{-1}^{1} dp \int_0^{2\pi} d\theta W(p, \theta) \, g_1(p, \theta) \, \overline{g_2(p, \theta)}. \tag{1.6}$$

We shall use the notation

$$f^{(B)}(\psi) = f(\cos \psi, \sin \psi), \tag{1.7}$$

for the *boundary function* of any function $f$ on $\mathbb{D}$, and the notation

$$f_\alpha(x, y) = f(x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha), \tag{1.8}$$

for the function obtained from $f$ by rotating the disk through an angle $\alpha$. Also, whenever $f$ has a Fourier decomposition with respect to $\psi$ for each fixed $r$, we define its $n$-th *Fourier component*, $\tilde{f}_n$, according to

$$\tilde{f}_n(x, y) = e^{in\psi} \tilde{f}^{(n)}(r), \tag{1.9}$$

for each signed integer, $n$, where

$$\tilde{f}^{(n)}(r) = \frac{1}{2\pi} \int_0^{2\pi} d\psi' e^{-in\psi'} f(r \cos \psi', r \sin \psi'), \tag{1.10}$$

is the usual $n$-th Fourier coefficient. As immediate consequences of these definitions we have the relations

$$f_\alpha^{(B)}(\psi) = f^{(B)}(\psi + \alpha) \tag{1.11}$$

and

$$f_\alpha(x, y) = \sum_{n=-\infty}^{\infty} e^{in\alpha} \tilde{f}_n(x, y), \tag{1.12}$$

for any rotation angle $\alpha$.

Now let $f$ be any absolutely integrable function on $\mathbb{D}$. (In particular, if $f \in L^2(\mathbb{D})$ then it has this property). Then it has a *Radon transform*, which we define as that function, $\hat{f}$, which assigns to each pair of parameters, $p$ and $\theta$, the line integral of $f$ along the associated chord, Eq. (1.2). We may express this in the convenient form

$$\hat{f}(p, \theta) = \iint_{\mathbb{D}} f(x, y)\, \delta(p - x \cos \theta - y \sin \theta)\, dx\, dy, \tag{1.13}$$

where $\delta$ is the usual (one-dimensional) delta function of Dirac. It is obvious that $\hat{f}(p, \theta)$ is periodic in $\theta$ with period $2\pi$, vanishes for $|p| > 1$, and satisfies the symmetry condition

$$\hat{f}(-p, \theta + \pi) = \hat{f}(p, \theta), \tag{1.14}$$

so that it can in fact be regarded as a symmetric function on $\mathbb{C}$. It can also be established without difficulty that

$$\hat{f}_\alpha(p, \theta) = \hat{f}(p, \theta + \alpha), \tag{1.15}$$

for any rotation angle $\alpha$, and that

$$\int_{-1}^{1} p^\ell \hat{f}(p, \theta)\, dp = \iint_{\mathbb{D}} (x \cos \theta + y \sin \theta)^\ell f(x, y)\, dx\, dy, \tag{1.16}$$

for any non-negative integer $\ell$. Also, the mapping, $f \to \hat{f}$, is obviously linear.

Analytic expressions for the inverse of this transformation, generalized to higher dimensions and to essentially arbitrary support regions, have been known for some time [3, 4, 5]. As previously indicated, however, direct use of these general results for practical purposes leads to numerical difficulties and tends to obscure the rather simple structure which is available in the case at hand. This structure is most easily elucidated by studying the effect of the mapping on *polynomial* functions of $x$ and $y$, restricted to $\mathbb{D}$. Before proceeding, therefore, we shall digress somewhat to accumulate certain useful facts concerning this class of functions.

## 2. Orthogonal Polynomials on $\mathbb{D}$

By $\mathscr{P}$ we shall mean the dense linear subspace of $L^2(\mathbb{D})$, consisting of the polynomials, $P(x, y)$, in two variables with complex coefficients. By the (*total*) *degree*, $D(P)$, of any non-zero polynomial $P \in \mathscr{P}$ we mean the largest value of $j + k$ for which $C_{j,k} \neq 0$ in the expansion

$$P(x, y) = \sum_{j,k \geqslant 0} C_{j,k} x^j y^k, \tag{2.1}$$

and by convention we also assign the degree, $D(0) = -1$, to the zero polynomial.

It is an easy exercise to show that the $n$-th Fourier component (Eq. (1.9)) of any polynomial $P$ is also a polynomial, which necessarily has a form given by

$$\tilde{P}_{\pm n}(x, y) = (x \pm iy)^n P_{\pm n}(x^2 + y^2) = r^n e^{\pm in\psi} P_{\pm n}(r^2), \tag{2.2}$$

for $n \geqslant 0$, where $P_{\pm n}(t)$ is some polynomial in the *single* variable, $t = r^2 = x^2 + y^2$. Clearly, if $P_n \neq 0$, the degree of $\tilde{P}_n$ is $|n| + 2k$, for some integer $k \geqslant 0$.

Let us call a polynomial $P$ *regular on* $\mathbb{D}$, or $\mathbb{D}$*-regular*, if $(P, Q)_{\mathbb{D}} = 0$ whenever $Q \in \mathscr{P}$ with $D(Q) < D(P)$. Then, for each $M \geqslant 0$, the set $\mathscr{P}_M$, consisting of the zero polynomial together with all $M$-th degree, $\mathbb{D}$-regular polynomials is an $M + 1$-dimensional subspace of $\mathscr{P}$. It is obvious that any two $\mathbb{D}$-regular polynomials, $P_1$ and $P_2$, of different degrees, are orthogonal—that is, $(P_1, P_2)_{\mathbb{D}} = 0$—and it is not difficult to show that any polynomial—hence any function, $f \in L^2(\mathbb{D})$—has a unique expansion of the form

$$f(x, y) = \sum_{M=0}^{\infty} P_M(x, y), \tag{2.3}$$

with $P_M \in \mathscr{P}_M$, all $M$. In other words, the $\mathscr{P}_M$ form a collection of mutually orthogonal subspaces spanning $L^2(\mathbb{D})$.

By a *regular orthogonal system of polynomials for* $\mathbb{D}$, we mean any orthogonal basis for $L^2(\mathbb{D})$ consisting entirely of polynomials regular on $\mathbb{D}$. Every such system can evidently be obtained by specifying for each $M$ a set of $M + 1$ mutually orthogonal $\mathbb{D}$-regular polynomials of degree $M$.

The most natural example of a regular orthogonal system of polynomials for $\mathbb{D}$ is the one obtained by Fourier decomposition of $\mathbb{D}$-regular polynomials. In this connection, let us define, for non-negative integers, $n$ and $k$, the polynomials of degree $n + 2k$, given by

$$F_{\pm n,k}(x, y) = (x \pm iy)^n Q_{n,k}(x^2 + y^2) = e^{\pm in\psi} r^n Q_{n,k}(r^2), \tag{2.4}$$

where $Q_{n,k}$ is the (unique) $k$-th degree polynomial in *one* variable satisfying the conditions:

$$\int_0^1 Q_{n,k}(t)\, t^{n+\ell}\, dt = 0, \qquad \text{for } \ell = 0, 1, \ldots, k - 1 \tag{2.5}$$

and

$$Q_{n,k}(1) = 1. \tag{2.6}$$

Condition (2.5) is equivalent to the assertion that, for each $n$, $\{Q_{n,k}\}$ is a system of orthogonal polynomials [6] for the weight function, $t^n$, on the interval (0, 1), whereas (2.6) is a standardization condition which fixes an otherwise arbitrary multiplicative factor in each $Q_{n,k}$.

LEMMA 1. *The set* $\{F_{n,k} \mid n = 0, \pm 1, \pm 2,...; k = 0, 1,...\}$ *is a regular orthogonal system of polynomials for* $\mathbb{D}$, *satisfying*

$$(F_{n,k}, F_{n',k'})_{\mathbb{D}} = \pi(|n| + 2k + 1)^{-1} \delta_{nn'}\delta_{kk'}. \tag{2.7}$$

*Proof.* First, it is clear that $F_{n,k} \in \mathscr{P}$, with $D(F_{n,k}) = |n| + 2k$. Next, consider the expression

$$\iint_{\mathbb{D}} (x + iy)^{\ell_1} (x - iy)^{\ell_2} F_{n,k}(x, y)\, dx\, dy$$
$$= \left(\int_0^{2\pi} e^{i(n+\ell_1-\ell_2)\psi}\, d\psi\right) \left(\int_0^1 Q_{|n|,k}(r^2)\, r^{|n|+\ell_1+\ell_2+1}\, dr\right),$$

for arbitrary non-negative integers $\ell_1$ and $\ell_2$ satisfying $\ell_1 + \ell_2 < |n| + 2k$. The first factor on the right vanishes unless $\ell_2 - \ell_1 = n$, in which case we must have $\ell_2 + \ell_1 = |n| + 2 \cdot m$, where $m = \min(\ell_1, \ell_2) < k$; the second factor then becomes

$$\tfrac{1}{2} \int_0^1 t^{|n|+m} Q_{|n|,k}(t)\, dt,$$

which vanishes because of (2.5). We may conclude that $(Q, F_{n,k})_{\mathbb{D}} = 0$ for any $Q \in \mathscr{P}$ with $D(Q) < |n| + 2k$. Hence, $F_{n,k}$ is regular on $\mathbb{D}$.

By expanding the polynomials $P_n$ appearing in (2.2) in terms of the complete set $\{Q_{n,k}\}$ for each $n$, an expansion of the Fourier components, $\tilde{P}_n$—hence of an arbitrary polynomial $P$—in terms of the $F_{n,k}$ is obtained. The $F_{n,k}$'s therefore form a basis for $L^2(\mathbb{D})$.

It is straightforward to establish the mutual orthogonality of the $F_{n,k}$'s—indeed, we find using (2.4) and (2.5), that

$$(F_{n,k}, F_{n',k'}) = \pi h_{|n|,k} \delta_{nn'} \delta_{kk'},$$

where (for $n \geqslant 0$)

$$h_{n,k} = \int_0^1 [Q_{n,k}(t)]^2\, t^n\, dt.$$

The proof is complete if we can demonstrate that the normalization factor $h_{n,k}$ has the value, $(n + 2k + 1)^{-1}$. For this, we refer to Eq. (A.9) of the Appendix, where we examine the polynomials $Q_{n,k}(t)$ in somewhat more detail.

It should be clear from the foregoing that the $n$-th Fourier component of a $\mathbb{D}$-regular polynomial, $P$, of degree $M$ vanishes unless the quantity, $k = (M - |n|)/2$, is a nonnegative integer, in which case $\tilde{P}_n$ is proportional to $F_{n,k}$.

We shall refer to $\{F_{n,k}\}$ as the *canonical basis* for $\mathbb{D}$, and to the associated expansion,

$$f(x, y) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \gamma_{n,k} F_{n,k}(x, y), \tag{2.8}$$

of an arbitrary function $f \in L^2(\mathbb{D})$ as the *canonical expansion* of $f$. In the latter, the coefficients are given by

$$\gamma_{\pm n,k} = \frac{(n + 2k + 1)}{\pi} \iint_{\mathbb{D}} F_{\mp n,k}(x, y) f(x, y)\, dx\, dy, \tag{2.9}$$

or, equivalently, by

$$\gamma_{\pm n,k} = 2(n + 2k + 1) \int_0^1 \tilde{f}^{(\pm n)}(r)\, Q_{n,k}(r^2)\, r^{n+1}\, dr, \tag{2.10}$$

for $n \geqslant 0$, where $\tilde{f}^{(n)}$ is defined in (1.10).

When the canonical expansion (2.8) is truncated by restricting the summation to terms with $|n| + 2k \leqslant M$, a polynomial, $P^{(M)}$, of degree $\leqslant M$ is obtained. This polynomial is the (unique) *least-squares* $M$-th *degree polynomial approximant* of $f$, in the sense that, within the space of all polynomials $P$ of degree $\leqslant M$, the quantity

$$\| f - P \|_{\mathbb{D}}^2 = \iint_{\mathbb{D}} | f(x, y) - P(x, y)|^2\, dx\, dy \tag{2.11}$$

is minimized when $P = P^{(M)}$.

Finally, we may note the following immediate but interesting consequence of (2.7) and the fact that the $F_{n,k}$ are standardized so that $F_{n,k}^{(B)}(\psi) = e^{in\psi}$, all $n$, $k$:

LEMMA 2. *Let $P$ and $Q$ be $\mathbb{D}$-regular polynomials of degree $M$. Then*

$$(P, Q)_{\mathbb{D}} = \tfrac{1}{2}(M + 1)^{-1} \int_0^{2\pi} P^{(B)}(\psi)\, \overline{Q^{(B)}(\psi)}\, d\psi. \tag{2.12}$$

In other words, the geometry of the subspace $\mathscr{P}_M$ is, apart from a constant factor, identical with that of the corresponding space of boundary functions on the unit circle. The latter space, it may be noted, consists of the trigonometric polynomials $q(\theta)$ of degree $\leqslant M$ satisfying the symmetry condition

$$q(\theta + \pi) = (-1)^M q(\theta);$$

that is, each $q(\theta)$ is a linear combination of the $M+1$ functions, $e^{in\theta}$, for $n = -M, -M+2, \ldots, M-2, M$.

## 3. Structure of the Radon Transformation on $L^2(\mathbb{D})$

The key to the structure of the mapping, $f \to \hat{f}$ defined in (1.13) lies in the following observation:

**Theorem 1.** *Let $P(x, y)$ be a $\mathbb{D}$-regular polynomial of total degree $M$, with boundary function $P^{(B)}(\psi)$. Then its Radon transform is given by*

$$\hat{P}(p, \theta) = 2(M+1)^{-1} (1-p^2)^{1/2}\, U_M(p)\, P^{(B)}(\theta), \tag{3.1}$$

*where $U_M$ is the $M$-th Chebyshev polynomial of the second kind, defined by*

$$U_M(\cos\varphi) = \frac{\sin(M+1)\varphi}{\sin\varphi}. \tag{3.2}$$

*Proof.* We first note, as an immediate consequence of (1.15) and the definition (1.13) that

$$\hat{P}(p, \theta) = \hat{P}_\theta(p, 0) = \int_{-(1-p^2)^{1/2}}^{+(1-p^2)^{1/2}} P_\theta(p, y)\, dy$$

for any function $P$. After a change of integration variable this becomes

$$\hat{P}(p, \theta) = (1-p^2)^{1/2} \int_{-1}^{1} P_\theta(p, (1-p^2)^{1/2} u)\, du. \tag{3.3}$$

Next, assume that $P(x, y)$ is any polynomial in $x$ and $y$ of total degree $\leqslant M$. Then so is the rotated function $P_\theta(x, y)$ for any fixed $\theta$, and, since any term proportional to an odd power of the integration variable $u$ does not contribute in (3.3), we find that the function

$$Q(p, \theta) \equiv (1-p^2)^{-1/2}\, \hat{P}(p, \theta) \tag{3.4}$$

is a polynomial in $p$ of degree $\leqslant M$, for each $\theta$. From (3.3), we also learn:

$$Q(1, \theta) = 2P_\theta(1, 0) = 2P^{(B)}(\theta). \tag{3.5}$$

Now suppose in addition that $P$ is regular on $\mathbb{D}$ with exact total degree $M$. Then, according to (1.16) and the definition of $\mathbb{D}$-regularity, we must have

$$\int_{-1}^{1} p^\ell Q(p, \theta)\, (1-p^2)^{1/2}\, dp = 0, \tag{3.6}$$

for any non-negative integer, $\ell < M$. In other words, $Q$ is a polynomial in $p$ of maximum degree $M$ which is orthogonal, with respect to the weight function, $(1-p^2)^{1/2}$, on the interval $(-1, 1)$, to all polynomials in $p$ of degree $<M$. As is well known [6], the only such polynomials are the multiples (including possibly 0) of $U_M$. Combining these results, and noting that $U_M(1) = M + 1$, we obtain (3.1).

COROLLARY. *Let $W_0$ be the weight function for $\mathbb{C}$ given by*

$$W_0(p, \theta) = (1 - p^2)^{-1/2}. \tag{3.7}$$

*Then, if $P$ and $Q$ are $\mathbb{D}$-regular polynomials, their Radon transforms satisfy*

$$(\hat{P}, \hat{Q})_{\mathbb{C}, W_0} = \begin{cases} 0, & \text{if} \quad D(P) \neq D(Q), \\ 4\pi(M+1)^{-1}(P, Q)_{\mathbb{D}}, & \text{if} \quad D(P) = D(Q) = M. \end{cases} \tag{3.8}$$

The proof is straightforward, if one makes use of Lemma 2 of the previous section and the orthogonalty relation,

$$\int_{-1}^{1} U_L(p)\, U_M(p)\, (1-p^2)^{1/2}\, dp = \frac{\pi}{2}\, \delta_{LM'}, \tag{3.9}$$

for the polynomials $U_M$. An analogous result holds, of course, for any scalar multiple of the weight $W_0$ defined in (3.7). It may be noted that the inner product associated with $W_0$ has the convenient representation

$$(g, h)_{\mathbb{C}, W_0} = \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta g(\cos\varphi, \theta)\, \overline{h(\cos\varphi, \theta)}. \tag{3.10}$$

Now suppose $\{\varphi_\mu\}$ is any orthogonal basis for $L^2(\mathbb{D})$. It is obvious from the above corollary that, if every $\varphi_\mu$ is a $\mathbb{D}$-regular polynomial, then the Radon transforms $\{\hat{\varphi}_\mu\}$ are mutually orthogonal in $L^2(\mathbb{C}, W_0)$. *The converse statement also holds*, as may be seen by considering the self-adjoint operator, $A$, on $L^2(\mathbb{D})$ defined by the condition

$$AP = 4\pi(M+1)^{-1} P, \qquad \text{for all } P \in \mathscr{P}_M; \tag{3.11}$$

then every eigenfunction of $A$ is a $\mathbb{D}$-regular polynomial, and, because of (3.8) and (2.3), we have

$$(\hat{f}, \hat{g})_{\mathbb{C}, W_0} = (Af, g)_{\mathbb{D}}, \tag{3.12}$$

for all functions, $f$ and $g$, in $L^2(\mathbb{D})$. Now, if the set $\{\hat{\varphi}_\mu\}$ consists of mutually orthogonal functions in $L^2(\mathbb{C}, W_0)$, we must have

$$(\hat{\varphi}_\mu, \hat{\varphi}_\nu)_{\mathbb{C}, W_0} - \lambda_\mu(\varphi_\mu, \varphi_\nu)_{\mathbb{D}} = 0,$$

for all indices $\mu$ and $\nu$, where

$$\lambda_\mu = (\hat{\varphi}_\mu, \hat{\varphi}_\mu)_{\mathbb{C},W_0}/(\varphi_\mu, \varphi_\mu)_{\mathbb{D}} .$$

This, together with (3.12) implies that every $\varphi_\mu$ is an eigenfunction of $A$ and, hence a $\mathbb{D}$-regular polynomial. We therefore have the following result:

THEOREM 2. *An orthogonal basis for $L^2(\mathbb{D})$ has mutually orthogonal Radon transforms in $L^2(\mathbb{C}, W_0)$ if and only if it is a regular orthogonal system of polynomials for $\mathbb{D}$, where $W_0$ is the weight function defined by* (3.7).

The foregoing results can be routinely applied to the polynomials, $F_{n,k}$, of the canonical basis defined in Section 2. In particular we find, for their Radon transforms, the formula

$$\hat{F}_{n,k}(\cos\varphi, \theta) = 2(|n| + 2k + 1)^{-1} \times \sin[(|n| + 2k + 1)\varphi] \times e^{in\theta}, \tag{3.13}$$

with the orthogonality relation

$$(\hat{F}_{n,k}, \hat{F}_{n',k'})_{\mathbb{C},W_0} = 4\pi^2(|n| + 2k + 1)^{-2}\,\delta_{nn'}\delta_{kk'} . \tag{3.14}$$

The coefficients, $\gamma_{n,k}$, in the canonical expansion, (2.8), of an arbitrary $f \in L^2(\mathbb{D})$ are therefore related to the Radon transform, $g = \hat{f}$, of $f$ according to the formula:

$$\begin{aligned}\gamma_{n,k} &= (2\pi)^{-2}(|n| + 2k + 1)^2 (g, \hat{F}_{n,k})_{\mathbb{C},W_0} \\ &= \frac{(|n| + 2k + 1)}{2\pi^2}\int_0^\pi d\varphi \sin[(|n| + 2k + 1)\varphi] \times \int_0^{2\pi} d\theta e^{-in\theta} g(\cos\varphi, \theta).\end{aligned} \tag{3.15}$$

Conversely, if $g$ is any function in $L^2(\mathbb{C}, W_0)$, Eqs. (3.15) and (2.8) define a function, $f_g \in L^2(\mathbb{D})$ *provided* the condition

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} |\gamma_{n,k}|^2 (|n| + 2k + 1)^{-1} < \infty \tag{3.16}$$

is satisfied. However, $\hat{f}_g$ is, in general, not $g$, but rather the orthogonal projection of $g$ onto the (proper) subspace of $L^2(\mathbb{C}, W_0)$ spanned by the functions, $\hat{F}_{n,k}$. It is easy to see that this subspace consists precisely of those functions, $g \in L^2(\mathbb{C}, W_0)$, which satisfy both the symmetry requirement,

$$g(-p, \theta + \pi) = g(p, \theta), \tag{3.17}$$

and the set of conditions

$$\int_{-1}^{1} dp \int_0^{2\pi} d\theta p^\ell e^{\pm in\theta} g(p, \theta) = 0, \tag{3.18}$$

for arbitrary integers, $\ell$ and $n$, such that $0 \leqslant \ell < n$. We may summarize these observations in the following theorem, the first part of which can also be obtained from published results [7, 8] applicable to a more general case.

THEOREM 3. *A function $g$ on $\mathbb{C}$ is the Radon transform of a function $f$ in $L^2(\mathbb{D})$ if and only if* (a) *it satisfies* (3.17) *and* (3.18), *and* (b) *the coefficients defined by* (3.15) *exist and satisfy* (3.16). *When this is the case, $f$ is given by the canonical expansion,* (2.8). *More generally, for any $g \in L^2(\mathbb{C}, W_0)$, and any $M \geqslant 0$, then within the space of polynomials, $P(x, y)$, of degree $\leqslant M$, the quantity,*

$$\| g - \hat{P} \|^2_{\mathbb{C}, W_0} = \int_0^\pi d\varphi \int_0^{2\pi} d\theta \, | g(\cos\varphi, \theta) - \hat{P}(\cos\varphi, \theta)|^2, \tag{3.19}$$

*is minimized when*

$$P(x, y) = \sum_{|n|+2k \leqslant M} \gamma_{n,k} F_{n,k}(x, y) \tag{3.20}$$

*with the $\gamma_{n,k}$ given by* (3.15).

A formal proof of these statements is straightforward and will be omitted.

Because of Theorems 2 and 3, the weight function $W_0$ (or any of its scalar multiples) would appear to be the most natural choice for defining an inner product between functions on $\mathbb{C}$. Moreover, as we demonstrate in the next section, it also leads in a natural way, to what is in effect a quadrature formula for the inverse Radon transform, ideally suited to the kind of experimental data described in the Introduction. It is nevertheless an interesting question whether analogus results can be obtained for other inner products, in particular, for other weight functions on $\mathbb{C}$. Regarding this question, we confine ourselves here to the following result:

THEOREM 4. *Let $u(p, \theta)$ be any essentially bounded, measurable function on $\mathbb{C}$, and let*

$$W(p, \theta) = (1 - p^2)^{-1/2} \, u(p, \theta) \tag{3.21}$$

*Then there exists a complete orthogonal set of functions, $\{\varphi_\mu\}$, on $\mathbb{D}$, with Radon transforms which are mutually orthogonal in $L^2(\mathbb{C}, W)$. A necessary and sufficient condition that $\{\varphi_\mu\}$ have this property is that each $\varphi_\mu$ be an eigenfunction of the integral operator, $A_W$, on $L^2(\mathbb{D})$, defined by the equation*

$$(A_W f)(x_1, y_1) = \iint_{\mathbb{D}} K_W(x_1 y_1 \mid x_2 y_2) f(x_2, y_2) \, dx_2 \, dy_2 , \tag{3.22}$$

*with kernel*

$$K_W(x_1 y_1 \mid x_2 y_2) = r_{12}^{-1} [W(p_{12}, \theta_{12}) + W(-p_{12}, \theta_{12} + \pi)] \tag{3.23}$$

*in which*

$$r_{12} = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2},$$
$$p_{12} = (x_1 y_2 - y_1 x_2)/r_{12}, \qquad \textit{and} \qquad \tan\theta_{12} = -(x_1 - x_2)/(y_1 - y_2). \tag{3.24}$$

It may be noted that $p_{12}$ and $\theta_{12}$ are parameters, in the sense of (1.2), for the chord passing through the two points, $(x_1, y_1)$ and $(x_2, y_2)$.

*Proof.* First, let $H$ be any separable Hilbert space of functions on $\mathbb{C}$, which contains $\hat{f}$, for every $f \in L^2(\mathbb{D})$. Then, the mapping, $R: f \to \hat{f}$, is a bounded linear transformation from $L^2(\mathbb{D})$ into $H$, and its *adjoint*, $R^*$, is a bounded linear transformation from $H$ into $L^2(\mathbb{D})$, uniquely defined by the requirement

$$(R^*g, f)_{\mathbb{D}} = (g, Rf)_H,$$

for all $g \in H$ and $f \in L^2(\mathbb{D})$. The product, $A = R^*R$ is then a self-adjoint operator on $L^2(\mathbb{D})$, such that

$$(\hat{f}_1, \hat{f}_2)_H = (Af_1, f_2)_{\mathbb{D}},$$

for all $f_1$ and $f_2$ in $L^2(\mathbb{D})$, and it is easily demonstrated that an orthogonal basis for $L^2(\mathbb{D})$ has mutually orthogonal transforms in $H$ if and only if it is a complete set of eigenfunctions for $A$.

Secondly, suppose in addition that $H = L^2(\mathbb{C}, W)$, for some weight function, $W$. Then, the adjoint, $R^* \equiv R_W{}^*$, can be explicitly represented by the relation

$$(R_W{}^*g)(x, y) = \int_{-1}^{1} dp \int_0^{2\pi} d\theta W(p, \theta)\, \delta(p - x\cos\theta - y\sin\theta)\, g(p, \theta),$$

which, in combination with (1.13), leads eventually to the explicit representation, (3.22)–(3.24), for the operator, $A_W = R_W{}^*R$.

To complete the proof, therefore, it is sufficient to show that, when $W$ has the form stated, the operator $A_W$ in fact has a complete set of eigenfunctions. But this follows from the stronger statement that $A_W$ is a compact operator. Because of the essential boundedness of the function, $u = W/W_0$, this in turn follows from the statement that $A_{W_0}$ is compact, and this last statement is apparent from the spectral properties of $A_{W_0}$, as indicated in (3.11).

## 4. The Discrete Case

Returning finally to the situation described at the outset, we consider a fixed set, $S_N$, of $N$ points spaced uniformly around the circumference of $\mathbb{D}$. We suppose that these points are numbered from 1 to $N$, in consecutive counterclockwise order, and for simplicity assume that point #1 lies along the positive $X$-axis. Then, the coordinates of the $I$-th point are given by

$$x_I + iy_I = e^{i\psi_I}, \tag{4.1}$$

where

$$\psi_I = 2\pi(I-1)/N, \qquad I = 1, 2,..., N. \tag{4.2}$$

The $N(N-1)/2$ possible chords joining pairs of points in $S_N$ correspond, via (1.2), to a mesh of $N(N-1)$ points on the cylinder, $\mathbb{C}$. To describe these, we define

$$p_J = \cos(\pi J/N) \tag{4.3}$$

and

$$\theta_{I,J} = \frac{\pi}{N}(2I + J - 2), \tag{4.4}$$

for $I = 1, 2,..., N$ and $J = 1, 2,..., N-1$. Then, the point $\langle p_J, \theta_{I,J}\rangle$ on $\mathbb{C}$ corresponds to the chord joining the two points in $S_N$ numbered $I_1$ and $I_2$, where

$$\begin{aligned} I_1 &= I < I_2 = I + J, && \text{if} \quad I + J \leqslant N, \\ I_1 &= I + J - N < I_2 = I, && \text{if} \quad I + J > N. \end{aligned} \tag{4.5}$$

For brevity, we shall denote the mesh point $\langle P_J, \theta_{I,J}\rangle$ simply by the index pair $\langle I, J\rangle$; in particular, if $g$ is a function on $\mathbb{C}$, it will be understood that $g(I, J) = g(P_J, \theta_{I,J})$. It should be noted that the symmetry condition, (3.17) or (1.14), when restricted to mesh points, becomes

$$g(I + J, N - J) = g(I, J) \tag{4.6}$$

for all $I$ and $J$ such that $I + J \leqslant N$.

Now it may be noted that the values, $p_J$, fall precisely at the zeroes of the polynomial, $U_{N-1}$:

$$U_{N-1}(p_J) = \frac{\sin N\left(\frac{\pi J}{N}\right)}{\sin \frac{\pi J}{N}} = 0, \qquad J = 1, 2,..., N-1. \tag{4.7}$$

It follows that the orthogonality condition, (3.9), for the $U_M$ has a discrete analog on the set $\{p_J\}$, namely,

$$\begin{aligned} &\sum_{J=1}^{N-1} (1 - p_J^2)\, U_L(p_J)\, U_M(p_J) \\ &\quad = \sum_{J=1}^{N-1} \sin\left[(L+1)\frac{\pi J}{N}\right] \times \sin\left[(M+1)\frac{\pi J}{N}\right] = \frac{N}{2}\,\delta_{LM}, \end{aligned} \tag{4.8}$$

provided $L < N-1$, $M < N-1$.

Because of the equal spacing of the $N$ angles, $\theta_{I,J}$, for each fixed $J$, we can also establish the identity

$$\sum_{I=1}^{N} e^{i(n-n')\theta_{I,J}} = \begin{cases} 0, & \text{unless } n - n' \equiv 0 \pmod{N}, \\ N, & \text{if } n - n' = 0, \\ N(-1)^J, & \text{if } |n - n'| = N. \end{cases} \tag{4.9}$$

Combining (4.8) and (4.9) and making use of (3.1), we eventually derive a discrete analog of (3.14),

$$\sum_{I,J} \hat{F}_{n,k}(I, J) \overline{\hat{F}_{n',k'}(I, J)} = 2N^2(|n| + 2k + 1)^{-2} \delta_{nn'} \delta_{kk'}, \tag{4.10}$$

which is valid if both $|n| + 2k$ and $|n'| + 2k'$ are $\leqslant N - 2$.

It should be mentioned that (4.10) is somewhat unusual, in that the allowed range of values for $n$ extends beyond the one ($|n| \leqslant [N/2]$) usually associated with $N$ point trigonometric interpolation. This is a reflection of the fact that the functions, $\hat{F}_{n,k}$, do not span the space of symmetric functions on $\mathbb{C}$, as indicated by (3.18). In fact, the only complication in the derivation of (4.10) is the need to deal separately with the possibility, $|n - n'| = N$, arising from (4.9); one can eliminate this possibility in the remaining summation over $J$ only by using the special form of $\hat{F}_{n,k}$. In any event, a simple count shows that the number of index combinations satisfying the stated restrictions is exactly $N(N - 1)/2$, as is the dimension of the space of polynomials in $x$ and $y$ of degree $\leqslant N - 2$.

THEOREM 5. *Given an indexed array of numbers,*

$$\{g(I, J) \mid 1 \leqslant I \leqslant N, 1 \leqslant J \leqslant N - 1\},$$

*let*

$$\begin{aligned} \gamma_{n,k} = \frac{|n| + 2k + 1}{N^2} \sum_{I,J} \sin\left[(|n| + 2k + 1)\frac{\pi J}{N}\right] \\ \times \exp\left[-\frac{in\pi}{N}(2I + J - 2)\right] g(I, J), \end{aligned} \tag{4.11}$$

*for each signed integer n and non-negative integer k such that* $|n| + 2k \leqslant N - 2$. *Then, for each* $M \leqslant N - 2$, *the quantity*

$$L(P) = \sum_{I,J} |g(I, J) - \hat{P}(I, J)|^2 \tag{4.12}$$

*is minimized within the class of polynomials,* $P(x, y)$, *of degree* $\leqslant M$, *when* $P = P^{(M)}$, *where*

$$P^{(M)}(x, y) = \sum_{|n|+2k \leqslant M} \gamma_{n,k} F_{n,k}(x, y). \tag{4.13}$$

This theorem can be easily proved from (4.10) and some of our earlier results, using standard methods.

In the practical situation, of course, the array $\{g(I, J)\}$ is determined from experimentally obtained data associated with the set of chords, so that the symmetry requirement, (4.6), is satisfied. Then $L(P^{(N-2)}) = 0$—that is, an exact fit to the data can always be obtained, though this would not normally be the most desirable one, because of the presence of "noise" in the data.

Finally, assuming *real* data $\{g(I, J)\}$, we can easily translate our results into a form with all complex quantities eliminated, by defining real coefficients $\alpha_{n,k}$ and $\beta_{n,k}$ according to the relations

$$\gamma_{\pm n,k} = \tfrac{1}{2}(\beta_{n,k} \mp i\alpha_{n,k}), \qquad \text{for } n \geqslant 1, \tag{4.14}$$

with

$$\beta_{0,k} = \gamma_{0,k} \qquad \text{and} \qquad \alpha_{0,k} = 0.$$

Then we have, in place of (4.11), the scheme:

$$\begin{aligned}
\beta_{0,k} &= \frac{2k+1}{N^2} \sum_{I,J} \sin\left[(2k+1)\frac{\pi J}{N}\right] g(I, J) \\
&\quad \cdots 0 \leqslant k \leqslant \left[\frac{N-2}{2}\right], \\
\begin{Bmatrix} \alpha_{n,k} \\ \beta_{n,k} \end{Bmatrix} &= \frac{2(n+2k+1)}{N^2} \sum_{I,J} \sin\left[(n+2k+1)\frac{\pi J}{N}\right] \\
&\quad \times \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[\frac{n\pi}{N}(2I+J-2)\right] g(I, J) \\
&\quad \cdots 1 \leqslant n \leqslant N-2, \qquad 0 \leqslant k \leqslant \left[\frac{N-n-2}{2}\right].
\end{aligned} \tag{4.15}$$

With these coefficients, the polynomial $P^{(M)}$ of (4.13), can finally be represented in the form

$$P^{(M)}(x, y) = \sum_{n=0}^{M} \sum_{k=0}^{[(M-n)/2]} (\alpha_{n,k} \sin n\psi + \beta_{n,k} \cos n\psi)\, r^n Q_{n,k}(r^2). \tag{4.16}$$

## APPENDIX. Some Properties of the Polynomials, $Q_{n,k}(t)$

The polynomials, $Q_{n,k}(t)$, defined in Section 2 can be expressed as shifted special cases of the classical *Jacobi polynomials*, $P_k^{(\alpha,\beta)}$ according to the relation

$$Q_{n,k}(t) = P_k^{(0,n)}(2t - 1). \tag{A.1}$$

This being the case, there are numerous relations involving the $Q_{n,k}$ which can be easily obtained from well-known formulas [6] pertaining to this more general class, and we shall confine ourselves here to listing only those few which seem most relevant.

A convenient starting point is the *Rodriguez formula*,

$$Q_{n,k}(t) = \frac{1}{k!t^n} \frac{d^k}{dt^k} \{t^{n+k}(t-1)^k\}, \tag{A.2}$$

which, as is easily demonstrated, yields a polynomial of degree $k$ satisfying (2.5) and (2.6), as required. By binomial expansion of $(t-1)^k$, followed by term-by-term differentiation in (A.2), we obtain the explicit representation

$$Q_{n,k}(t) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{n+k+j}{k} t^j, \tag{A.3}$$

with leading $(j = k)$ coefficient

$$a_{n,k} = \binom{n+2k}{k}, \tag{A.4}$$

and with constant $(j = 0)$ term

$$Q_{n,k}(0) = (-1)^k \binom{n+k}{n}. \tag{A.5}$$

We are interested specifically in the *normalization factor*, $h_{n,k}$, defined by

$$h_{n,k} = \int_0^1 [Q_{n,k}(t)]^2 \, t^n \, dt. \tag{A.6}$$

For this, we first use (A.3) and (2.5) to obtain

$$h_{n,k} = a_{n,k}\lambda_{n,k}, \tag{A.7}$$

where

$$\lambda_{n,k} = \int_0^1 Q_{n,k}(t) \, t^{n+k} \, dt.$$

The latter quantity can be evaluated by substituting (A.2) and integrating by parts $k$ times, with the result

$$\begin{aligned} \lambda_{n,k} &= \int_0^1 t^{n+k}(1-t)^k \, dt \\ &= \frac{k!(n+k)!}{(n+2k+1)!} \\ &= \left[(n+2k+1)\binom{n+2k}{k}\right]^{-1}, \end{aligned} \tag{A.8}$$

from which we infer, finally,

$$h_{n,k} = (n + 2k + 1)^{-1}. \tag{A.9}$$

Of the variety of *recursion formulas* which are available, we mention only the following two, without proof:

$$(n + 2k + 3) Q_{n,k+1} = (k + 1) Q_{n+1,k} + (n + k + 2) Q_{n+1,k+1}, \tag{A.10}$$

$$(n + 2k + 2) tQ_{n+1,k} = (n + k + 1) Q_{n,k} + (k + 1) Q_{n,k+1}. \tag{A.11}$$

We may express the $n = 0$ case of (A.11) in the simple form

$$Q_{0,k+1} = 2tQ_{1,k} - Q_{0,k}, \tag{A.12}$$

and rearrange (A.10) to the form,

$$Q_{n+1,k+1} = Q_{n,k+1} + \frac{k + 1}{n + k + 2} [Q_{n,k+1} - Q_{n+1,k}]. \tag{A.13}$$

Together with the "initial conditions"

$$Q_{n,0}(t) = 1, \qquad \text{all } n, \text{ all } t, \tag{A.14}$$

equations (A.12) and (A.13) provide an efficient computational scheme for generating the entire array of values, $\{Q_{n,k}(t) \mid n + 2k \leqslant M\}$, for some $M$, and some fixed $t$, specified in advance. This is extremely useful, in practical implementations of the procedure described in Section 4, for evaluating the expression (4.16) at any point $(x, y)$, once the coefficients $\alpha_{n,k}$ and $\beta_{n,k}$ have been determined.

*Note added in proof.* Since the manuscript for this paper was submitted for publication, the following facts have come to the author's attention: (1) The polynomials defined in (2.4) are apparently identical with the ones first introduced by F. Zernike [*Physica* **1** (1934), 689] and well known in the diffraction theory of optical aberrations [cf., e.g., Born and Wolf, "Principles of Optics," p. 767, Pergamon Press, New York, 1964, and references therein]; (2) the expression (3.13) for the transforms of these polynomials has been previously pointed out by S. I. Herlitz [*Arkiv Fysik* **23** (1963), 571, with Addendum], and independently by A. M. Cormack [*J. Appl. Phys.* **35** (1964), 2908], using different methods of derivation.

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