A numerical scheme for unsteady flow of a viscous fluid between elliptic plates

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ABSTRACT

A new continuation method has been developed to solve the nonlinear eigenvalue problem describing the unsteady, squeezing flow of a viscous fluid between elliptic plates. Unlike the numerical schemes previously used (e.g. homotopy algorithm), the present scheme is conceptually simple, noniterative, insensitive to the first approximation and works for all values of squeeze number S characterizing the flow. The numerical results compare extremely well with those obtained with sophisticated schemes. Since existing numerical data are limited to three sparsely spaced values of S, additional data are reported for systematically spaced values of squeeze number S and ellipticity parameter β. Although the scheme has been applied to a specific problem, it appears potentially capable of handling a variety of nonlinear eigenvalue problems.

1. INTRODUCTION

The prediction of flow field when a viscous fluid is squeezed between parallel plates is of fundamental importance in the study of unsteady loading of bearings and in viscometry. The flow is usually characterised by squeeze number S, which indicates the relative importance of inertial and viscous forces. At low squeeze rates ($S \ll 1$), the inertial effects can be ignored and the classical Reynolds' equation can be used [1]. However, at high squeeze rates ($S \gg 1$), the inertial terms dominate the viscous term and one must solve the unsteady Navier-Stokes equations to describe the flow satisfactorily. The only feasible approach is to use a numerical scheme as typified by the work of Grimm [2].

Considering the flow between elliptic plates, Wang and Watson [3] recently found that, if the normal velocity is specified as proportional to $(1-\alpha t)^{-1/2}$, where t is time and α is a constant, then a similarity solution is possible. The analysis leads to a difficult eigenvalue problem involving two coupled third-order, nonlinear boundary value problems. The equations contain two parameters : squeeze number S and ellipticity parameter β. To obtain a numerical solution, Wang and Watson introduced a new homotopy algorithm, which is globally convergent and does not necessitate a good initial approximation to ensure convergence. Despite this, the scheme failed to converge beyond S = 20, and a standard imbedding technique had to be employed to generate solutions for S > 20.

In the present paper, we examine the case of elliptic plates and solve it by a new continuation method. The scheme has the attractive feature of being noniterative and gives accurate solutions for all values of S. Since Wang and Watson [3] present limited numerical data (because of high computational cost of their scheme) we present data for systematically spaced values of S (in the range 0 to 25) and β (in the range 0 to 1). The computational efficiency of the scheme can be judged from the fact that all the data presented here were generated within 15 seconds of CPU time on Amdahl 470V/7 system.

2. PROBLEM

For squeezing flow of a viscous fluid between elliptic plates, with normal velocity proportional to $(1-\alpha t)^{-1/2}$, the unsteady Navier-Stokes equations can be reduced to the similarity form as [3]

$$f''' + k = S[2f' + \eta f'' + \frac{1}{2}(f')^2 - \frac{1}{2}f''(f+g)] \quad (1)$$

$$g''' + \beta k = S[2g' + \eta g'' + \frac{1}{2}(g')^2 - \frac{1}{2}g''(f+g)] \quad (2)$$

$$f(0) = g(0) = f''(0) = g''(0) = 0 \quad (3)$$

$$f'(1) = g'(1) = 0, \; f(1) + g(1) = 2 \quad (4)$$

where the symbols are as defined in [3] and need not be repeated. The primes denote differentiation with respect to η.

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3. SOLUTION METHOD

Since the eigenvalue k is a constant in equations (1, 2) we introduce an additional differential equation for k as in [4],

$$k' = 0 \tag{5}$$

Let us assume the first approximation as

$$f_0(\eta) = g_0(\eta) = \sin\left(\frac{\pi}{2}\eta\right), \quad k_0 = 1 \tag{6}$$

where f_0 and g_0 have been chosen as they satisfy the boundary conditions (3, 4). Indeed, any functions that satisfy the boundary conditions can be chosen as a first approximation and the method would eventually converge to the same solution. The value of $k_0 = 1$ is arbitrary. Next we introduce a variable τ and write equations (1, 2, 5) as

$$\begin{aligned} &f''' + k - S[2f' + \eta f'' + \tfrac{1}{2}(f')^2 - \tfrac{1}{2}f''(f+g)] \\ &= (1-\tau)\{f_0''' + k_0 - S[2f_0' + \eta f_0'' + \tfrac{1}{2}(f_0')^2 \\ &- \tfrac{1}{2}f_0''(f_0 + g_0)]\} \end{aligned} \tag{7}$$

$$\begin{aligned} &g''' + \beta k - S[2g' + \eta g'' + \tfrac{1}{2}(g')^2 - \tfrac{1}{2}g''(f+g)] \\ &= (1-\tau)\{g_0''' + \beta k_0 - S[2g_0' + \eta g_0'' + \tfrac{1}{2}(g_0')^2 \\ &- \tfrac{1}{2}g_0''(f_0 + g_0)]\} \end{aligned} \tag{8}$$

$$k' = (1-\tau)k_0' \tag{9}$$

It is reasonable to expect that if τ changes from 0 to 1 the functions f, g and k will change from their first approximations f_0, g_0 and k_0 to their true values. To establish the dependence of the solution on τ, we differentiate equations (7, 8, 9) and boundary conditions with respect to τ. The result is

$$\begin{aligned} &F''' + K - S[2F' + \eta F'' + f'F' - \tfrac{1}{2}F''(f+g) - \tfrac{1}{2}f''(F+G)] \\ &= -\{f_0''' + k_0 - S[2f_0' + \eta f_0'' + \tfrac{1}{2}(f_0')^2 - \tfrac{1}{2}f_0''(f_0+g_0)]\} \end{aligned} \tag{10}$$

$$\begin{aligned} &G''' + \beta K - S[2G' + \eta G'' + g'G' - \tfrac{1}{2}G''(f+g) - \tfrac{1}{2}g''(F+G)] \\ &= -\{g_0''' + \beta k_0 - S[2g_0' + \eta g_0'' + \tfrac{1}{2}(g_0')^2 - \tfrac{1}{2}g_0''(f_0+g_0)]\} \end{aligned} \tag{11}$$

$$K' = 0 \tag{12}$$

$$F(0) = G(0) = F''(0) = G''(0) = 0 \tag{13}$$

$$F'(1) = G'(1) = 0, \quad F(1) + G(1) = 0 \tag{14}$$

where

$$F^n = \frac{d}{d\tau}(f^n), \quad G^n = \frac{d}{d\tau}(g^n), \quad K = \frac{dk}{d\tau} \tag{15}$$

and n (primes) stands for nth derivative with respect to η.

The solution scheme now proceeds as follows :

(1) A first approximation as given by equation (6) is assumed.

(2) The functions f, f', f'' on the left hand side of equation (10) are replaced by f_0, f_0' and f_0'' respectively. Similarly, the functions g, g' and g'' on the left-hand side of equation (11) are replaced by g_0, g_0' and g_0''.

(3) It is noted that equations (10-14) now constitute linear boundary value problems in F and G and can be solved noniteratively by the method of superposition [5] as follows. Since there are three missing initial conditions, we introduce three constants a, b, c and write the solutions as

$$F = F_0 + aF_1 + bF_2 + cF_3 \tag{16}$$

$$G = G_0 + aG_1 + bG_2 + cG_3 \tag{17}$$

$$K = K_0 + aK_1 + bK_2 + cK_3 \tag{18}$$

Substituting equations (16, 17) into equations (10, 11), a total of eight equations for F_0, F_1, F_2, F_3 and G_0, G_1, G_2, G_3 can be readily obtained. Further, if we set the missing initial conditions as

$$F'(0) = a, \quad G'(0) = b, \quad K(0) = c \tag{19}$$

it follows from the boundary condition (13)

$F_0(0) = 0$	$F_1(0) = 0$	$F_2(0) = 0$	$F_3(0) = 0$	(20)
$F_0'(0) = 0$	$F_1'(0) = 1$	$F_2'(0) = 0$	$F_3'(0) = 0$	(21)
$F_0''(0) = 0$	$F_1''(0) = 0$	$F_2''(0) = 0$	$F_3''(0) = 0$	(22)
$G_0(0) = 0$	$G_1(0) = 0$	$G_2(0) = 0$	$G_3(0) = 0$	(23)
$G_0'(0) = 0$	$G_1'(0) = 0$	$G_2'(0) = 1$	$G_3'(0) = 0$	(24)
$G_0''(0) = 0$	$G_1''(0) = 0$	$G_2''(0) = 0$	$G_3''(0) = 0$	(25)
$K_0(0) = 0$	$K_1(0) = 0$	$K_2(0) = 0$	$K_3(0) = 1$	(26)

Now that all initial conditions are specified, the eight equations for F_0, F_1, F_2, F_3, G_0, G_1, G_2, and G_3 can be integrated using a Runge-Kutta scheme in one sweep without any iteration. The terminal values (at $\eta = 1$) of F_0, F_1, F_2, F_3 and G_0, G_1, G_2 and G_3 and their derivatives can now be used in the boundary condition at $\eta = 1$, that is, equation (14), to give three simultaneous algebraic equations. The solution gives a, b and c as follows.

$$a = \Delta_a/\Delta, \quad b = \Delta_b/\Delta, \quad c = \Delta_c/\Delta \tag{27}$$

where the determinants are

$$\Delta_a = \begin{vmatrix} -F_0'(1) & F_2'(1) & F_3'(1) \\ -G_0'(1) & G_2'(1) & G_3'(1) \\ -[F_0(1) + G_0(1)] & F_2(1) + G_2(1) & F_3(1) + G_3(1) \end{vmatrix} \tag{28}$$

$$\Delta_b = \begin{vmatrix} F_1'(1) & -F_0'(1) & F_3'(1) \\ G_1'(1) & -G_0'(1) & G_3'(1) \\ F_1(1)+G_1(1) & -[F_0(1)+G_0(1)] & F_3(1)+G_3(1) \end{vmatrix} \tag{29}$$

$$\Delta_c = \begin{vmatrix} F_1'(1) & F_2'(1) & -F_0'(1) \\ G_1'(1) & G_2'(1) & -G_0'(1) \\ F_1(1)+G_1(1) & F_2(1)+G_2(1) & -[F_0(1)+G_0(1)] \end{vmatrix} \tag{30}$$

$$\Delta = \begin{vmatrix} F_1'(1) & F_2'(1) & F_3'(1) \\ G_1'(1) & G_2'(1) & G_3'(1) \\ F_1(1)+G_1(1) & F_2(1)+G_2(1) & F_3(1)+G_3(1) \end{vmatrix} \tag{31}$$

(4) Once a, b, c are known, F, G and K and their derivatives can be calculated from equations (16-18).

(5) We now integrate equation (15) with respect to τ to give the values at $\tau = \Delta\tau$ as

$$f^n|_{\tau=\Delta\tau} = f^n|_{\tau=0} + F^n|_{\tau=0} \cdot \Delta\tau \tag{32}$$

$$g^n|_{\tau=\Delta\tau} = g^n|_{\tau=0} + G^n|_{\tau=0} \cdot \Delta\tau \tag{33}$$

$$k|_{\tau=\Delta\tau} = k|_{\tau=0} + K|_{\tau=0} \cdot \Delta\tau \tag{34}$$

(6) To obtain f^n, g^n and k at $\tau = 2\Delta\tau$, the functions f, f′, f″ in equation (10) and the functions g, g′ and g″ in equation (11) are replaced by their new values from step 5. Steps 3, 4 and 5 are repeated.

(7) The process is continued until $\tau = 1$. The functional values at $\tau = 1$ are the desired solutions of equations (1-4).

4. NUMERICAL RESULTS

Numerical results were generated for a range of values of S and β. The step sizes used in the computation were $\Delta\eta = 0.002$ and $\Delta\tau = 0.05$. A detailed sample solution for S = 25 and $\beta = 0$ is given in table 1. It shows the progress of the solution as τ changes from 0 to 1. The present value of k = 173.1767 compares extremely well with the values of 175.34 and 173.95 obtained respectively from imbedding technique and matched asymptotic expansions [3].
Table 2 compares the present values of k with those reported in [3] for specific values of β and S. Note that the present results are quoted only for cases where the corresponding data from [3] are available. As can be seen, the agreement is good for all values of S and β.
Since the data presented in [3] are limited, we present (table 3) additional data for k for systematically spaced values of S and β covering the entire range of practical interest. However, the detailed solutions for f and g have to be excluded due to space limitations but the authors, if requested, would be pleased to furnish them to the readers.

5. DISCUSSION

A continuation method has been developed to solve the nonlinear eigenvalue problem describing the squeezing flow of a viscous fluid between elliptic plates. The attractive features are that the method is conceptually simple and dispenses with the usual iteration. In a previous numerical study of the problem [3], a new homotopy method had to be used for S < 20, and a standard imbedding technique for S > 20. In contrast, the present scheme worked for all values of S and gave results of the same accuracy in a simpler and more efficient manner.
One striking feature of this method is that the method is insensitive to the first approximation. In fact, the same first approximation is chosen for the solutions for all values of S and β. To demonstrate, an inspection of equation (6) shows that the first approximation of the eigenvalue k is chosen to be $k_0 = 1$ for all cases. However, the same first approximation proceeded to the final solution automatically in each case. The most severe case is the one shown in table 1, where the final solution is k = 173.1767. With iterative methods, one can realize the difficulties in obtaining the required solution by starting from a first approximation, which is 1/173 of its final value.
It is interesting to observe that for S < 0, which corresponds to the case of separating plates, the homotopy method [3] gave two solutions, one with positive eigenvalue and the other with negative eigenvalue. However, the perturbation method [3] gave only one solution with positive eigenvalue. The present scheme falls in line with the perturbation solution in the sense that it too produced one solution with positive eigenvalue.
In all probability, the negative eigenvalue gives rise to an unstable solution, which is not likely to occur in practice. However, a stability analysis is needed to resolve the situation completely.

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TABLE 1. Sample solution for $\beta = 0$, $S = 25$

τ	η	f	f′	f″	g	g′	g″	k
0.00	0.0	0.0000	1.5708	0.0000	0.0000	1.5708	0.0000	1.0000
	0.2	0.3090	1.4939	- 0.7625	0.3090	1.4939	- 0.7625	1.0000
	0.4	0.5878	1.2708	1.4503	0.5878	1.2708	- 1.4503	1.0000
	0.6	0.8090	0.9233	- 1.9962	0.8090	0.9233	- 1.9962	1.0000
	0.8	0.9511	0.4854	- 2.3467	0.9511	0.4854	- 2.3467	1.0000
	1.0	1.0000	0.0000	- 2.4674	1.0000	0.0000	- 2.4674	1.0000
0.25	0.0	0.0000	1.6751	0.0000	0.0000	1.2511	0.0000	37.2379
	0.2	0.3315	1.6227	- 0.5224	0.2461	1.1900	- 0.6070	37.2379
	0.4	0.6422	1.4666	- 1.0387	0.4682	1.0117	- 1.1629	37.2379
	0.6	0.9111	1.2040	- 1.6113	0.6440	0.7324	- 1.6056	37.2379
	0.8	1.1144	0.7955	- 2.6225	0.7563	0.3816	- 1.8669	37.2379
	1.0	1.2054	0.0000	- 6.2219	0.7946	0.0000	- 1.9201	37.2379
0.50	0.0	0.0000	1.8129	0.0000	0.0000	0.8980	0.0000	77.1069
	0.2	0.3605	1.7813	- 0.3174	0.1766	0.8539	- 0.4380	77.1069
	0.4	0.7081	1.6838	- 0.6707	0.3359	0.7249	- 0.8432	77.1069
	0.6	1.0284	1.5000	- 1.2364	0.4617	0.5221	- 1.1652	77.1069
	0.8	1.2960	1.1230	- 2.8816	0.5413	0.2690	- 1.3355	77.1069
	1.0	1.4318	0.0000	-10.2820	0.5682	0.0000	- 1.3358	77.1069
0.75	0.0	0.0000	1.9923	0.0000	0.0000	0.4983	0.0000	121.7197
	0.2	0.3975	1.9780	- 0.1448	0.0980	0.4734	- 0.2474	121.7197
	0.4	0.7891	1.9309	- 0.3466	0.1862	0.4005	- 0.4767	121.7197
	0.6	1.1658	1.8193	- 0.8758	0.2554	0.2860	- 0.6564	121.7197
	0.8	1.5024	1.4752	- 3.1303	0.2988	0.1447	- 0.7351	121.7197
	1.0	1.6869	0.0000	-14.7493	0.3131	0.0000	- 0.7031	121.7197
1.00	0.0	0.0000	2.2282	0.0000	0.0000	0.0279	0.0000	173.1767
	0.2	0.4456	2.2281	- 0.0024	0.0054	0.0258	- 0.0197	173.1767
	0.4	0.8910	2.2235	- 0.0667	0.0101	0.0203	- 0.0351	173.1767
	0.6	1.3325	2.1772	- 0.5334	0.0134	0.0122	- 0.0436	173.1767
	0.8	1.7459	1.8661	- 3.3793	0.0150	0.0040	- 0.0347	173.1767
	1.0	1.9848	0.0000	-19.8179	0.0152	0.0000	- .02022	173.1767

TABLE 2. Comparison of values of k.
PM = present method, HM = homotopy method [3], RP = regular perturbation [3]
IM = imbedding method [3], MAE = matched asymptotic expansions [3]

β	S = -0.5			S = 0			S = 1.0			S = 25.0		
	PM	HM	RP	PM	HM	RP	PM	HM	RP	PM	IM	MAE
0	1.8099	1.7865	1.7982	6.0063	6.0000	6.0000	13.8984	13.9443	13.8785	173.1767	175.34	173.95
0.2	1.7683	1.7527	1.8330									
0.25				4.8050	4.8000	4.8000	10.3809	10.3699	10.2472			
0.50	1.6311	1.6248	1.6662	4.0104	4.0000	4.0000	8.4530	8.4336	8.3791	99.8439	99.979	98.401
0.70	1.5048	1.5018	1.5163	3.5386	3.5294	3.5294				87.1966	87.261	85.850
1.00	1.3036	1.3023	1.3045	3.0078	3.0000	3.0000	6.2778	6.2603	6.2465	74.0414	73.865	72.655

TABLE 3. Additional data for k.

S \ β	0	0.2	0.4	0.6	0.8	1.0
-0.5	1.8099	1.7683	1.6857	1.5701	1.4374	1.3036
0	6.0063	5.0052	4.2902	3.7539	3.3368	3.0078
0.5	10.0227	8.0075	6.7409	5.8510	5.1839	4.6618
1	13.8984	10.9030	9.1165	7.8904	6.9829	6.2778
5	42.6763	32.5195	26.9326	23.2204	20.5186	18.4401
10	76.4796	57.9693	47.9310	41.2975	36.4829	32.7854
15	109.35 30	82.7145	68.3460	58.8715	52.0027	46.7310
25	173.1767	131.1877	108.3300	93.2887	82.3960	74.0414