Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER
Linear Algebra and its Applications 371 (2003) 209–224
www.elsevier.com/locate/laa

On maximum principles for M-operators

Anke Kalauch

Institut für Analysis, Technische Universität Dresden, D-01062 Dresden, Germany

Received 1 July 2002; accepted 8 February 2003

Submitted by M. Fiedler

Abstract

For M-matrices a condition to satisfy the "maximum principle for inverse column entries" is known. We generalize this result (concerning a more general maximum principle) for M-operators on $\mathbb{R}^n$, ordered by some cone, as well as, to a certain extent, for M-operators on infinite-dimensional ordered normed spaces.

Keywords: M-operator; Ordered normed space; Maximum principle

1. Introduction

In the present paper we study certain maximum principles for positive operators on ordered normed spaces, in particular for the (positive) inverses of M-operators. In the most of our results the underlying space is $\mathbb{R}^n$ ordered by some convex cone K. We define M-operators as some generalization of M-matrices. Recall that a matrix $A = (a_{ij})_{n,n}$ is an *M-matrix* if $A = sI - B$, where I denotes the identity, B is a positive matrix and $s > r(B)$. [1] The last inequality ensures the invertibility of A, one has $A^{-1} = s^{-1}I + s^{-2}B + \cdots + s^{-k}B^{k-1} + \cdots$, where the series converges in the operator norm, and A^{-1} is positive. A matrix A is an M-matrix iff A has a positive inverse and $a_{ij} \leqslant 0$ for $i \neq j$ (see e.g. [2]). In an arbitrary normed space, ordered by some cone, for operators the "negative-off-diagonal" property is known (see Definition 4.1). We call an operator A on some ordered normed space an *M-operator* if A is

E-mail address: kalauch@math.tu-dresden.de (A. Kalauch).

[1] Here $r(B)$ denotes the spectral radius of the operator B. We consider only *nonsingular* M-matrices. *Singular* M-matrices, i.e. matrices of the form $A = r(B)I - B$, are not relevant for our topic.

0024-3795/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0024-3795(03)00518-4

negative-off-diagonal and has a positive inverse. Note that any operator $A = sI - B$, where B is positive and $s > r(B)$, is an M-operator.[2]

The considered maximum principles for a given linear operator equation in some real [3] vector space X can roughly be described as follows: A positive input causes a positive output, and the maximum response takes place in that part of the system where the influence is nonzero. Notice the following examples:

1. In connection with the study of discrete approximations for differential equations in [10] the following maximum principle for an $n \times n$-matrix $A = (a_{ij})$ is introduced: For any $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ with $Ax = y$ from $y \geqslant 0$, $y \neq 0$ there follows $x \geqslant 0$ and, moreover,
$$\max_{i \in N} x_i = \max_{i \in N^+(y)} x_i, \tag{1}$$
where $N = \{1, \ldots, n\}$ and $N^+(y) = \{i \in N: y_i > 0\}$. In [11] the following proposition is shown: If A is an M-matrix and $Ae \in \mathbb{R}^n_+$, where $e = (1, 1, \ldots, 1)^{\mathrm{T}}$, then A satisfies (1).
In [15, Definition 3.31] the above maximum principle is called the *maximum principle for inverse column entries*. For $y = e^{(i)}$, the ith unit vector, the preimage $x = A^{-1}y$ is the ith column of $A^{-1} = (\bar{a}_{ij})$, and if A is positively invertible and satisfies (1) then A^{-1} is weakly diagonally dominant of its column entries, i.e. $\bar{a}_{ii} \geqslant \bar{a}_{ji}$ for all $i, j \in N$.
2. Let $X = C(T)$ be the space of all continuous functions on some compact topological Hausdorff space T ordered by the cone of all nonnegative functions of $C(T)$. Let $A : C(T) \to C(T)$ be a linear operator which possesses a positive continuous inverse. Then for a positive nonzero function y the question arises, whether there exists a point $t \in T$ at which the function $x = A^{-1}y$ attains its maximum and $y(t) > 0$ holds as well.

One approach to obtain a maximum principle that covers both examples is to replace the values of the "components" of a vector x in (1) by the values of certain "extreme functionals" on an element x of X. In [7] the maximum principle (1) is generalized to linear positively invertible operators $A : \mathbb{R}^n \to \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with a finitely generated cone or a circular cone, respectively. In [5] a generalization is considered for linear positively invertible operators on a normed space $(X, \|\cdot\|)$ ordered by a closed cone K which has a nonempty interior. This covers the both examples above. To avoid the positive inverse of some operator, in [4] the maximum principle MP is formulated for positive operators (see Definition 3.4).

In the present paper we generalize the statement on M-matrices and the maximum principle (1) in the first example to a certain extent for M-operators, concerning MP

[2] In the numerical analysis the notion *M-operator* is sometimes used for an operator $A = sI - B$ on some ordered normed space, where B is positive and $s > r(B)$. Our notion "M-operator" is more general.

[3] Throughout only real vector spaces are considered.

with respect to some interior point of the given cone. It turns out that the facial structure of the cone is crucial. Furthermore, we consider special M-operators, namely operators of the form $A = I - B$, where B is positive and $\|B\| < 1$, and provide sufficient conditions for A^{-1} to satisfy MP.

2. Preliminaries

Let X be a real vector space and M a nonempty subset of X. The linear, positive-linear, affine, convex hull of M is denoted by $\text{lin}(M)$, $c(M)$, $\text{aff}(M)$, $\text{co}(M)$, respectively. For $x \in X \setminus \{0\}$ let $r(x) = \{\lambda x : \lambda \geqslant 0\}$ be the ray generated by x. The set of extreme points of a nonempty convex set $C \subset X$ is denoted by $\text{ext}(C)$ (where $x \in C$ is an *extreme point* of C if $x = \lambda y + (1 - \lambda)z$ implies $x = y = z$ whenever $y, z \in C$ and $\lambda \in (0, 1)$). A set $K \subset X$ is a *wedge* in X if $K = c(K)$. K is a called a *cone* in X if in addition one has $K \cap (-K) = \{0\}$. For a cone K in X we use the notations $x \in K$ and $x \geqslant 0$ synonymously and write $x > 0$ instead of $x \geqslant 0$, $x \neq 0$. A cone K is *generating* if each $x \in X$ can be represented as $x = y - z$ with $y, z \in K$. A subset D of a cone K is a *base* of K if D is a nonempty convex set such that every $x > 0$ has a unique representation $x = \lambda y$ with $y \in D$ and $\lambda > 0$.

If (X, τ) is a real locally convex Hausdorff space and C an arbitrary subset of X, then by $\text{int}(C)$ and ∂C we denote the sets of all interior and boundary points of C, respectively. The *Krein–Milman Theorem* guarantees for any nonempty convex compact subset $C \subseteq X$ both the condition $\text{ext}(C) \neq \emptyset$ and the representation

$$C = \overline{\text{co}}^{\tau}(\text{ext}(C)), \tag{2}$$

where $\overline{\text{co}}^{\tau}(M)$ denotes the τ-closure of the convex hull of $M \subseteq X$.

In the following let $(X, K, \|\cdot\|)$ be a normed space ordered by a cone K. The norm $\|\cdot\|$ is *semi-monotone* if there exists a constant N such that for every $0 \leqslant x \leqslant y$ one has $\|x\| \leqslant N\|y\|$. As usual, X' denotes the space of all continuous linear functionals on X. On X' the weak* topology $\sigma(X', X)$ is considered. The subset

$$K' = \left\{ f \in X' : f(x) \geqslant 0 \quad \text{for every } x \in K \right\}$$

is always a wedge in X'. If K' turns out to be a cone in X', then we call it the *dual cone* to K and introduce an order in X' by means of $f \geqslant 0$ iff $f \in K'$. Note that K' is a cone in X' if and only if X is the norm closure of $K - K$. The condition $\text{int}(K) \neq \emptyset$ implies that K is generating, hence, in this case, K' is a cone. Furthermore, if $\text{int}(K) \neq \emptyset$, then $x \in \text{int}(K)$ if and only if for each $f \in K'$, $f \neq 0$ one has $f(x) > 0$ (see [13, Theorem II.2.2]). If K is a closed cone, then for any $x > 0$ there exists a functional $f \in K'$ such that $f(x) > 0$ (see [13, Theorem II.4.1]).

A *face* of a cone K is a (proper) nonempty subcone $H \neq \{0\}$ of K such that $x \in H$ and $0 \leqslant y \leqslant x$ imply $y \in H$. Note that $H \subset \partial K$. An element $x \in K$ is an *extremal* of K if $r(x)$ is a face of K (i.e. $r(x)$ is an *extreme ray*). If K has a base D with extreme points, then every $x \in \text{ext}(D)$ is an extremal of K. A (nonempty) subset $H \neq \{0\}$ of a cone K is *exposed* if there exists a functional $f \in K'$ such that

$H = f^{-1}(0) \cap K$. Note that every exposed subset is a face, but not vice versa (cf. Example 4.2). If an exposed subset is a ray, we will call it an *exposed ray*.

For some $x \in X$ and $r > 0$ let $B(x, r) = \{v \in X: \|x - v\| \leqslant r\}$ and $B_1 = B(0, 1)$. Let $M \subseteq X$ be a convex set; an element $x \in M$ is a *relative interior point* of M, written $x \in \mathrm{ri}(M)$, if x is an interior point of M in $\mathrm{aff}(M)$, i.e. if there exists a number $r > 0$ such that $B(x, r) \cap \mathrm{aff}(M) \subseteq M$. For a subset $S \subseteq X$ and an element $x \in X$ let $\mathrm{dist}(S, x) = \inf\{\|x - v\|: v \in S\}$.

Note that a cone K in $\mathbb{R}^n$ is generating if and only if $\mathrm{int}(K) \neq \emptyset$. We call a cone K in $\mathbb{R}^n$ *finitely generated* if there exists a nonempty finite set $S \subseteq \mathbb{R}^n$ such that $K = c(S)$ (for more details see [7]). A finitely generated cone is the intersection of a finite number of closed halfspaces and therefore itself is closed. A subset $K \in \mathbb{R}^n$ is called a *circular* cone if there exists an element $z \in \mathbb{R}^n$ with $\|z\| = 1$ and a number $r > 0$ such that

$$K = \left\{\lambda x: x \in \mathbb{R}^n, \langle x, z\rangle = 1, \|x - z\| \leqslant r, \lambda \geqslant 0\right\}. \tag{3}$$

As usual, a linear operator $A : X \to X$ on an ordered vector space (X, K) is *positive* if $AK \subseteq K$. In this case we write $A \geqslant 0$. Denote the set of all continuous linear operators on a normed space $(X, \|\cdot\|)$ by $\mathscr{L}(X)$.

3. Maximum principles: definition and examples

We define the maximum principles for an operator in an arbitrary ordered normed space $(X, K, \|\cdot\|)$, although operators in such a general space only appear occasionally in the following. In the most of our results operators in $(\mathbb{R}^n, K, \|\cdot\|)$ are considered. Let $(X, K, \|\cdot\|)$ be an ordered normed space, where K is a closed cone with a nonempty interior. The following investigations essentially depend on a fixed element of $\mathrm{int}(K)$. The relationship between the maximum principles for different fixed interior points of K are presented in [5] and for a special case in [7]. First we recall some definitions and basic consequences of the condition $\mathrm{int}(K) \neq \emptyset$; for an extended version see [4]. Let u be a fixed element of $\mathrm{int}(K)$. Then the set

$$F := F_u := \{f \in K': f(u) = 1\} \tag{4}$$

is a $\sigma(X', X)$-compact base of K' (see [14, Theorem II.3.2]) and because of (2) one has $\mathrm{ext}(F) \neq \emptyset$ and

$$F = \overline{\mathrm{co}}^{\sigma(X',X)}(\mathrm{ext}(F)).$$

Consider the situation in $\mathbb{R}^n$, equipped with the standard cone:

Example 3.1. Let $X = \mathbb{R}^n$, $K = \mathbb{R}^n_+$ (hence $K' = \mathbb{R}^n_+$) and $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in K$. The corresponding base is

$$F := F_e = \left\{(x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1\right\}$$

and the set $\mathrm{ext}(F)$ consists of the unit vectors

$$e^{(i)} = (0, \dots, 0, 1, 0, \dots, 0)^{\mathrm{T}}$$

with 1 at the ith position, $i = 1, \dots, n$. The value of the functional $e^{(i)}$ at a point $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ corresponds to the component x_i of the vector x.

Example 3.2. In $X = C[0, 1]$ let $K = C_+[0, 1]$ be the cone of all nonnegative functions and for u take the function $\mathbf{1} \in \mathrm{int}(K)$. Then F is the set of all normalized Borel measures on [0, 1], and the set $\mathrm{ext}(F)$ is the collection of the evaluation maps ε_t determined by the points $t \in [0, 1]$, where $\varepsilon_t(x) = x(t)$ for each $x \in C[0, 1]$ (see e.g. [6, Section 10.3]).

In $(X, K, \|\cdot\|)$ for each $x \in X$ the number

$$\alpha(x) = \max\{f(x)\colon f \in F\}$$

is correctly defined since F is $\sigma(X', X)$-compact and the function $\hat{x} : X' \to \mathbb{R}$ defined by means of $f \mapsto f(x)$ is $\sigma(X', X)$-continuous. $\hat{x}$ attains its maximum on F at an extreme point of F (according to the *Bauer maximum principle*), hence

$$\alpha(x) = \max\{f(x)\colon f \in \mathrm{ext}(F)\}. \tag{5}$$

Since K is closed, for any $x > 0$ there is a functional $f \in K'$ such that $f(x) > 0$, and therefore $x > 0$ yields $\alpha(x) > 0$. Note that $\alpha(u) = 1$. For $x \in X$ the set

$$F^{\max}(x) = \{f \in F\colon f(x) = \alpha(x)\}$$

is a nonempty $\sigma(X', X)$-compact extreme convex subset of F containing extreme points of F. In [4] the representation

$$F^{\max}(x) = \overline{\mathrm{co}}^{\sigma(X',X)}\big(\mathrm{ext}(F) \cap F^{\max}(x)\big) \tag{6}$$

is shown. Note that $f \in F^{\max}(x)$, $g \in F$ imply $g(x) \leqslant f(x)$. Furthermore $F^{\max}(u) = F$.

For $H \subseteq K$ let $F^0(H) = \{f \in F\colon f(x) = 0 \text{ for all } x \in H\}$. For $x \in K$ we will abbreviate $F^0(\{x\})$ by $F^0(x)$. We denote $F^+(x) = F \setminus F^0(x)$. Note that for $x > 0$ we have

$$F^{\max}(x) \subseteq F^+(x). \tag{7}$$

We will make use of the following statement.

Proposition 3.3. *For every $x \in K$ there exists a representation $x = \alpha(x)u - z$ where $z \in K$ and $F^{\max}(x) = F^0(z)$.*

Proof. Fix $x \in K$ and consider $z = \alpha(x)u - x$. Assume $z \notin K$. Since K is closed, the sets K and $\{z\}$ can be separated by a hyperplane, i.e. there exists a functional $g \in F$ with $g(z) < 0$. In particular one has $g(x) \geqslant 0$. Due to

$$g(x) = \alpha(x)g(u) - g(z) = \alpha(x) - g(z) > \alpha(x)$$

a contradiction is obtained. □

Definition 3.4. Let $(X, K, \|\cdot\|)$ be an ordered normed space with a closed cone K that has a nonempty interior, let u be a fixed interior point of K and let $A \in \mathscr{L}(X)$ be positive. A is said to satisfy

- the maximum principle MP with respect to u, if for every $x > 0$ one has
  $F^{\max}(Ax) \cap F^+(x) \neq \emptyset$;
- the (strong) maximum principle sMP with respect to u, if for every $x > 0$ one has
  $F^{\max}(Ax) \subseteq F^+(x)$.

A positive continuous linear operator on an ordered normed space that satisfies MP with respect to $u \in \operatorname{int}(K)$ is called an *MP-operator* with respect to u.

Remark. Let $A \in \mathscr{L}(X)$ be positive. If A satisfies sMP with respect to u, then $\{x \in K\colon Ax = u\} \subseteq \operatorname{int}(K)$, since $x \in K$ with $Ax = u$ implies $x \neq 0$, and $F = F^{\max}(u) = F^{\max}(Ax) \subseteq F^+(x)$ yields $x \in \operatorname{int}(K)$.

The maximum principle MP generalizes the maximum principles 1 and 2 discussed in Section 1. Indeed, consider $X = \mathbb{R}^n$, $K = \mathbb{R}^n_+$ and an $n \times n$-matrix A that possesses a positive inverse A^{-1}. Using the description of the extreme points of F in Example 3.1, A satisfies the maximum principle 1 of Section 1 if and only if A^{-1} satisfies MP with respect to $u = (1, 1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. Furthermore, considering an operator $A : C(T) \to C(T)$ that has a positive inverse A^{-1}, then A satisfies the maximum principle 2 of Section 1 if and only if A^{-1} satisfies MP with respect to the function $u \equiv \mathbf{1}$.

To illustrate the dependence of MP on the fixed interior point of K consider $X = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then A satisfies sMP with respect to $u = (1, 1)^{\mathrm{T}}$, but A does not satisfy even MP, e.g. with respect to $v = (1, 3)^{\mathrm{T}}$, since $\operatorname{ext}(F_v) = \{(1, 0)^{\mathrm{T}}; (0, \frac{1}{3})^{\mathrm{T}}\}$, and e.g. $x = (0, 3)^{\mathrm{T}}$ yields $Ax = (3, 6)^{\mathrm{T}}$, $\alpha(Ax) = 3$ and $F_v^{\max}(Ax) = \{(1, 0)^{\mathrm{T}}\} \subseteq F_v^0(x)$.

Due to (7) the identity operator I in an ordered normed space $(X, K, \|\cdot\|)$ always satisfies sMP (and hence MP) with respect to an arbitrary element $u \in \operatorname{int}(K)$. For an MP-operator A with respect to u the operator λA for some $\lambda > 0$ is an MP-operator with respect to u as well. The sum and the composition of MP-operators need not be MP-operators even in a finite-dimensional space equipped with the standard cone, as the following example shows.

Example 3.5. Let $X = \mathbb{R}^3$, $K = \mathbb{R}^3_+$,

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

We consider the maximum principles MP and sMP with respect to the element $u = (1, 1, 1)^{\mathrm{T}}$ (cf. Example 3.1). Both matrices A and B satisfy MP, but neither A nor B satisfies sMP. For $x = (1, 0, 1)^{\mathrm{T}}$ one has $F^{+}(x) \cap \mathrm{ext}(F) = \{e^{(1)}, e^{(3)}\}$. Since $(A + B)x = (3, 4, 3)^{\mathrm{T}}$ yields $F^{\max}((3, 4, 3)^{\mathrm{T}}) \cap \mathrm{ext}(F) = \{e^{(2)}\}$, the operator $A + B$ does not satisfy MP. This also points out that a convex combination of MP-operators, e.g. $\frac{1}{2}A + \frac{1}{2}B$, in general is not an MP-operator. Furthermore, a simple calculation shows that for any $k \in \mathbb{N}$, $k \geqslant 2$, one has $F^{\max}(A^k x) \cap \mathrm{ext}(F) = \{e^{(2)}\}$, hence no power of A satisfies MP. Finally, the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies $I \leqslant C \leqslant A$, i.e. C is bounded from both sides by MP-operators. However, C is not an MP-operator since $Cx = (1, 2, 1)^{\mathrm{T}}$ implies that $F^{\max}\{Cx\} = \{e^{(2)}\}$.

4. *M*-operators

In the following let $(X, K, \|\cdot\|)$ again be an ordered normed space.

Definition 4.1. An operator $A \in \mathscr{L}(X)$ is called negative-off-diagonal (nod) if $x \in K$ and $f \in K'$ with $f(x) = 0$ imply $f(Ax) \leqslant 0$. A nod operator A that possesses a positive inverse A^{-1} is called an M-operator.

Obviously, an operator $A = sI - B$, where B is positive and $s \geqslant 0$, is a nod operator, and it is an M-operator if $s > r(B)$. In general, a nod operator can not be represented as $sI - B$, where B is a positive operator (see e.g. [9]). For an example of an M-operator that is not representable see e.g. [1, 4(1.6)].

If the cone K is closed and satisfies $\mathrm{int}(K) \neq \emptyset$, then we fix an arbitrary element $u \in \mathrm{int}(K)$ and define a base F of the cone K' corresponding (4). In Definition 4.1 we can restrict ourselves to elements $x \in \partial K$ and to functionals of the set $F^0(x) \cap \mathrm{ext}(F)$, since

$$F^0(x) = \overline{\mathrm{co}}^{\sigma(X', X)}\left\{F^0(x) \cap \mathrm{ext}(F)\right\} \tag{8}$$

(see e.g. [4]). Hence A is a nod operator if and only if $x \in \partial K$ and $f \in \mathrm{ext}(F)$ with $f(x) = 0$ imply $f(Ax) \leqslant 0$.

If we consider the space $(\mathbb{R}^n, \mathbb{R}^n_+, \|\cdot\|)$, then a matrix $A = (a_{ij})_{n,n}$ is a nod [4] operator if and only if $a_{ij} \leqslant 0$ for $i \neq j$. Indeed, let A be a nod operator and suppose $a_{ij} > 0$ for some pair (i, j) such that $i \neq j$. Then the jth unit vector $x = e^{(j)}$

[4] In the matrix theory sometimes the notion *cross-positive* for a matrix A is used provided $(-A)$ is negative-off-diagonal.

yields $x_i = 0$ and $y_i = (Ax)_i = a_{ij} > 0$, a contradiction. Vice versa, let $a_{ij} \leqslant 0$ for $i \neq j$ and $x \in \mathbb{R}^n_+$ with $x_i = 0$ for some fixed i. Then $y_i = (Ax)_i = \sum_j a_{ij}x_j = \sum_{j\neq i} a_{ij}x_j \leqslant 0$ shows that A is a nod operator.

In $(\mathbb{R}^n, \mathbb{R}^n_+, \|\cdot\|)$ a matrix A is an M-operator if and only if A is a (nonsingular) M-matrix.

In analogy to the Example 1 from Section 1 we examine the following question:

(Q) *Let $(X, K, \|\cdot\|)$ be an ordered normed space, $u \in \operatorname{int}(K)$ a fixed element and $A \in \mathscr{L}(X)$ an M-operator. Is A^{-1} an MP-operator with respect to u provided $Au \in K$?*

In the space $X = \mathbb{R}^n$ equipped with the standard cone $K = \mathbb{R}^n_+$ a confirmative answer for (Q) is given in [12]. It turns out that the stronger condition $Au \in \operatorname{int}(K)$ yields a confirmative answer for arbitrary ordered normed spaces, as follows from Theorem 4.3. It remains to examine the condition $Au \in \partial K$. In this case (Q) can in general not be answered confirmatively (even if X is finite dimensional; see Example 4.2). A confirmative answer is obtained in $X = \mathbb{R}^n$ if the cone K has a special facial structure (see Theorem 4.8). In the case of an infinite dimensional space X no satisfactory results are known.

The next example shows that there exists an M-operator A, satisfying $Au \in K$ for some $u \in \operatorname{int}(K)$, which is not an MP-operator with respect to u.

Example 4.2. In the ordered normed space $(\mathbb{R}^3, K, \|\cdot\|)$ let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^3$ and $K = K_1 \cup K_2$, where

$$K_1 = \left\{t(x_1, x_2, 1)^{\mathrm{T}}: x_1^2 + x_2^2 \leqslant 1, t \geqslant 0\right\}$$

is a circular cone and

$$K_2 = \left\{t(x_1, x_2, 1)^{\mathrm{T}}: 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1, t \geqslant 0\right\}$$

is finitely generated. We illustrate the following considerations in Fig. 1 where we use $v = (0, 0, 1)^{\mathrm{T}} \in \operatorname{int}(K)$ and the corresponding bases $D_v = \{x \in K: \langle x, v\rangle = 1\}$ of K and $F_v = \{x \in K': \langle x, v\rangle = 1\}$ of K', i.e. the intersections of K and K' with the plane $x_3 = 1$ for elements $(x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$. The operator

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

is an M-operator in the space $(\mathbb{R}^3, K, \|\cdot\|)$, since

(i) A is a nod operator with respect to K. Indeed, consider the following cases of extreme points of F_v and elements $x = (x_1, x_2, 1)^{\mathrm{T}} \in K$:
 1. Let $\tilde{f} = (w_1, w_2, 1)^{\mathrm{T}}$ with $w_1^2 + w_2^2 = 1$ and at least one $w_i > 0$. Then $\langle x, \tilde{f}\rangle = 0$ iff $x_1 = -w_1$ and $x_2 = -w_2$. This yields $\tilde{f}(Ax) = \tilde{f}((-w_1 - 2w_2 - 2,\ 2w_1 - w_2,\ 2w_1 + 1)^{\mathrm{T}}) = -(w_1^2 + w_2^2) = -1 \leqslant 0$.

2. For $\tilde{f}_1 = (-1, 0, 1)^{\mathrm{T}}$ we get $\langle x, \tilde{f}_1\rangle = 0$ iff $x_1 = 1$ and $0 \leqslant x_2 \leqslant 1$. Hence $\tilde{f}_1(Ax) = \tilde{f}_1((2x_2 - 1, x_2 - 2, -1)^{\mathrm{T}}) = -2x_2 \leqslant 0$.
3. Finally, let $\tilde{f}_2 = (0, -1, 1)^{\mathrm{T}}$. Then $\langle x, \tilde{f}_2\rangle = 0$ iff $x_2 = 1$ and $0 \leqslant x_1 \leqslant 1$. We conclude $\tilde{f}_2(Ax) = \tilde{f}_2((x_1, 1 - 2x_1, 1 - 2x_1)^{\mathrm{T}}) = 0$.

(ii) the operator

$$A^{-1} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 4 \\ 2 & -4 & 5 \end{pmatrix}$$

is positive with respect to K (see Fig. 1, where the base G_v of the cone $A^{-1}K$ is illustrated).

We consider the element $u = (3, 6, 7)^{\mathrm{T}} \in \mathrm{int}(K)$ and show that A^{-1} does not satisfy MP with respect to u, although $Au \in K$. Observe that $z = Au = (1, 0, 1)^{\mathrm{T}} \in \partial K$ and that the ray $r(z)$ is a face of K which is not an exposed subset of K. (The elements $\tilde{u} = \frac{1}{7}u \in D_v$ and $z \in D_v$ are illustrated in Fig. 1.) The base F_u of K' possesses the following set of extreme points:

$$\mathrm{ext}(F_u) = \left\{ \frac{1}{3w_1 + 6w_2 + 7}(w_1, w_2, 1)^{\mathrm{T}} \colon w_1^2 + w_2^2 = 1,\, w_1 \geqslant 0 \text{ or } w_2 \geqslant 0 \right\}.$$

In particular, $f_1 = \frac{1}{4}\tilde{f}_1$ and $f_2 = \tilde{f}_2$ are extreme points of F_u. Note that $Az = (-1, -2, -1)^{\mathrm{T}}$ and $f_1(z) = f_1(Az) = 0$. Now put $x := u - z = (2, 6, 6)^{\mathrm{T}} \in K$, then $y := Ax = (1, 1, 1)^{\mathrm{T}} \in K$. (The elements $\tilde{x} = \frac{1}{6}x \in D_v$ and $y \in D_v$ are illustrated in Fig. 1.) Because of Proposition 3.3 one has $F^{\max}(x) = \{f_1\}$, furthermore $f_1(y) = 0$. Hence A^{-1} does not satisfy MP with respect to u.

The next theorem answers (Q) in the case $Au \in \mathrm{int}(K)$.

Theorem 4.3. *Let $(X, K, \|\cdot\|)$ be an ordered normed space with a closed cone K that has a nonempty interior and let $u \in \mathrm{int}(K)$ be fixed. If $A \in \mathscr{L}(X)$ is an M-operator that satisfies $Au \in \mathrm{int}(K)$, then A^{-1} satisfies sMP with respect to u.*

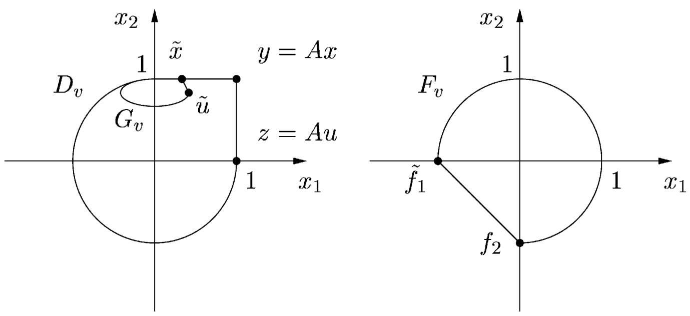

Fig. 1. Illustration of Example 4.2.

Proof. For an element $y > 0$ we get $x = A^{-1}y > 0$ and hence $\alpha(x) > 0$. Fix an arbitrary functional $f \in F^{\max}(x)$, then Proposition 3.3 yields $x = \alpha(x)u - z$ where $z \in K$ and $f(z) = 0$. The assumption $Au \in \operatorname{int}(K)$ ensures $f(Au) > 0$, furthermore $f(Az) \leqslant 0$ since A is a nod operator. Hence $f(Ax) = \alpha(x)f(Au) - f(Az) > 0$ and A^{-1} satisfies sMP. □

Now we examine (Q) if $Au \in \partial K$. Example 4.2 already showed that in general (Q) can be answered negatively in this case. To get a confirmative answer for (Q) we need some more assumptions on the structure of ∂K. We confine ourselves to $X = \mathbb{R}^n$, equipped with a closed generating cone K (i.e. K has nonempty interior). As above, fix some element $u \in \operatorname{int}(K)$ and the base F of K' corresponding (4). For a set $H \subseteq K$ put

$$E_H = \bigcap \left\{ f^{-1}(0) \colon f \in F^0(H) \right\}.$$

Note that E_H is a closed linear subspace of X. For an element $z \in K$ we will abbreviate $E_{\{z\}}$ by E_z. If $z \in H$, then $F^0(H) \subseteq F^0(z)$ and $E_z \subseteq E_H$. If H is a face of K, then the inclusions $H \subseteq K \cap E_H$ and $H - H \subseteq E_H$ are obvious. As an example such that $H - H \neq E_H$ see Example 4.2 with $H := r(z)$.

Lemma 4.4. *Let K be a closed generating cone in $\mathbb{R}^n$ and let H be a face of K with $H - H = E_H$. Then H is an exposed subset of K.*

Proof. First notice that $H = (H - H) \cap K$. Indeed, if $x = x_1 - x_2 \geqslant 0$ with x_1, $x_2 \in H$, then $x_1 \geqslant x \geqslant 0$ and, since H is a face, $x \in H$.

For $\varepsilon > 0$ put $S_\varepsilon = \{x \in \mathbb{R}^n \colon \operatorname{dist}(E_H, x) < \varepsilon\}$. For the closed unit ball B_1 in $\mathbb{R}^n$ one has

$$\begin{aligned} B_1 &= (B_1 \cap E_H) \cup \left[B_1 \Big\backslash \bigcap_{f \in F^0(H)} f^{-1}(0) \right] \\ &\subseteq (B_1 \cap S_\varepsilon) \cup \bigcup_{f \in F^0(H)} \left(B_1 \setminus f^{-1}(0) \right) \end{aligned}$$

which provides an open covering for B_1 in the induced topology. Since B_1 is compact we get a finite covering, i.e. there exist functionals $f_i \in F^0(H)$, $i = 1, \ldots, k$, such that

$$B_1 \subseteq (B_1 \cap S_\varepsilon) \cup \bigcup_{i=1}^{k} \left(B_1 \setminus f_i^{-1}(0) \right) = (B_1 \cap S_\varepsilon) \cup \left(B_1 \Big\backslash \bigcap_{i=1}^{k} f_i^{-1}(0) \right).$$

This implies

$$B_1 \cap \bigcap_{i=1}^{k} f_i^{-1}(0) \subseteq B_1 \cap S_\varepsilon$$

for every $\varepsilon > 0$. Hence $\bigcap_{i=1}^{k} f_i^{-1}(0) \subseteq \overline{E_H}$ and, since E_H is closed,

$$E_H = \bigcap_{i=1}^{k} f_i^{-1}(0).$$

Put $g = \frac{1}{k}\sum_{i=1}^{k} f_i$ and observe that $g^{-1}(0) \cap K = E_H \cap K$. Now, by assumption, $H = (H - H) \cap K = E_H \cap K = g^{-1}(0) \cap K$, hence H is exposed. □

Recall that for a face H of K and an element $z \in H$ one has $z \in \mathrm{ri}(H)$ if and only if there is a number $\varepsilon > 0$ such that $B(z, \varepsilon) \cap (H - H) \subseteq H$. The smallest face G of K that contains z is the set

$$G = \{x \in X\text{: there is a number } \lambda \geqslant 0 \text{ such that } 0 \leqslant x \leqslant \lambda z\}.$$

This implies $F^0(z) = F^0(G)$ and hence

$$E_z = E_G. \tag{9}$$

Lemma 4.5. *Let K be a closed generating cone in $\mathbb{R}^n$, H a face of K and $z \in H$. Then the properties*

(i) $z \in \mathrm{ri}(H)$,
(ii) *H is the smallest face of K that contains z*

are equivalent. If, in addition, every face of K is an exposed subset of K, then (i) *and* (ii) *are equivalent to the property*

(iii) $H = K \cap E_z$.

Proof. For the equivalence of (i) and (ii) see [3, Theorem 5.6].

(ii) ⇒ (iii): Because of (9) we immediately get $H \subseteq K \cap E_H = K \cap E_z$. Vice versa, since H is exposed, there exists a functional $g \in K'$ such that $H = g^{-1}(0) \cap K$. From $g(z) = 0$ we get $g \in F^0(z)$ and $E_z \subseteq g^{-1}(0)$. Hence $K \cap E_z \subseteq K \cap g^{-1}(0) = H$.

(iii) ⇒ (ii): Let $H = K \cap E_z$ and let G be the smallest face of K that contains z. The inclusion $G \subseteq H$ is obvious. G is exposed, i.e. there exists a functional $g \in K'$ such that $G = g^{-1}(0) \cap K$. Since $z \in G$ we get $g \in F^0(z)$ and $E_z \subseteq g^{-1}(0)$, hence $H \subseteq K \cap g^{-1}(0) = G$. □

Note that for a face H of K with $\dim(H) = n - 1$ the set $F^0(H)$ contains only one element, say f, and $f^{-1}(0) = H - H$. Furthermore, $f^{-1}(0) \cap K = H$, hence H is exposed. For example, if K is a closed generating cone in $\mathbb{R}^3$ and every extreme ray of K is exposed, then K has the property required in Lemma 4.5 that every face of K is an exposed subset of K. If K is a circular cone in $\mathbb{R}^n$, one easily shows that every face H of K is an exposed ray.

Lemma 4.6. *Let K be a finitely generated cone in $\mathbb{R}^n$ that is generating. Then every face H of K is exposed and $H - H = E_H$.*

Proof. The cone K has a base D with $\mathrm{ext}(D) = \{x_1, x_2, \ldots, x_r\}$, and K' possesses a base F with $\mathrm{ext}(F) = \{f_1, f_2, \ldots, f_s\}$ (for details see e.g. [7]). If G is a face of K, one has $G = c\{x_{i_1}, \ldots, x_{i_k}\}$ for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$, $F^0(G) = \mathrm{co}\{f_{j_1}, \ldots f_{j_l}\}$ for some $\{j_1, \ldots, j_l\} \subseteq \{1, \ldots, s\}$ and $G = f^{-1}(0) \cap K$ for $f = \frac{1}{l}\sum_{m=1}^{l} f_{j_m} \in F$. Hence every face of K is an exposed subset of K.

Now let H be a face of K, $z \in \mathrm{ri}(H)$ and $v \in E_H \setminus \{0\}$, we show $v \in H - H$. Observe that $F^0(z) = F^0(H) \subseteq F^0(v)$. Since $F^+(z) \cap \mathrm{ext}(F)$ is a finite set one has $b(z) := \min\{g(z)\colon g \in F^+(z) \cap \mathrm{ext}(F)\} > 0$. Put $a(v) := \max\{|f(v)|\colon f \in F\}$, then $a(v) > 0$ since K is generating and closed. We get $1 \geqslant g(v)/a(v) \geqslant -1$ for each $g \in F$. Put $\varepsilon := b(z)/[2a(v)]$ and $w := z + \varepsilon v$ and note that $\varepsilon > 0$. For every $g \in F^+(z) \cap \mathrm{ext}(F)$ one has

$$g(w) = g(z) + \frac{b(z)}{2} \cdot \frac{g(v)}{a(v)} \geqslant b(z) - \frac{b(z)}{2} > 0.$$

If $g \in F^0(z)$ then $g \in F^0(v)$ and we get $g(w) = 0$. Consequently, $g(w) \geqslant 0$ for every $g \in F$ and, since K is closed, $w \in K$. We conclude $w \in K \cap E_H$ and apply Lemma 4.5 and (9). Since every face of K is exposed we get $w \in K \cap E_z = H$. This yields

$$v = \frac{w}{\varepsilon} - \frac{z}{\varepsilon} \in H - H,$$

and hence $E_H = H - H$. □

Before we come to the main theorem notice the following simple statement (without proof).

Proposition 4.7. *Let E be a linear subspace of $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ a regular matrix such that $A(E) \subseteq E$. Then $A|_E : E \to E$ is surjective.*

Theorem 4.8. *Let K be a closed generating cone in $\mathbb{R}^n$. Let every extreme ray of K be exposed, and for every face H of K with $\dim(H) > 1$ let $H - H = E_H$. Furthermore, let $u \in \mathrm{int}(K)$ be fixed and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an M-operator in $(\mathbb{R}^n, K, \|\cdot\|)$. If $Au \in K$, then A^{-1} is an MP-operator with respect to u.*

Proof. Let A be an M-operator with $Au \in K$ and suppose that A^{-1} does not satisfy MP with respect to u. Then there exists an element $y > 0$ such that for every functional $f \in F^{\max}(A^{-1}y)$ one has $f(y) = 0$. Now the idea is to find a linear subspace E of $\mathbb{R}^n$ that is invariant with respect to A, i.e. $A(E) \subseteq E$, such that $u \notin E$ and

$Au \in E$. Then Proposition 4.7 implies that there exists an element $e \in E$ such that $Ae = Au$. Since A is injective we get a contradiction.

Put $x = A^{-1}y$. Hence $\alpha(x) > 0$. Due to Proposition 3.3 there exists an element $z \in K$ such that $x = \alpha(x)u - z$ and $F^0(z) = F^{\max}(x)$. Note that $z \neq 0$, otherwise $F = F^0(z) = F^{\max}(x) \subseteq F^0(y)$ which contradicts $y \neq 0$. Let $f \in F^0(z)$. Since A is a nod operator, we get $f(Az) \leqslant 0$. The condition $Au \in K$ implies $f(Au) \geqslant 0$. Since $0 = f(y) = \alpha(x)f(Au) - f(Az)$, $f(Au) \geqslant 0$ and $-f(Az) \geqslant 0$ one has $f(Au) = f(Az) = 0$ for every $f \in F^0(z)$. The set $H := K \cap E_z$ is a face of K with $r(z) \subseteq H$ and $Au \in H$. Consider the following cases:

(i) $H = r(z)$. Since $y \in H$ one has $Az = \alpha(x)Au - y \in H - H$. If v is an arbitrary element of $H - H$, then $v = \lambda z$ for some $\lambda \in \mathbb{R}$ and $Av = \lambda Az$, i.e. Av belongs to $H - H$. Hence the linear subspace $E := H - H$ of $\mathbb{R}^n$ is invariant with respect to A, furthermore $Au \in E$ and, obviously, $u \notin E$.

(ii) $H \neq r(z)$. Lemma 4.4 and the assumption ensure that every face of K is exposed. Hence, due to Lemma 4.5, the element z is a relative interior point of H, i.e. there exists a number $\varepsilon > 0$ such that $B(z, \varepsilon) \cap (H - H) \subseteq H$. Furthermore $E_z = E_H$, and, by assumption, $E_z = H - H$. Consider an element $v \in E_z$ with $\|v\| \leqslant \varepsilon$. We get $z \pm v \in B(z, \varepsilon) \cap E_z = B(z, \varepsilon) \cap (H - H) \subseteq H$ and, in particular, $f(z \pm v) = 0$ for every $f \in F^0(z)$. Since A is a nod operator one has $0 \geqslant f(A(z \pm v)) = f(Az) \pm f(Av) = \pm f(Av)$ and hence $f(Av) = 0$ for each $f \in F^0(z)$. This yields $Av \in E_z$. Hence $E := E_z$ is an invariant linear subspace of $\mathbb{R}^n$ with respect to the operator A with $Au \in E$ and $u \notin E$. □

Due to Lemma 4.6 and the remarks on the facial structure of a circular cone in $\mathbb{R}^n$ we get the following.

Corollary 4.9. *Let K be a circular or a finitely generated cone in $\mathbb{R}^n$ that is generating, let $u \in \operatorname{int}(K)$ and $A: \mathbb{R}^n \to \mathbb{R}^n$ an M-operator. If $Au \in K$ then A^{-1} is an MP-operator with respect to u.*

5. Operators $I - B$ with positive B

In this section we examine the above maximum principles for a special class of M-operators on an arbitrary ordered normed space $(X, K, \|\cdot\|)$ (where the cone K is closed and has a nonempty interior), namely operators $A = I - B$ where $B \geqslant 0$ and $\|B\| < 1$.

We start with a simple consequence of Theorem 4.3. Let $u \in \operatorname{int}(K)$ and

$$\gamma = \sup\left\{t \in \mathbb{R}_+ : B(u, t) \subset K\right\}. \tag{10}$$

Then $\gamma > 0$, and γ is finite due to $K \cap (-K) = \{0\}$. For $x \in K \setminus \{0\}$ one has

$$u \pm \gamma \frac{x}{\|x\|} \geqslant 0 \tag{11}$$

and, in particular, $u - \gamma Bu/\|Bu\| \geqslant 0$. If

$$\gamma > \|Bu\| \tag{12}$$

then $Au = u - Bu \in \mathrm{int}(K)$. Consequently, due to Theorem 4.3, the condition (12) ensures that A^{-1} satisfies sMP with respect to u.

For fixed $u \in \mathrm{int}(K)$ let F be the base of K' corresponding (4), and for $x \in K$ let $\alpha(x)$ be defined corresponding (5). Due to (11) for every $x \in K \setminus \{0\}$ and every $f \in F$ one has

$$0 \leqslant f\left(u \pm \gamma \frac{x}{\|x\|}\right) = 1 \pm \gamma \frac{f(x)}{\|x\|}$$

which implies $|f(x)| \leqslant \frac{1}{\gamma}\|x\|$ and $\alpha(x) \leqslant \frac{1}{\gamma}\|x\|$ (which also holds for $x = 0$). Put

$$M = \frac{1}{\gamma}.$$

Then $M > 0$ and $\alpha(x) \leqslant M\|x\|$ for every $x \in K$. If, in addition, we have an estimate $m\|x\| \leqslant \alpha(x)$ for every $x \in K$ and some constant $m > 0$, then the following theorem provides a condition for an operator $I + C$, $C \geqslant 0$, to satisfy sMP.

Theorem 5.1. *Let $(X, K, \|\cdot\|)$ be an ordered normed space, where K is a closed cone that has a nonempty interior, and let $u \in \mathrm{int}(K)$ be fixed. Let $C \in \mathscr{L}(X)$ be a positive operator. If there exists a constant $m > 0$ such that $m\|x\| \leqslant \alpha(x)$ for each $x \in K$ and $\|C\| < m/M$, then the operator $I + C$ satisfies sMP with respect to u.*

Proof. We fix $x > 0$ and have to show $f(x) > 0$ for each $f \in F^{\max}(x + Cx)$. First note that for some $g \in F^{\max}(x)$ one has

$$0 < \|x\| \leqslant \frac{\alpha(x)}{m} = \frac{g(x)}{m} \leqslant \frac{g(x)}{m} + \frac{g(Cx)}{m} = \frac{g(x + Cx)}{m}. \tag{13}$$

Fix $f \in F^{\max}(x + Cx)$ and suppose $f(x) = 0$. Then $g(x + Cx) \leqslant f(x + Cx) = f(Cx)$, and from (13) we conclude

$$0 < \|x\| \leqslant \frac{f(Cx)}{m} \leqslant \frac{\alpha(Cx)}{m} \leqslant \frac{M\|Cx\|}{m} \leqslant \frac{M}{m}\|C\|\|x\| < \|x\|,$$

which is a contradiction. Hence $I + C$ satisfies sMP. □

For $A = I - B$, where $B \geqslant 0$ and $\|B\| < 1$, the inverse operator

$$(I - B)^{-1} = I + B + B^2 + \cdots$$

is positive and can be represented as $(I - B)^{-1} = I + C$, where $C = B + B^2 + \cdots$ and

$$\|C\| = \|B + B^2 + \cdots\| \leqslant \|B\| + \|B\|^2 + \cdots = \frac{\|B\|}{1 - \|B\|}\,. \tag{14}$$

Corollary 5.2. *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies all assumptions of Theorem 5.1 and let $B \in \mathscr{L}(X)$ be a positive operator with*

$$\|B\| < \frac{m}{m+M}.$$

Then $(I - B)^{-1}$ *satisfies* sMP *with respect to* u.

Proof. By assumption,

$$\frac{\|B\|}{1-\|B\|} < \frac{m}{M}.$$

If we apply (14) and Theorem 5.1, then the result is immediate. □

Now we show how the constant m can be determined, where we are guided by the following simple.

Example 5.3. Let $X = C[0, 1]$ be equipped with the maximum norm and the cone $K = C_+[0, 1]$. Assume $u \in \operatorname{int}(K)$ and define F corresponding (4), then $\operatorname{ext}(F)$ is the set of maps ε_t, determined by the points $t \in [0, 1]$, with $\varepsilon_t(x) = x(t)/u(t)$ for each $x \in C[0, 1]$, and $\alpha(x) = \max\{\varepsilon_t(x): t \in [0, 1]\}$. For every $x \in K$ one has

$$\frac{1}{\|u\|}\|x\| = \max_{t\in[0,1]} \frac{x(t)}{\|u\|} \leqslant \max_{t\in[0,1]} \frac{x(t)}{u(t)} = \alpha(x),$$

hence we get $m = 1/\|u\|$. Note that $\gamma = \min\{u(t): t \in [0, 1]\}$ and, obviously,

$$\alpha(x) \leqslant \max_{t\in[0,1]} \frac{x(t)}{\gamma} = \frac{1}{\gamma}\|x\|. \qquad \square$$

The constant m can be ascertained if in addition to the assumptions of Theorem 5.1 the norm $\|\cdot\|$ is assumed to be semi-monotone. [5]

Proposition 5.4. *Let* $(X, K, \|\cdot\|)$ *be an ordered normed space, where* K *is a closed cone with a nonempty interior, and fix* $u \in \operatorname{int}(K)$. *Let the norm* $\|\cdot\|$ *be semi-monotone with the constant* N. *Then for each* $x \in K$ *one has*

$$\frac{1}{N\|u\|}\|x\| \leqslant \alpha(x).$$

Proof. Fix $x \in K \setminus \{0\}$ and consider $z = \alpha(x)u - x$. Due to Proposition 3.3 one has $z \in K$ and therefore $x + z \geqslant x$. The semi-monotony of the norm yields

$$\|u\| = \left\| \frac{1}{\alpha(x)}(x+z) \right\| \geqslant \frac{1}{\alpha(x)} \cdot \frac{1}{N}\|x\|,$$

hence

$$\frac{1}{N\|u\|}\|x\| \leqslant \alpha(x). \qquad \square$$

[5] If the interval $[0, u]$ is norm bounded, then the norm is semi-monotone (see e.g. [13, Theorem IV.2.3]).

Remark. If (X, K) is a vector lattice equipped with a lattice norm, i.e. $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$, then $N = 1$ and $m = 1/\|u\|$.

Summarizing the above results, we get the following conclusion: If the ordered normed space $(X, K, \|\cdot\|)$ satisfies all assumptions of Proposition 5.4 and if $B \in \mathscr{L}(X)$ is a positive operator with

$$\|B\| < \frac{1}{1 + \frac{N \cdot \|u\|}{\gamma}},$$

then by Corollary 5.2 the operator $(I - B)^{-1}$ satisfies sMP with respect to u.

Acknowledgements

The author thanks Prof. M.R. Weber (Technical University Dresden, Institute for Analysis) and Prof. B.M. Makarov (Saint-Petersburg State University, Faculty of Mathematics and Mechanics) for helpful comments.

References

[1] A. Berman, M. Neumann, R. Stern, Nonnegative Matrices in Dynamic Systems, John Wiley and Sons, New York, 1989.
[2] A. Berman, J. Plemmons, Nonnegative Matrices in Mathematical Sciences, Siam, Philadelphia, 1994.
[3] A. Brøndsted, An Introduction to Convex Polytopes, Springer-Verlag, New York, Heidelberg, Berlin, 1983.
[4] A. Kalauch, M.R. Weber, On a certain maximum principle for positive operators in an ordered normed space, Positivity 4 (2000) 179–195.
[5] A. Kalauch, M.R. Weber, On a certain maximum principle for positively invertible matrices and operators in an ordered normed space, Lecture Notes in Pure and Applied Mathematics: Function Spaces—the Fifth Conference, vol. 213, 2000, pp. 217–230.
[6] L. Narici, E. Beckenstein, Topological Vector Spaces, Marcel Dekker Inc., New York, Basel, 1985.
[7] H. Pühl, W. Schirotzek, M.R. Weber, On matrices satisfying a maximum principle with respect to a cone, Linear Algebra Appl. 277 (1–3) (1998) 337–356.
[8] H.H. Schaefer, Banach Lattices and Positive Operators, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
[9] H. Schneider, M. Vidyasagar, Cross-positive matrices, SIAM J. Numer. Anal. 7 (1970) 508–519.
[10] G. Stoyan, On a maximum principle for matrices and on conservation of monotonicity with applications to discretization methods, ZAMM 62 (1982) 375–381.
[11] G. Stoyan, On the maximum principles for monotone matrices, Linear Algebra Appl. 78 (1986) 147–161.
[12] C. Türke, M.R. Weber, On a maximum principle for inverse monotone matrices, Linear Algebra Appl. 218 (1995) 47–57.
[13] B.Z. Vulikh, Introduction to the theory of cones in normed spaces (Russian), Izdat. Kalinin Univ., Kalinin, 1977.
[14] B.Z. Vulikh, Special topics of the geometry of cones in normed spaces (Russian). Izdat. Kalinin Univ., Kalinin, 1978.
[15] G. Windisch, M-matrices in numerical analysis, Teubner-Texte Math. 115, Leipzig, 1989.