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On Simultaneous Approximation by Modified Lupas Operators

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1. INTRODUCTION

Lupas proposed a family of linear positive operators mapping $C[0, \infty)$ into $C[0, \infty)$, the class of all bounded and continuous functions on $[0, \infty)$, namely,

$$(L_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f(k/n),$$

where $x \in [0, \infty)$.

Motivated by Derriennic [1], we propose modified Lupas operators defined, for functions integrable on $[0, \infty)$, as

$$(M_n f)(x) = (n-1) \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} P_{n,k}(y) f(y)\, dy,$$

where

$$P_{n,k}(t) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}.$$

The object of the present paper is to study the problem of simultaneous approximation by these operators. Throughout this paper, the superscript (r) and $\|\cdot\|$ stand for the rth derivative of the function and the sup norm on $[0, a]$, respectively. Also, $\sum$ stands for $\sum_{k=0}^{\infty}$.

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2. Preliminary Results

We shall need the following lemmas.

Lemma 1. *Let*

$$T_{n,m} = (n-r-1) \sum P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y)(y-x)^m \, dy$$

then

$$T_{n,0} = 1 \tag{2.1}$$

$$T_{n,1} = \frac{(r+1)(1+2x)}{(n-r-2)}, \qquad n > r+2 \tag{2.2}$$

$$\begin{aligned}(n-m-r-2)\, T_{n,m+1} &= x(1+x)(T_{n,m}^{(1)} + 2m T_{n,m-1}) \\ &\quad + (m+r+1)(1+2x)\, T_{n,m},\end{aligned} \tag{2.3}$$

where $n > m+r+2$.

Proof. We can easily verify (2.1) and (2.2), while proof of (2.3) follows. We have

$$T_{n,m}^{(1)} = (n-r-1) \sum P_{n+r,k}^{(1)}(x) \int_0^\infty P_{n-r,k+r}(y)(y-x)^m \, dy - m \cdot T_{n,m-1}.$$

Using $t(1+t)\, P_{u,v}^{(1)}(t) = (v-ut)\, P_{u,v}(t)$ twice and integrating by parts, we get

$$\begin{aligned} &x(1+x)(T_{n,m}^{(1)} + m \cdot T_{n,m-1}) \\ &\quad = (n-r-1) \sum P_{n+r,k}(x) \int_0^\infty y(1+y)\, P_{n-r,k+r}^{(1)}(y)(y-x)^m \, dy \\ &\quad\quad - r(1+2x)\, T_{n,m} + (n-r) \cdot T_{n,m+1} \end{aligned}$$

or

$$\begin{aligned} &x(1+x)(T_{n,m}^{(1)} + m \cdot T_{n,m-1}) + r(1+2x) \cdot T_{n,m} - (n-r) \cdot T_{n,m+1} \\ &\quad = (n-r-1) \sum P_{n+r,k}(x) \int_0^\infty ((1+2x)(t-x) + (t-x)^2 + x(1+x)) \\ &\quad\quad \times P_{n-r,k+r}^{(1)}(y) \cdot (y-x)^m \, dy \\ &\quad = -(m+1)(1+2x) \cdot T_{n,m} - (m+2) \cdot T_{n,m+1} - mx \cdot (1+x) \cdot T_{n,m-1}. \end{aligned}$$

This leads to (2.3).

Remark. In particular (2.1) and (2.2) in (2.3) gives

$$T_{n,2}=\frac{2(n-1)\cdot x\cdot(1+x)}{(n-r-2)(n-r-3)}+\frac{(r+1)(r+2)(1+2x)^2}{(n-r-2)(n-r-3)}, \tag{2.4}$$

where $n>r+3$. Also, (2.3) leads us to

$$T_{n,m}=O\left(\frac{1}{n^{[\frac{m+1}{2}]}}\right) \tag{2.5}$$

where $[\lambda]$ stands for the maximum integer less than λ.

LEMMA 2. *For* $r=0, 1, 2,...$

$$(M_n^{(r)}f)(x)=\frac{(n-r-1)!\,(n+r-1)!}{(n-1)!\,(n-2)!}\sum P_{n+r,k}(x)$$
$$\times\int_0^\infty P_{n-r,k+r}(y)\,f^{(r)}(y)\,dy. \tag{2.6}$$

Proof. By Leibnitz' theorem

$$(M_n^{(r)}f)(x)=\frac{(n+r-1)!}{(n-2)!}\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}(-1)^{r-i}P_{n+r,k-i}(x)$$
$$\times\int_0^\infty P_{n,k}(y)f(y)\,dy$$
$$=\frac{(n+r-1)!}{(n-2)!}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\sum_{k=0}^{\infty}P_{n+r,k}(x)$$
$$\times\int_0^\infty P_{n,k+i}(y)f(y)\,dy$$
$$=\frac{(n+r-1)!}{(n-2)!}\sum P_{n+r,k}(x)\int_0^\infty\sum_{i=0}^{r}(-1)^{r-i}$$
$$\times\binom{r}{i}P_{n,k+i}(y)\,f(y)\,dy.$$

Again, by Leibnitz' theorem

$$P_{n-r,k+r}^{(r)}(y)=\frac{(n-1)!}{(n-r-1!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}P_{n,k+i}(y).$$

Hence,

$$(M_n^{(r)} f)(x) = \frac{(n-r-1! \,(n+r-1)!}{(n-1)! \,(n-2)!} \sum P_{n+r,k}(x)$$

$$\times \int_0^\infty P_{n-r,k+r}^{(r)}(y) \cdot (-1)^r f(y)\, dy.$$

Further, integration by parts r times gets us to the desired result.

3. Main Results

THEOREM 1. *If f is integrable in $[0, \infty)$, admits its $(r+1)$th and $(r+2)$th derivatives, which are bounded at a point $x \in [0, \infty)$, and $f^{(r)}(x) = O(x^\alpha)$ (α is a positive integer $\geqslant 2$) as $x \to \infty$, then*

$$\lim_{n \to \infty} n((M_n^{(r)} f)(x) - f^{(r)}(x))$$

$$= (r+1)(1+2x) f^{(r+1)}(x) + x(1+x) f^{(r+2)}(x).$$

Proof. By the Taylor formula,

$$f^{(r)}(y) - f^{(r)}(x) = (y-x) f^{(r+1)}(x) + \frac{(y-x)^2}{2} f^{(r+2)}(x) + \frac{(y-x)^2}{2} \eta(y, x), \tag{3.1}$$

where

$$\eta(y, x) = \frac{f^{(r)}(y) - f^{(r)}(x) - (y-x) \cdot f^{(r+1)}(x) - (y-x)^2/2\, f^{(r+2)}(x)}{(y-x)^2/2}$$

if $x \neq y$

$= 0$

if $x = y$.

Now, for arbitrary $\varepsilon > 0$, $A > 0$ there exists a $\delta > 0$ such that

$$|\eta(y, x)| \leqslant \varepsilon \qquad \text{for } |y-x| \leqslant \delta,\ x \leqslant A. \tag{3.2}$$

Use of (2.6) in (3.1) and further use of (2.2) and (2.4) leads to

$$\frac{(n-1)! \,(n-2)!}{(n-r-2)! \,(n+r-1)!} (M_n^{(r)} f)(x) - f^{(r)}(x)$$

$$= T_{n,1} \cdot f^{(r+1)}(x) + T_{n,2} \cdot f^{(r+2)}(x) + E_{n,r}(x),$$

where

$$E_{n,r}(x)=\frac{(n-r-1)}{2}\sum P_{n+r,k}(x)\int_0^{\infty} P_{n-r,k+r}(y)(y-x)^2\cdot\eta(y,x)\,dy.$$

We shall now show that $n\cdot E_{n,r}(x)\to 0$ as $n\to\infty$. Let

$$R_{n,r,1}(x)=\frac{n\cdot(n-r-1)}{2}\sum P_{n+r,k}(x)\int_{|y-x|\leqslant\delta} P_{n-r,k+r}(y)$$
$$\cdot(y-x)^2\cdot\eta(y,x)\,dy$$

and

$$R_{n,r,2}(x)=\frac{n\cdot(n-r-1)}{2}\sum P_{n+r,k}(x)$$
$$\times\int_{|y-x|>\delta} P_{n-r,k+r}(y)\cdot(y-x)^2\cdot\eta(y,x)\,dy,$$

so that

$$n\cdot E_{n,r}(x)=R_{n,r,1}(x)+R_{n,r,2}(x).$$

It follows from (3.2) and (2.4)

$$|R_{n,r,1}(x)|\leqslant\varepsilon\cdot\frac{n\cdot(n-r-1)}{2}\sum P_{n+r,k}(x)$$
$$\times\int_{|y-x|\leqslant\delta} P_{n-r,k+r}(y)\cdot(y-x)^2\cdot dy$$
$$\leqslant\varepsilon\cdot x(1+x),\qquad \text{as } n\to\infty. \tag{3.3}$$

Further, from the assumption of the theorem,

$$R_{n,r,2}(x)=O\left(\frac{n\cdot(n-r-1)}{2}\sum P_{n+r,k}(x)\right.$$
$$\left.\times\int_{|y-x|>\delta} P_{n-r,k+r}(y)\cdot y^{\alpha}\cdot dy\right)$$
$$=O\left(\frac{n\cdot(n-r-1)}{2}\sum P_{n+r,k}(x)\int_{|y-x|>\delta} P_{n-r,k+r}(y)\right.$$
$$\left.\cdot\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(y-x)^i\cdot x^{\alpha-i}\right)dy\right)$$

$$= O\left(\frac{n\cdot(n-r-1)}{2}\sum P_{n+r,k}(x)\int_{|y-x|>\delta} P_{n-r,k+r}(y)\right.$$

$$\left.\cdot\frac{(y-x)^3}{\delta^3}\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(y-x)^i\cdot x^{\alpha-i}\right)dy\right)$$

$$= O\left(\frac{n\cdot(n-r-1)}{2\delta^3}\cdot\sum P_{n+r,k}(x)\int_0^{\infty} P_{n-r,k+r}(y)\right.$$

$$\left.\times\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(y-x)^{i+3}\cdot x^{\alpha-i}\right)dy\right)$$

$$= O\left(\frac{1}{n}\right), \qquad \text{in view of (2.5).} \tag{3.4}$$

Hence, from (3.3) and (3.4)

$$\lim_{n\to\infty} |n\, E_{n,r}(x)| \leqslant \varepsilon\cdot x(1+x).$$

However, as ε is arbitrary, $\lim_{n\to\infty}(n\cdot E_{n,r}(x))=0$.

This completes the proof.

Remark. We may note here that

$$\frac{(n-1)!\,(n-2)!}{(n+r-1)!\,(n-r-2)!}\to 1, \qquad \text{as } n\to\infty.$$

THEOREM 2. *Let $f\in C^{(r+1)}[0,a]$. and let $\omega(f^{(r+1)};\cdot)$ be the modulus of continuity of $f^{(r+1)}$. Then for $r=0, 1, 2,\ldots$*

$$\|(M_n^{(r)}f)(x)-f^{(r)}(x)\| \leqslant \frac{(r+1)(1+2a)}{(n-r-2)}\cdot\|f^{(r+1)}(x)\|$$

$$+C(n,r)\cdot\left(\sqrt{\lambda}+\frac{\lambda}{2}\right)\cdot\omega(f^{(r+1)};C(n,r)),$$

where $\lambda = 2(n-1)\cdot a(1+a) + (r+1)(r+2)(1+2a)^2$; $C(n,r) = 1/(n-r-2)\cdot(n-r-3)$.

Proof. We may write

$$f^{(r)}(y)-f^{(r)}(x)=(y-x)f^{(r+1)}(x)+\int_x^y (f^{(r+1)}(t)-f^{(r+1)}(x))\,dt.$$

Hence,

$$\frac{(n-1)!\,(n-2)!}{(n-r-2)!\,(n+r-1)!}\,(M_n^{(r)}f)(x)-f^{(r)}(x)$$

$$=(n-r-1)\sum P_{n+r,k}(x)\int_0^{\infty} P_{n-r,k+r}(y)(f^{(r)}(y)-f^{(r)}(x))\,dy$$

$$=(n-r-1)\sum P_{n+r,k}(x)\int_0^{\infty} P_{n-r,k+r}(y)\left((y-x)f^{(r+1)}(x)\right.$$

$$\left.+\int_x^y (f^{(r+1)}(t)-f^{(r+1)}(x))\,dt\right)dy.$$

Also,

$$|f^{(r+1)}(t)-f^{(r+1)}(x)| \leqslant \left(1+\frac{|t-x|}{\delta}\right)\cdot \omega(f^{(r+1)};\delta).$$

Therefore,

$$\left|\frac{(n-1)!\,(n-2)!}{(n-r-2)!\,(n+r-1)!}\,(M_n^{(r)}f)(x)-f^{(r)}(x)\right|$$

$$\leqslant |T_{n,1}|\cdot|f^{(r+1)}(x)|+\left(|\sqrt{T_{n,2}}|+\frac{|T_{n,2}|}{2\delta}\right)\omega(f^{r+1},\delta),$$

in view of Schwarz's inequality. Further, choosing $\delta=C(n,r)$ and using (2.2) and (2.4) we get the required result.

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References

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