A differentiable manifold with noncoinciding dimensions

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Abstract
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A differentiable $n$-manifold $M^n_m$, $4 \leq n < m$, with dimensions $n = \text{ind } M^n_m < m = \dim M^n_m < \text{Ind } M^n_m = m + n - 2$ is constructed under Jensen's principle $\Diamond$. The space $M^n_m$ is perfectly normal, countably compact and hereditarily separable.

Keywords: Differentiable manifold; Dimension; Large inductive dimension; Jensen’s principle.

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Introduction

The main result of this paper is

**Theorem 0.1** ($\Diamond$). For any integers $m$ and $n$ such that $4 \leq n < m$, there exists a differentiable, countably compact, perfectly normal, hereditarily separable $n$-manifold $M^n_m$ with dimensions

\[ n = \text{ind } M^n_m < m = \dim M^n_m < m + n - 2 = \text{Ind } M^n_m. \]

Recall that $\Diamond$ is a set-theoretical principle introduced by Jensen [11]. We use the following variant of $\Diamond$:

There is a sequence $\{J_\alpha : \alpha < \omega_1\}$, where $J_\alpha \subset \alpha$, such that $\{\alpha : J_\alpha = K \cap \alpha\}$ is stationary for any $K \subset \omega_1$. A set $A \subset \omega_1$ is said to be **stationary** if $A \cap B \neq \emptyset$ for each closed unbounded $B \subset \omega_1$.

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Let us remind that $\Diamond$ follows from the constructibility axiom ($V = L$), and the continuum hypothesis follows from $\Diamond$.

It should be mentioned that recently Filippov and the author [10], using the continuum hypothesis, for any $n \geq 3$ constructed an example of a normal, countably compact topological $n$-manifold $M^n$ with:

$$\dim M^n = n < 2n - 2 = \text{Ind} M^n.$$ 

And before that the author constructed [8] a "real" example of a normal, countably compact topological 2-manifold $M^2$, which has nowhere dense closed subset $N$ with $\text{Ind} N = \text{Ind} M^2 = 2$.

We use the technique of continuous spectra. Recall that an inverse spectrum $S = \{X_\alpha, \pi^\alpha_\beta: \alpha < \tau\}$, where $\tau$ is an ordinal, is said to be continuous if for any limit ordinal $\alpha < \tau$ the space $X_\alpha$ is naturally homeomorphic to $\lim(S \upharpoonright \alpha)$.

We will need also a notion of fully closed map which was introduced in [5] (see also [6]).

**Definition.** A continuous map $f : X \to Y$ is said to be fully closed if for each point $y \in Y$ the set $\{y\} \cup f^* U_1 \cup f^* U_2$ is open in $Y$ for any two open subsets $U_1, U_2 \subset X$ such that $f^{-1}(y) \subset U_1 \cup U_2$.

Here for an arbitrary map $f : X \to Y$ and for an arbitrary set $A \subset X$ by $f^* A$ we denote the small image of $A$, i.e.,

$$f^* A = \{y \in Y: f^{-1} y \subset A\} = Y \setminus f(X \setminus A).$$

For a map $f : X \to Y$ we set $\text{supp} f = \{y \in Y: |f^{-1} y| > 2\}$, where by $|A|$ we denote the cardinality of $A$. This set $\text{supp} f$ we will call a supporter of $f$. We will say that a family of maps $f_\alpha : X_\alpha \to Y$. $\alpha \in A$, is independent if the family $\{\text{supp} f_\alpha: \alpha \in A\}$ is disjoint. A proper map $f : X \to Y$ is said to be simple if its supporter is empty or consists of one point. For the definition of the fibre product of maps $f_\alpha : X_\alpha \to Y$, see [9].

**Theorem A** [7]. A proper map $f : X \to Y$ between Tychonoff spaces is fully closed iff it is a fibre product of some independent family of simple maps.

**Theorem B** [5]. If $f : X \to Y$ is a fully closed proper map between paracompact spaces, then

$$\dim Y \leq \max\{\dim X, \dim f\}.$$ 

**Theorem C.** Let $S = \{X_\alpha, \pi^\alpha_\beta: \alpha < \tau\}$ be a continuous spectrum, consisting of bicompress, and let all neighbouring projections $\pi^\alpha_{\alpha + 1}$ be fully closed with $\dim \pi^\alpha_{\alpha + 1} \leq n$. Then for $X = \lim S$ we have

$$\dim X \leq \max\{\dim X_0, n\}.$$
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One can prove Theorem C by a transfinite induction on the length $\tau$ of the spectrum $S$ using Theorem B.

1. Main lemma

Recall that a continuous map $f : X \to Y$ is said to be a near homeomorphism if for any open covering $\mathcal{U}$ of $Y$ there is a homeomorphism $g : X \to Y$, which is $\mathcal{U}$-close to $f$, i.e., for any $x \in X$ there is $U \in \mathcal{U}$ such that $f(x), g(x) \in U$.

**Lemma 1.1.** Let $X$ be a bicompactum, and let $K$ be a closed subset of $X$, and $x_0 \in K$. Let for any neighbourhoods $Ox_0$ and $OK$ there is a homeomorphism $h : X \to X$ such that

1. $h(K) \subset Ox_0$;
2. $h|X\setminus OK = \text{id}$.

Then every map $f : X \to X$, having the only one nontrivial inverse image $f^{-1}(x_0) = K$, is a near homeomorphism.

**Proof.** By the Bing's shrinking criterium [3] it is sufficient for any open coverings $\mathcal{U}$ and $\mathcal{V}$ of $X$ to find a homeomorphism $h : X \to X$ such that $h(f^{-1}x_0)$ is contained in some element of $\mathcal{U}$, and $fh$ is $\mathcal{V}$-close to $f$. We denote by $Ox_0$ some element of $\mathcal{U}$, containing $x_0$, and let $Of^{-1}x_0 = f^{-1}V$, where $x_0 \in V \in \mathcal{V}$. Let us take a homeomorphism $h$, satisfying conditions (1) and (2) of the lemma. Then it is easy to check that $h$ satisfies the Bing's shrinking criterion and the lemma is proved.

**Lemma 1.2.** Let $B^n$ be a closed $n$-ball with the boundary $S^{n-1}$. Let $L \subset S^{n-1}$ be some nonempty compactum, and let $g : L \times [0, 1] \to B^n$ be an imbedding such that:

1. $g(l,0) = l$ for any $l \in L$;
2. $g(L \times (0, 1]) \subset B^n \setminus S^{n-1}$.

Let $X$ be a quotient space of $B^n$ with respect to the decomposition, the only nontrivial member of which is $L$, and let $q : B^n \to X$ be the corresponding quotient map, and $x_0 = q(L)$.

Then an arbitrary map $f : X \to X$ onto $X$, which is identical onto $q(S^{n-1})$ and has the only nontrivial inverse image $f^{-1}x_0 = qg(L \times [0, 1])$, is a near homeomorphism. Moreover, $f$ is approximated by homeomorphisms, which are identical onto $q(S^{n-1})$.

**Proof.** We use the following trivial topological

**Fact.** For arbitrary neighbourhoods $OL$ and $Og(L \times [0, 1])$ there is a homeomorphism $\overline{h} : B^n \to B^n$ such that:

1. $h(g(L \times [0, 1])) \subset OL$;
2. $\overline{h} | B^n \setminus Og(L \times [0, 1]) = \text{id}$;
3. $\overline{h} | S^{n-1} = \text{id}$. 
This homeomorphism generates a homeomorphism $h : X \to X$, which satisfies the conditions of Lemma 1.1 for $K = qg(L \times [0, 1])$. To get the last property of $f$, we take an approximating homeomorphism from the proof of Lemma 1.1. The lemma is proved. □

Let $A$ be a closed subset of a topological space $X$, and let $\varphi : A \to B$ be a quotient map onto some space $B$. We denote by $X_{\varphi}$ the quotient space of $X$ with respect to the decomposition, the elements of which are fibres $\varphi^{-1} b$ of the map $\varphi$ and singletons from $X \setminus A$. The quotient map $X \to X_{\varphi}$ we denote by $p_{\varphi}$.

Let $f : X \to Y$ be a continuous map such that $f \mid A$ is a homeomorphism. The set $f(A)$ is mapped onto $B$ by the map $\varphi f^{-1}$. There is a unique map $f_{\varphi} : X_{\varphi} \to Y_{\varphi f^{-1}}$ such that $f_{\varphi} p_{\varphi} = p_{\varphi f} \circ f$. Indeed, for every point $x \in X_{\varphi}$, the set $p_{\varphi f^{-1}} f_{\varphi}^{-1} x$ consists of one point, which is the image $f_{\varphi}(x)$ of $x$.

**Definition.** A continuous map $f : X \to Y$ is called a near $A$-homeomorphism, where $A$ is closed in $X$, if for any open covering $\mathcal{U}$ of $Y$ there is a homeomorphism $h : X \to Y$ such that $h$ is $\mathcal{U}$-close to $f$ and coincides with $f$ onto $A$.

**Lemma 1.3.** If $f : X \to Y$ is a near $A$-homeomorphism between bicompacta, then for any quotient map $\varphi : A \to B$ the map $f_{\varphi} : X_{\varphi} \to Y_{\varphi f^{-1}}$ is a near $B$-homeomorphism.

**Proof.** Let $\mathcal{U}$ be an open covering of $Y_{\varphi f^{-1}}$. For the covering $p_{\varphi f^{-1}}^{-1}(\mathcal{U})$ of $Y$ we take a homeomorphism $h : X \to Y$ from the definition of a near $A$-homeomorphism. Then the map $g = p_{\varphi f^{-1}} h p_{\varphi}^{-1}$ is the desired homeomorphism between $X_{\varphi}$ and $Y_{\varphi f^{-1}}$. □

**Definition.** We will say that a countable subset $A$ of a topological space $X$ converges to a set $F \subset X$ if for any neighbourhood $OF$ the set $A \setminus OF$ is finite. We will say that the set $F$ is enveloped by the countable set $A$ if $A$ converges to $F$ and $F \subset \overline{A}$.

**Main lemma.** Let $B^n$ be a closed $n$-ball, $n \geq 2$, and $S^{n-1}$ be its boundary. Let $A$ be a closed subset of $S^{n-1}$, and let $\varphi : A \to B$ be a continuous map onto some compactum $B$. Suppose that for some point $b_0 \in B$ the set $L = \varphi^{-1}(b_0)$ is connected and nowhere dense in $S^{n-1}$. Suppose further that $h : L \times [0, 1] \subset B^n$ is an imbedding, satisfying the following conditions:

1. $h(l, 0) = l$ for any $l \in L$;
2. $h(L \times (0, 1]) \subset B^n \setminus S^{n-1}$.

Then for an arbitrary countable family $\mathcal{C} = \{C_i : i \in \omega\}$ of countable sets $C_i \subset O^n = B^n \setminus S^{n-1}$ such that $y = p_{\varphi}^{-1} \varphi^{-1}(b_0)$ is a limit point of $C_i$ for any $i$ there exists a continuous map $g = g(y, \varphi) : B^n \to B^n$ with the following properties:

1. $g$ is a near $S^{n-1}_{\varphi}$-homeomorphism;
2. $g$ has the only nontrivial fibre $g^{-1}y = p_{\varphi} h(L \times [0, 1]) \equiv K$;
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(5) \( g | S^n = \text{id} \);
(6) \( g : O^n \setminus K \to O^n \) is a diffeomorphism;
(7) \( K \subset [g^{-1}C_i] \) for any \( i \).

**Proof.** Since (3) and (5), it suffices to construct \( g \) for the case \( B = \{b_0\} \). Then applying Lemma 1.3 we get a general case from this particular one. We will construct the map \( g \) so that it will be identical outside some neighbourhood of \( K \). So, we assume that the halfspace \( \mathbb{R}^n_+ = \mathbb{R}^n \setminus [0, \infty) \) is imbedded into \( B^n \) with \( L \times \{0\} \subset \mathbb{R}^n \setminus \{0\} \subset S^{n-1} \). After this it is enough to construct a diffeomorphism

\[ f : \mathbb{R}^n_+ \setminus L \times [0, 1] \to \mathbb{R}^n_+ \setminus L \times \{0\}, \]

which is identical onto \( (\mathbb{R}^n_+ \setminus L) \times [0] \) and outside some neighbourhood \( O(L \times \{0\}) \), and satisfies the condition

(7') \( L \times [0, 1] \subset f^{-1}(C_i) \) for any \( i \in \omega \) (here \( C_i \subset \mathbb{R}^n \setminus \{0\} \) is a countable set such that \( \bigcap_i (L \times \{0\}) = \emptyset \)).

We will get a map \( g \) as an extension of the map \( p \circ f^{-1} \) such that \( g \) is identical onto \( B^n_+ \setminus (\mathbb{R}^n_+ \setminus L) \) and transfers the set \( K \) into the point \( y \). This map will satisfy the condition (3) of the Main lemma according to Lemma 1.2.

**Construction of the map \( f \).** Without lost of generality we can assume that the set \( C = \bigcup \{C_i : i \in \omega\} \) is a sequence, converging to the compactum \( L \times \{0\} \), and \( C_i \cap C_{i'} = \emptyset \) for \( i \neq i' \). There exists a fundamental sequence of neighbourhoods \( U_j \) of \( L \) in \( \mathbb{R}^n_+ \) satisfying for any \( j \in \omega \) the following conditions:

(a) \( U_{j+1} \subset U_j \);
(b) \( U_j \) is a connected \((n-1)\)-manifold with boundary;
(c) \((\text{Bd } U_j) \times \mathbb{R}^+ \cap C = \emptyset \).

Let \( V_j = U_j \times [0, 1 + 1/(j + 1)] \) and \( W_j = U_j \times [0, \epsilon_j] \), where positive numbers \( \epsilon_j \) will be defined later. The sequence \( \{\epsilon_j : j \in \omega\} \) will be strictly decreasing and converging to 0. So, \( \{V_j : j \in \omega\} \) and \( \{W_j : j \in \omega\} \) will be fundamental sequences of neighbourhoods of \( L \times [0, 1] \) and \( L \times \{0\} \) in \( \mathbb{R}^n_+ \). These neighbourhoods evidently satisfy the condition

(d) \( \bar{V}_{j+1} \subset V_j, \bar{W}_{j+1} \subset W_j \).

The following fact is trivial.

**Statement.** Let \( X \) and \( Y \) be connected spaces, and let \( X_0 \) and \( Y_0 \) be their proper subspaces. Then \( X \times Y \setminus X_0 \times Y_0 \) is connected.

This statement and condition (b) imply the condition

(e) the sets \( \bar{V}_j \setminus V_{j+1} \) and \( \bar{W}_j \setminus W_{j+1} \) are connected \( n \)-manifolds with boundary.

Since the set \( L \) is nowhere dense in \( \mathbb{R}^n_+ \), for each \( j \in \omega \) we fix a sequence

\[ D_j \subset V_j \setminus L \times [0, 1] \cup \mathbb{R}^n \setminus \{0\}, \]

which envelopes \( L \times [0, 1] \) and meets no boundary \( \text{Bd } V_j, \ i \geq j \). Moreover, we suppose that \( D_j \cap D_{j'} = \emptyset \) for \( j \neq j' \).
We will construct the diffeomorphism \( f \) so that \( f(D_i) \subset C_j \). This will imply condition (7'). We begin with a definition of \( \varepsilon_j \). For a pair \((i, j)\) of integers, satisfying the condition \( 0 \leq i \leq j \), we set

\[
N(i, j) = \left| D_i \cap \left( V_j \setminus \overline{V}_{j+1} \right) \right|.
\]

We will inductively define numbers \( \varepsilon_j \) such that for \( 0 \leq i \leq j \):

\[
N(i, j) \leq \left| C_i \cap \left( W_j \setminus \overline{W}_{j+1} \right) \right|,
\]

\[
(i, j + 1)
\]

\[
C_i \cap \text{Bd} \ W_j = \emptyset.
\]

We can begin with \( \varepsilon_0 \). Since \( C_0 \) converges to \( L \times \{0\} \), we can assume, reducing \( C_0 \) if needed, that \( C_0 \cap (\mathbb{R}_+^{n-1} \times \{\varepsilon_0\}) = \emptyset \). Then \( (c_0) \) implies \((0, 0)\).

Suppose we have \( \varepsilon_k \), \( k \leq j \), such that the conditions \((i, k)\) and \((i, k)\) are fulfilled for \( 0 \leq i \leq k \leq j \). For positive \( \varepsilon < \varepsilon_j \) we set

\[
W_{j+1}^\varepsilon = U_{j+1} \times \{0, \varepsilon\},
\]

\[
L_{\varepsilon}(i, j) = \left| C_i \cap \left( W_j \setminus \overline{W}_{j+1}^\varepsilon \right) \right|.
\]

If \( \varepsilon \) converges to 0, then \( L_{\varepsilon}(i, j) \) converges to \( \infty \), since

\[
\bigcap \left\{ W_{j+1}^\varepsilon : 0 < \varepsilon < \varepsilon_j \right\} = U_{j+1} \times \{0\} \subset \mathbb{R}_+^{n-1} \setminus C_i.
\]

Hence, there is a positive number \( \varepsilon_{j+1} < \varepsilon_j \) such that the conditions \((i, j + 1), i \leq j\), are fulfilled. But then these conditions are fulfilled for all \( \varepsilon_{j+1} \) from some neighbourhood of \( \varepsilon_{j+1} \). So, we can get \((i, j + 1)\) by a small shift of \( \varepsilon_{j+1} \) and using \( (c_{j+1}) \).

Now we are going to construct a diffeomorphism

\[
f_1 : \mathbb{R}_+^{n-1} \setminus L \times \{0, 1\} \rightarrow \mathbb{R}_+^{n-1} \setminus L \times \{0\},
\]

identical outside \( V_0 = W_0 \) and transforming the set \( \overline{V}_j \setminus V_{j+1} \) into the set \( \overline{W}_j \setminus W_{j+1}, j \in \omega \). For this we need some notations. Let \( U_{j-} \) and \( U_{j+1}^- \) be open subsets of \( \mathbb{R}_+^{n-1} \) such that

\[
U_{j+1}^- \subset U_{j+1}^+ \subset U_{j+1}^- \subset U_{j}^- \subset U_{j} \subset U_j
\]

for all \( j \in \omega \). Let

\[
\alpha_j : \left[ 1 + \frac{1}{j + 2}, 1 + \frac{1}{j + 1} \right] \rightarrow [\varepsilon_{j+1}, \varepsilon_j]
\]

be some diffeomorphism, which is linear on small neighbourhoods of the points \( 1 + 1/(j + 2) \) and \( 1 + 1/(j + 1) \) with the coefficients \( \varepsilon_{j+1}/(1 + 1/(j + 2)) \) and \( \varepsilon_{j}/(1 + 1/(j + 1)) \) accordingly. This diffeomorphism is extended to the diffeomorphism

\[
\tilde{\alpha}_j : \left[ 0, 1 + \frac{1}{j + 1} \right] \rightarrow [0, \varepsilon_j],
\]
which is linear on the interval \([0, 1 + 1/(j + 2)]\). Let

\[ \beta_j : \left[0, 1 + \frac{1}{j+1} \right] \rightarrow [0, \epsilon_j] \]

be a linear map, and let \(\psi_j : \mathbb{R}^{n-1} \rightarrow [0, 1]\) be some smooth function such that 
\(\psi_j(U^+_{j+1}) = 1\) and \(\psi_j(\mathbb{R}^{n-1} \setminus U^-_j) = 0\). Now for a point 
\((r, t) \in \overline{V}_j \setminus V_{j+1} \subset \mathbb{R}^{n-1} \times [0, \infty)\)
we put

\[ f_j(r, t) = \left(r, \psi_j(r) \alpha_j(t) + (1 - \psi_j(r)) \beta_j(t) \right). \]

Setting \(f_1|\mathbb{R}^n \setminus V_0 = \text{id}\), we obtain, easy to verify, the homeomorphism \(f_1\), with the 
described above properties.

For \(0 \leq i \leq j\) we set

\[ D_i = D_i \cap \left(V_j \setminus V_{j+1}\right). \]

Denote by \(C_i\) some subset of \(C_i \cap (W_j \setminus \overline{W}_{j+1})\), which consists of 
\(N(i, j) = |D_i|\) points. Such a subset exists in view of \((i, j + 1)\).

Now we will construct a diffeomorphism

\[ f_2 : \mathbb{R}^n_+ \times L \times \{0\} \rightarrow \mathbb{R}^n_+ \times L \times \{0\} \]

with the following properties:

1. \(f_2(W_j \setminus \overline{W}_{j+1}) = W_j \setminus \overline{W}_{j+1}\);
2. \(f_2(f_1(D_i)) = C_i\);
3. \(f_2(\mathbb{R}^{n-1}_+ \setminus L) \times \{0\} \cup (\mathbb{R}^n_+ \setminus W_0) = \text{id}\).

This is possible, because of the following evident property of the Euclidian space:

*Let \(G\) be a domain of the Euclidean space \(\mathbb{R}^n\) with a boundary \(F\). Let \(A\) and \(B\) be some finite subsets of \(G\) such that \(|A| = |B|\). Then any bijection \(h_0 : A \rightarrow B\) can be extended to a diffeomorphism \(h : G \rightarrow G\), which is identical in a neighbourhood of \(F\).*

During the construction of \(f_2\) the sets \(\text{Int}_{\mathbb{R}^n}(W_j \setminus \overline{W}_{j+1})\) will successively play the role of the domain \(G\), and the sets \(A_j = \bigcup\{f_i(D_i) : 0 \leq i \leq j\}\) and \(B_j = \bigcup\{C_i : 0 \leq i \leq j\}\) will play roles of the sets \(A\) and \(B\). The sets \(A_j\) and \(B_j\) have equal cardinalities, since the definition of \(C_i\) and the conditions \(C_i \cap C_{i'} = \emptyset\), \(D_i \cap D_{i'} = \emptyset\) for \(i \neq i'\).

Now we set \(f = f_2f_1\). Condition (9) will guarantee that \(f(D_i) \subset C_j\), and this 
inclusion will imply the main property (7') of the diffeomorphism \(f\). The lemma is 
proved. \(\Box\)
2. Manifold $M^n_m$

2.1. The space $Y^n_m, m > n \geq 4$

Let $M_1$ be the Menger curve. We imbed the product $M_1 \times I^{n-4}$ into the sphere $S^{n-1}$, which is the boundary of the closed ball $B^n$. Denote by $f_1 : M_1 \rightarrow Q$ an open map onto the Hilbert cube $Q$ such that all fibres $f_1^{-1}(q)$ are homeomorphic to the Menger curve. Such map was constructed by Anderson [2]. For us only one thing is essential: $f_1$ is a monotone map with nontrivial fibres. By $f : M_1 \times I^{n-4} \rightarrow Q$ we denote the composition $f_1 \text{pr}$, where $\text{pr} : M_1 \times I^{n-4} \rightarrow M_1$ is the projection. For each $m = n + 1, n + 2, \ldots, \infty$ we fix some imbedding $I^m \subset Q$ (we can assume that $I^m \subset Q$). By $\mathcal{D}_m^n$ we denote the following decomposition of $B^n$:

$$\mathcal{D}_m^n = \{f^{-1}_q : q \in I^m\} \cup \{\{b\} : b \in B^n \setminus f^{-1}_q I^m\}.$$ 

The quotient space $B^n \setminus \mathcal{D}_m^n$ we denote by $Y_m^n$, the quotient map $B^n \rightarrow Y_m^n$ we denote by $\varphi$. Let $H^n_m = \varphi(S^{n-1})$. A point $y \in H^n_m$ is called a point of the first genus if $\varphi^{-1}y$ consists of one point, and $y$ is called a point of the second genus if $\varphi^{-1}y$ is homeomorphic to $M_1 \times I^{n-4}$.

**Proposition 2.1.** For an arbitrary homeomorphism $f : Y^n_m \rightarrow Y^n_m$ we have $f(O^n) = O^n$, $f(H^n_m) = H^n_m$.

**Proof.** If $y \in H^n_m$ is a point of the first genus, then near this point the space $Y^n_m$ looks like the halfspace $\mathbb{R}^+_n$. If $y$ is a point of the second genus, then $\dim Oy = m$ for an arbitrary neighbourhood $Oy$ in $Y^n_m$. \(\square\)

2.2. Spectrum $S$

Let $H^n_m$, $Z_0$, $Z_{(\alpha, \beta)}$, where $\alpha \leq \beta < \omega_1$, $\beta$ is an isolated ordinal, be disjoint sets of the cardinality of the continuum. For a countable ordinal $\gamma \geq 1$ we set

$$Z_\gamma = Z_0 \cup \left( \bigcup \{Z_{(\alpha, \beta)} : \alpha \leq \beta \leq \gamma \} \right).$$

Since $\beta$ is isolated, for each limit ordinal $\gamma$ we have $Z_\gamma = \cup \{Z_\delta : \delta < \gamma \}$. Now we set

$$X_\gamma = Z_\gamma \cup H^n_m$$

and define maps $\pi_\delta : X_\gamma \rightarrow X_\delta$ so that the system

$$S = \{X_\gamma, \pi_\delta : \delta \leq \gamma < \omega_1\}$$

is a continuous spectrum of sets. For that it suffices to define neighbouring projections $\pi_\gamma^{\gamma+1}$. Assuming the continuum hypothesis, fix some bijection $\xi : H^n_m \rightarrow \omega_1$. By a definition the map $\pi_\gamma^{\gamma+1}$ is identical onto $X_\gamma$ and takes $Z_{(\alpha, \gamma+1)}$ into the point $\xi^{-1}\alpha$ for all $\alpha \leq \gamma + 1$. 

Now we topologize the spectrum $S$. We set

$$Z_{\omega_1} = \cup \{ Z_\gamma: \gamma < \omega_1 \}$$

and fix some bijection $\eta: Z_{\omega_1} \to \omega_1$. Assuming $\triangle$, we take a sequence $\{ J_\alpha: \alpha < \omega_1 \}$ such that the set $\{ \alpha: J_\alpha = K \cap \alpha \}$ is stationary for any set $K \subseteq \omega_1$. A topology on $X_\gamma$ we define inductively such that for all $\gamma < \omega_1$:

1. $X_\gamma$ is homeomorphic to $Y^n_m$;
2. $\pi_{\gamma}^{\delta}$ is a near homeomorphism for $\gamma' < \gamma$.

We begin with $X_0$. Take some bijection $Z_0 \to O^n$. After this

$$X_0 = Z_0 \cup H^n_m = O^n \cup H^n_m = Y^n_m.$$ 

Suppose that we have $X_\delta$ and $\pi_{\delta}^{\delta}$, satisfying (1) and (2) for all $\delta' < \delta < \gamma < \omega_1$.

**Case 1:** $\gamma$ is a limit ordinal. Let $X_\gamma$ be the limit of the inverse spectrum $S_\gamma = (X_\delta, \pi_{\delta}^{\delta}: \delta < \gamma)$, and let $\pi_{\gamma}^{\delta}: X_\gamma \to X_\delta$ be the composite projection of the spectrum $S_\gamma$. Then $\pi_{\gamma}^{\delta}$ is a near homeomorphism by the approximation lemma of Brown [4]. Hence, $X_\gamma$ is homeomorphic to $Y^n_m$.

**Case 2:** $\gamma = (\gamma - 1) + 1$. Our task is to topologize the map $\pi_{\gamma}^{\gamma - 1}$, which was defined above, as a map between the sets $X_\gamma$ and $X_{\gamma - 1}$. Let $\pi_{\gamma}^{\gamma - 1}$ be a fibre product of maps $\rho_{\delta}^{\delta}$, $\alpha < \gamma$. Here $\rho_{\delta}^{\delta}$ is a map $g - \xi_{(\delta, \gamma)}$ from Main lemma where $B_\delta^n = Y^n_m$, $S_\delta^n = H^n_\delta$, $y = \xi^{-1}_\delta$ and the family $\mathcal{E}$ consists of all sets $\eta^{-1}J_\beta \cap Z_{\gamma - 1}$, $\beta < \gamma - 1$, and such that $\gamma \in \eta^{-1}J_\beta \cap Z_{\gamma - 1}$. As for the inverse image $(\rho_{\delta}^{\delta})^{-1}y = K$, it is homeomorphic to $I$ or to the cone over $M_1 \times I^{n-4}$, depending on a genus of the point $y$.

Now we give another description of $\pi_{\gamma}^{\gamma - 1}$. By [7, Theorem 2] $\pi_{\gamma}^{\gamma - 1}$ can be represented as a limit of an inverse sequence of maps, which are homeomorphic to $\rho_{\delta}^{\delta}$, $\alpha < \gamma$. We describe this representation in detail. By a definition we have

$$Z_\gamma = Z_{\gamma - 1} \cup (\bigcup \{ Z_{(\alpha, \gamma)}: \alpha < \gamma \}).$$

Let us number the countable family $\{ Z_{(\alpha, \gamma)}: \alpha < \gamma \}$ by integers $i \in \omega$:

$$\{ Z_{(\alpha, \gamma)}: \alpha < \gamma \} = \{ Z^{(i, \gamma)}: i \in \omega \}.$$

We set

$$Z_i^\gamma = Z_{\gamma - 1} \cup \left( \bigcup \{ Z^{(k, \gamma)}: k < i - 1 \} \right), \quad X_i^\gamma = Z_i^\gamma \cup H^n_m.$$ 

Now for $i \geq 1$ we define a map $\pi_i^\gamma: X_i^\gamma \to X_{i - 1}^\gamma$ setting $\pi_i^\gamma(Z^{(i - 1, \gamma)}) = \xi^{-1}_\gamma(\alpha)$, where $Z^{(i - 1, \gamma)} = Z_{(\alpha, \gamma)}$, and $\pi_i^\gamma$ is identical onto $X_{i - 1}^\gamma$. Applying Main lemma, we inductively define a topology on $X_i^\gamma$ such that:

3. $X_i^\gamma$ is homeomorphic to $Y^n_m$;
4. $\pi_i^\gamma$ coincides with the map $g = g_{(\delta, \gamma)}$ from Main lemma, where $y = \xi^{-1}_\delta$ and $\mathcal{E}$ consists of all sets $\eta^{-1}J_\beta$, $\beta < \gamma - 1$, lying in $Z_{\gamma - 1}$ and such that $y$ is a limit point of $\eta^{-1}J_\beta$ in $X_{i - 1}^\gamma$.

Let us note that the limit point of $\eta^{-1}J_\beta$ in $X_{i - 1}^\gamma$ iff $y$ is a limit point of $\eta^{-1}J_\beta$ in $X_{\gamma - 1}$, since we get $X_{i - 1}^\gamma$ from $X_{\gamma - 1}$ by a local changing of the space $X_{\gamma - 1}$ in finitely many points $y' \neq y$. The mentioned above [7, Theorem 2] implies
that $X_\gamma = \lim \{X^i_\gamma, \pi^i_\gamma; i \in \omega\}$ ($\pi^0_\gamma$ is the identity map of $X^0_\gamma = X_{\gamma-1}$), and $\pi^{\gamma-1}_\gamma$ is the composite projection of the inverse sequence $\{X^i_\gamma, \pi^i_\gamma; i \in \omega\}$. Since all maps $\pi^i_\gamma$ are near homeomorphisms, the map $\pi^{\gamma-1}_\gamma$ is a near homeomorphism by the approximation lemma of Brown.

**Remark.** Actually we need more than an existence of a homeomorphism between $X_\gamma$ and $X_{\gamma-1}$. To provide an equality $\pi^{\gamma-1}_\gamma | H^n_m = \text{id}$ we need the unique representation $X_\gamma = O^n \cup \cap H^n_m$. But Proposition 2.1 gives us this uniqueness.

So, the map $\pi^{\gamma-1}_\gamma$ is constructed, the properties (1) and (2) are fulfilled, and the spectrum $S$ is topologized.

Because all maps $\pi^{\gamma-1}_\gamma | H^n_m$ are identical, the bicomplex $X_\omega = \lim S$ is naturally represented as a disjoint union of $Z_\omega_i$ and $H^n_m$. Each $Z_\gamma = X_\gamma \setminus H^n_m$, $\gamma < \omega_1$, is homeomorphic to the open ball $O^n$ by Proposition 2.1. So, each $Z_\gamma$ is open in $Z_\delta$ for $\gamma < \delta$ by the Brouwer theorem on open subsets of the Euclidean space $\mathbb{R}^n$. Hence, $Z_\omega$ is a topological $n$-manifold, which we denote by $M^n_m$.

**Proposition 2.2.** The space $M^n_m$ is a differentiable $n$-manifold.

In the proof we will use

**Theorem of Kozlowski and Zenor** [12]. *If a differentiable manifold $M$ has an atlas $\{(U_i, \varphi_i): i \in \omega\}$ such that $U_i \subset U_{i+1}$ and $\varphi_i(U_i) = \mathbb{R}^n$ for all $i \in \omega$, then $M$ is diffeomorphic to $\mathbb{R}^n$.*

As in the paper of Kozlowski and Zenor [12], we will inductively construct a differentiable structure $\mathcal{D}_\gamma$ on $Z_\gamma$ such that:

1. $\{Z_\gamma, \mathcal{D}_\gamma\}$ is diffeomorphic to $\mathbb{R}^n$: i.e., the atlas $\mathcal{D}_\gamma$ contains a chart $(Z_\gamma, \varphi_\gamma)$ with $\varphi_\gamma(Z_\gamma) = \mathbb{R}^n$;
2. if $\beta < \gamma$, then $(Z_\beta, \varphi_\beta) \in \mathcal{D}_\gamma$.

Let $\mathcal{D}_0$ be the usual differentiable structure on $Z_0 = O^n$ generated by the atlas consisting of the single chart $(Z_0, \varphi_0)$, where $\varphi_0: O^n \to \mathbb{R}^n$ is some diffeomorphism.

Suppose the $\mathcal{D}_\gamma$ satisfying (1) and (2) are constructed for all $\gamma < \delta < \omega_1$. If $\delta$ is a limit ordinal, then let $\mathcal{D}_\delta$ be the differentiable structure generated by $\{(Z_\gamma, \varphi_\gamma): \gamma < \delta\}$. The manifold $(Z_\delta, \mathcal{D}_\delta)$ is diffeomorphic to $\mathbb{R}^n$ by the theorem of Kozlowski and Zenor.

Now let $\delta = \gamma + 1$. We know that $Z^i_{\gamma+1} = \bigcup \{Z^i_{\gamma+1}; i \in \omega\}$, where $Z^0_{\gamma+1} = Z_\gamma$ and each $Z^i_{\gamma+1}$ is homeomorphic to $\mathbb{R}^n$. So, our task is to construct inductively a differentiable structure $\mathcal{D}_i$ on $Z^i_{\gamma+1}$ such that $(Z^i_{\gamma+1}, \mathcal{D}_i)$ is diffeomorphic to $\mathbb{R}^n$ and $\mathcal{D}_i \subset \mathcal{D}_i^{i+1}$. Then $\{Z^i_{\gamma+1}, \bigcup \{\mathcal{D}_i; i \in \omega\}\}$ will be diffeomorphic to $\mathbb{R}^n$ by the theorem of Kozlowski and Zenor. If $\mathcal{D}_i$ is already constructed and contains a chart $(Z^i_{\gamma+1}, \varphi^i_{\gamma+1})$ with $\varphi^i_{\gamma+1}(Z^i_{\gamma+1}) = \mathbb{R}^n$, then $\mathcal{D}_i^{i+1}$ is generated by $\mathcal{D}_i$ and the
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single chart \((Z_{y+1}^i, \varphi_{y+1}^i)\), where \(\varphi_{y+1}^i = \varphi_{i+1}^j g_{i+1}^{-1}\) and \(g_{i+1} = \pi_{y+1}^i Z_{y+1}^i \setminus Z_{y+1}^i\). The family \(\mathcal{D}^i \cup \{(Z_{y+1}^i, \varphi_{y+1}^i)\}\) is an atlas, since the map \(g_{i+1} : Z_{y+1}^i \setminus Z_{y+1}^i \to Z_{y+1}^{i+1}\) is a diffeomorphism between \(Z_{y+1}^i\) and \(Z_{y+1}^{i+1}\).

So the atlases \(\mathcal{D}_y\) satisfying (1) and (2) are constructed for all \(y < \omega_1\). Hence, the differentiable structure \(\mathcal{D} = \cup \{\mathcal{D}_y : y < \omega_1\}\) onto \(Z_m = M_m^\alpha\) is defined. The proposition is proved. \(\square\)

In what follows we need

Main property of sets \(\eta^{-1} J\). If \(\beta < \alpha\), and \(x \in X_\alpha\) is a limit point of \(\eta^{-1} J \cap X_\alpha = H\), then for any \(\alpha' \geq \alpha\) every point \(y \in (\pi_\alpha)\eta^{-1} x\) is a limit point of \(H\) in \(X_\alpha\).

**Proof.** By transfinite induction: Assume that we proved our assertion for all \(\alpha' < \delta\). If \(\delta\) is a limit ordinal, then it follows from the continuity of \(S\) that for any neighbourhood \(O_y\) there is a neighbourhood \(U\) of \(\pi_\delta^\alpha(y)\) for some \(\alpha', \alpha < \alpha' < \delta\), such that \((\pi_\delta^\alpha)^{-1} U \subseteq O_y\). By the assumption we have \(H \cap U \neq \emptyset\). But \(H = (\pi_\delta^\alpha)^{-1} H\), since \(H \subseteq X_\alpha\) and \(\alpha < \alpha'\). Hence \(H \cap O_y \neq \emptyset\).

Now let \(\delta = (\delta - 1) + 1\). Then \(\beta < \alpha < \delta - 1\), and \(z = \pi_{\delta-1}^\delta(y)\) is a limit point of \(H\) by the inductive assumption. If \((\pi_\delta^\alpha)^{-1} z\) consists of one point \(y\), then \(y\) is a limit point of \(H\), since \(y\) is a limit point of any set \(A \subseteq X_\delta\) such that \(z\) is a limit point of \(\pi_{\delta-1}^\delta(A)\). If \((\pi_\delta^\alpha)^{-1} z\) consists of more than one point, then by the definition of \(\pi_{\delta-1}^\delta\) we have \(z \in H_m, \xi(z) < \delta\), and each point of \((\pi_{\delta-1}^\delta)^{-1} z\) is a limit point of the set \(H\). \(\square\)

**Proposition 2.3.** The space \(M_m^\alpha\) is normal and \(\beta M_m^\alpha = X_{\omega_1}\).

To prove Proposition 2.3 it suffices to check that if \(F_1\) and \(F_2\) are disjoint closed subsets of \(M_m^\alpha\), then their closures in \(X_{\omega_1}\) are disjoint as well. And for this it is enough to prove

**Proposition 2.4.** If \(F\) is an arbitrary subset of \(M_m^\alpha\) and \(y \in \overline{F}_{X_{\omega_1}}\), then \(\pi_\beta^{-1}\pi_\alpha(y) \subset \overline{F}_{X_{\omega_1}}\) for some \(\beta < \omega_1\), where \(\pi_\beta : X_{\omega_1} \to X_\beta\) is the composite projection of \(S\).

**Lemma 2.5.** The set \(I = \{y : \pi_\gamma(y) \in \overline{F \cap Z_\gamma^\alpha}\}\) is closed and unbounded in \(\omega_1\).

**Proof.** Let \(\{y_i : i \in \omega\}\) be an increasing sequence of elements from \(I\). We will show that \(y = \sup y_i \in I\). Take an arbitrary neighbourhood \(U\) of \(\pi_\gamma(y)\). Since \(S\) is a continuous spectrum, there is a neighbourhood \(V\) of \(\pi_i(y)\) for some \(i < \omega\) such that \((\pi_\gamma^i)^{-1} V \subseteq U\). By the definition of \(\gamma_i\), we have \(V \cap (F \cap Z_{\gamma_i}) \neq \emptyset\). On the other hand, \((\pi_\gamma^i)^{-1} Z_{\gamma_i} = Z_{\gamma_i}\). Hence, \(\emptyset \neq (\pi_\gamma^i)^{-1} (V \cap (F \cap Z_{\gamma_i})) = ((\pi_\gamma^i)^{-1} V) \cap (F \cap Z_{\gamma_i}) \subset U \cap (F \cap Z_{\gamma_i})\). Therefore, \(y \in I\).

Now we will prove that \(I \cap \{y_\gamma, \omega_1\} \neq \emptyset\) for an arbitrary \(\gamma_0 < \omega_1\). Let \(\{U_{\gamma_0}^i : i \in \omega\}\) be a fundamental sequence of neighbourhoods of \(\pi_{\gamma_0}(y)\) in the compactum
For each \( i \) we take a point \( y_{\gamma_i}^0 \in F \cap \pi_{\gamma_i}^{-1}(U_{\gamma_i}) \). Since \( F = \bigcup \{ F \cap Z_{\beta} : \beta < \omega_1 \} \), there is \( \gamma_i > \gamma_0 \) such that \( y_{\gamma_i}^0 \in F \cap Z_{\gamma_i} \) for all \( i \in \omega \). In the same way we inductively construct an increasing sequence of countable ordinals \( \gamma_k \) and sequences \( \{ y_{\gamma_i}^k : i \in \omega, \ k \in \omega \} \), with

\[
y_{\gamma_i}^{k-1} \in F \cap Z_{\gamma_k} \cap \pi_{\gamma_i}^{-1}(U_{\gamma_i}^{k-1}).
\]

where \( \{ U_{\gamma_i}^k : i \in \omega \} \) is a fundamental sequence of neighbourhoods of \( \pi_{\gamma_k}(y) \). Let \( \gamma = \sup \gamma_k \). It follows from the continuity of \( S \) that \( \{(\pi_{\gamma_k})^{-1}U_{\gamma_k} : i, \ k \in \omega \} \) is a fundamental sequence of neighbourhoods of \( \pi_{\gamma_k}(y) \). Since \( \pi_{\gamma_k}^{-1}(U_{\gamma_k}) = \{ \} \), we have \( y_{\gamma_k}^{\gamma_k} \in (\pi_{\gamma_k})^{-1}U_{\gamma_k} \). So, \( \pi_{\gamma_k}(y) \) is in the closure of \( \{ y_{\gamma_i}^k : i, \ k \in \omega \} \subset F \cap Z_y \). Hence, \( \gamma \in I \). The lemma is proved.

**Proof of Proposition 2.4.** If \( y \in Z_{\omega_1} \), then \( y \in Z_{\beta} \) for some \( \beta < \omega_1 \), consequently \( y = \pi_{\beta}^{-1}(\pi_{\beta}(y)) \). So \( \pi_{\beta}^{-1}(\pi_{\beta}(y)) = y \in \overline{F \times \omega_1} \). It remains to consider the case \( y \in H_m^n = X_{\omega_1} \setminus Z_{\omega_1} \).

In case it is needed we can change \( F \) for its dense subset. Hence, without loss of generality we can assume that \( F \cap Z_y \) is countable for each \( \gamma < \omega_1 \). Then by [13, Lemma 3.12] the set

\[
A = \{ \lambda : \eta^{-1}\lambda \cap F = F \cap Z_\lambda \}
\]

is closed and unbounded in \( \omega_1 \). It follows from \( \diamond \) that the set

\[
B = \Gamma \cap A \cap \{ \beta : J_{\beta} = \eta(F) \cap \beta \}
\]

is unbounded in \( \omega_1 \). Let \( \beta \in B \cap [\xi(y), \omega_1) \). Then

1. \( \pi_{\beta}(y) \in \overline{F \cap Z_{\beta}} \); 
2. \( \overline{Z_{\beta} \cap \eta^{-1}J_{\beta}} = (\text{since } J_{\beta} = \eta(F) \cap \beta = Z_{\beta} \cap F \cap \eta^{-1}\beta = (\because \beta \in A) = Z_{\beta} \cap F). \)

It follows from (1), (2) and Main property of sets \( \eta^{-1}J \) that for \( \gamma > \beta \):

\[
(\pi_{\gamma}^{-\beta})^{-1}(\pi_{\beta}(y)) \subset \overline{Z_{\beta} \cap F_{\gamma}}. \tag{1,\gamma}
\]

Since \( y \in H_m^n \), we have \( y = \pi_{\gamma}(y) \) and

\[
(\pi_{\gamma}^{-\beta})^{-1}(\pi_{\beta}(y)) \setminus \{ \pi_{\gamma}(y) \} \subset Z_{\gamma}. \tag{2,\gamma}
\]

In what follows we need the following elementary fact.

**Statement.** If \( U \) is an open subset of a topological space \( X \) and \( A \subset U \), then \( \overline{A}^U = \overline{A} \cap U \).

It follows from (1,\gamma), (2,\gamma) and the Statement that

\[
(\pi_{\gamma}^{-\beta})^{-1}(\pi_{\beta}(y)) \setminus \{ \pi_{\gamma}(y) \} \subset \overline{Z_{\beta} \cap F_{\gamma}} \cap X_y. \tag{3,\gamma}
\]

By the definition of the spectrum \( S \) we have

\[
\pi_{\beta}^{-1}(\pi_{\beta}(y)) \setminus \{ y \} = U \{ (\pi_{\gamma}^{-\beta})^{-1}(\pi_{\beta}(y)) \setminus \{ \pi_{\gamma}(y) \} : \beta < \gamma < \omega_1 \}. \tag{1}
\]
Hence, (3,γ) implies
\[ \pi_{\beta}^{-1}\pi_{\beta}(y) \setminus \{y\} \subset \bigcup \{ \mathbb{Z}_\beta \cap F^{\mathbb{Z}_\gamma}: \beta < \gamma < \omega_1 \}, \]
and, consequently,
\[ \pi_{\beta}^{-1}\pi_{\beta}(y) \setminus \{y\} \subset \mathbb{Z}_\beta \cap F^{\omega_1} \subset F^{\omega_1}. \]
Thus, Propositions 2.4 and 2.3 are proved. □

**Proposition 2.6.** If \(|(\pi_{\gamma}^{\beta+1})^{-1}\pi_{\gamma}(y)| \geq 2\) for some \(y \in H^n_m\) and \(\gamma < \omega_1\), then \(|(\pi_{\delta}^{\beta})^{-1}\pi_{\delta}(y)| \geq 2\) for \(\gamma < \delta < \delta' < \omega_1\).

**Proof.** It suffices to show that \(|(\pi_{\delta}^{\gamma+1})^{-1}\pi_{\delta}(y)| \geq 2\) for any \(\delta \geq \gamma + 1\). By the definition of \(\pi_{\gamma}^{\gamma+1}\), the point \(y = \pi_{\gamma}(y)\) has nontrivial inverse image with respect to \(\pi_{\gamma}^{\gamma+1}\) if the following condition is fulfilled:
\[ \xi(y) \leq \gamma + 1 \text{ and } \pi_{\gamma}(y) \text{ is a limit point of the set } \eta^{-1}J_\beta \cap Z_\gamma \text{ for } \beta \leq \gamma. \]
But since Main property of sets \(\eta^{-1}J\), this condition being fulfilled for \(\pi_{\gamma}(y)\) is fulfilled for \(\pi_{\gamma}(y)\) when \(\delta \geq \gamma\). The proposition is proved. □

**Proposition 2.7.** If \(F\) is an arbitrary closed subset of \(M^n_m\), then there is \(y < \omega_1\) such that
\[ \pi_{\gamma}^{-1}\left(\pi_{\gamma}(F^{X_{\omega_1}}) \setminus F\right) \subset F^{X_{\omega_1}}. \]

**Proof.** In the closed subset \(\Phi = F^{X_{\omega_1}} \setminus F\) of the compactum \(H^n_m \subset X_{\omega_1}\), there is a countable dense subset \(\Phi_0 = \{y_i: i \in \omega\}\). In view of Proposition 2.4 for each \(i \in \omega\) there is \(\beta_i < \omega_1\) such that
\[ \pi_{\gamma}^{-1}\pi_{\gamma}(y) \subset F^{X_{\omega_1}}. \]
Increasing \(\beta_i\), if necessary, we may assume that \(|(\pi_{\beta_i}^{\beta_i+1})^{-1}\pi_{\beta_i}(y_i)| \geq 2\). But the spectrum \(S\) has monotone projections. Hence, Proposition 2.6 implies that for \(\beta_i \leq \alpha < \alpha' < \omega_1\) we have
\[ \pi_{\alpha}(y_i) \in (\pi_{\alpha'}^{\alpha'})^{-1}\pi_{\alpha}(y_i) \setminus \{\pi_{\alpha}(y_i)\} \in F^{X_{\omega_1}}. \]
Let \(\beta = \sup \beta_i\).

**Lemma 2.8.** For every \(\gamma \geq \beta\) there exists a countable set \(D_\gamma \subset Z_\gamma \cap F\) such that:
\[ \pi_{\gamma}(\Phi_0) \subset D_\gamma \subset F_{\gamma}, \]
\[ D_\gamma = Z_\gamma \cap D_\gamma \quad \text{for } \gamma' > \gamma. \]

**Proof.** We will construct \(D_\gamma\) inductively, starting with \(D_\beta\). Let us note that for any \(y \in H^n_m\) and \(\alpha < \alpha' < \omega_1\) we have
\[ Z_{\alpha'} \cap \pi^{-1}_{\alpha'} \pi_{\alpha}(y) = (\pi_{\alpha'}^{\alpha'})^{-1}\pi_{\alpha}(y) \setminus \{\pi_{\alpha}(y)\}. \]
Hence, it follows from (2) that
\[ (\pi_{\beta}^{-1})_{\gamma} \{ \pi_{\beta}(y_i) \} \subset Z_\beta \cap F. \]
The point \( \pi_{\beta}(y_i) \) is a limit point of \((\pi_{\beta}^{-1})_{\gamma} \{ \pi_{\beta}(y_i) \} \). Let \( C_i^\beta \) be a sequence, converging to \( \pi_{\beta}(y_i) \) and lying in \((\pi_{\beta}^{-1})_{\gamma} \{ \pi_{\beta}(y_i) \} \). The set \( D_\beta = \cup \{ C_i^\beta : \gamma \in \omega \} \) is in \( Z_\beta \cap F \) and, evidently, satisfies for \( \gamma = \beta \) the conditions (4) and (5) (the last of them means in this case that \( D_\beta \subset Z_\beta \)). An inductive step from \( \gamma \) to \( \gamma + 1 \) is realized in the same way by applying the condition (3) for \( \alpha = \gamma \) and \( \alpha' = \gamma + 1 \) (we add to \( D_\gamma \) countably many sets \( C_i^{\gamma+1} \subset (\pi_{\gamma}^{-1})_{\gamma+1} \{ \pi_{\gamma+1}(y_i) \} \)). For a limit ordinal \( \gamma' \) we put \( D_{\gamma'} = \cup \{ D_\gamma : \gamma < \gamma' \} \). The lemma is proved. \( \Box \)

Coming back to the proof of Proposition 2.7 we set \( D = \cup \{ D_\gamma : \beta \leq \gamma < \omega_1 \} \). By [13, Lemma 3.12] the set
\[ \Lambda = \{ \lambda : \eta^{-1}\lambda \cap D = D \cap Z_\lambda \} \]
is closed and unbounded in \( \omega_1 \). Then by \( \Diamond \) the set
\[ \Gamma = \Lambda \cap \{ \alpha : J_\alpha = \eta(D) \cap \alpha \} \]
is unbounded in \( \omega_1 \). We will show that any \( \gamma \in \Gamma \) is the desired (in Proposition 2.7) number.

Since \( J_\gamma = \eta(D) \cap \gamma \), we have \( \eta^{-1}J_\gamma = D \cap \eta^{-1}\gamma = (\text{since } \gamma \in \Lambda) \cap D \cap Z_\gamma = (\text{by (5)} \text{ and the definition of } D) = D_\gamma \).

Therefore, in view of (4) any point from \( \pi_\gamma(\Phi) \) is a limit point of the set \( \eta^{-1}J_\gamma \subset Z_\gamma \). From Main property of sets \( \eta^{-1}J \) it follows that for any \( \gamma' > \gamma \) we have
\[ (\pi_\gamma)^{-1} \pi_{\gamma'}(\Phi) \subset \eta^{-1}J_{\gamma'}, \]
hence,
\[ \pi_{\gamma - 1} \pi_{\gamma'}(\Phi) \subset \eta^{-1}J_{\gamma'} = \overline{D_{\gamma'}} \subset \overline{\overline{D_{\gamma'}}}. \]

The last inclusion according to \( D_{\gamma'} \subset Z_{\gamma'} \cap F \subset F \) proves Proposition 2.7. \( \Box \)

**Proposition 2.9.** If \( F \) is an arbitrary closed subset of \( M_m^n \), then there is \( \beta < \omega_1 \) such that
\[ \pi_{\beta}^{-1} \pi_{\beta} F_{X_{\omega_1}} = F_{X_{\omega_1}}. \tag{7} \]

**Proof.** Since the inclusion \( \supset \) takes place for any spectrum of bicompacta, it suffices to check the inclusion \( \subset \). The open set \( Z_\beta \subset X_\beta \) is a set of one-to-one correspondence of the map \( \pi_{\beta} \). Thus, it is enough to find \( \beta \) such that
\[ \pi_{\beta}^{-1}(\pi_{\beta} F_{X_{\omega_1}} \cap Z_\beta) \subset F_{X_{\omega_1}}. \]
And for it that suffices according to Proposition 2.7 to find \( \beta \geq \gamma \) such that

\[
\pi_\beta^{-1}\left(\pi_\beta \bar{F}^X_{\omega_1} \setminus Z_\beta\right) \subset \pi_\gamma^{-1}\pi_\gamma(\Phi),
\]

(8)

where \( \Phi = \bar{F}^X_{\omega_1} \setminus F \). Assume that the inclusion (8) does not take place for all \( \beta \geq \gamma \). Then for each \( \beta \geq \gamma \) there is a point.

\[
x_\beta \in \pi_\beta^{-1}\left(\pi_\beta \bar{F}^X_{\omega_1} \setminus Z_\beta\right) \setminus \pi_\gamma^{-1}\pi_\gamma(\Phi).
\]

Let \( \{U_i : i \in \omega\} \) be a fundamental sequence of neighbourhoods of \( \pi_\gamma^{-1}\pi_\gamma(\Phi) \). There is a number \( i \) such that \( B = \{\beta : x_\beta \notin U_i\} \) is uncountable. Let \( x \) be a condensation point of the set \( \{x_\beta : \beta \in B\} \). Clearly, \( x \notin U_i \). To get a contradiction, it remains to show that \( x \in \Phi \).

If \( \beta \geq \gamma \), then \( \pi_\beta^{-1}\pi_\beta \left(A \right) \subset \pi_\gamma^{-1}\pi_\gamma \left(A \right) \) for an arbitrary set \( A \subset X_{\omega_1} \). Consequently, \( x_\beta \notin \pi_\beta^{-1}\pi_\beta(\Phi) \) and, hence, \( x_\beta \in \pi_\beta^{-1}\pi_\beta(F) \). Therefore, for any \( \beta \geq \gamma \) there is a point \( y_\beta \in F \) such that:

\[
x_\beta \in \pi_\beta^{-1}\pi_\beta(y_\beta).
\]

(9)

Since \( x_\beta \notin \pi_\beta^{-1}Z_\beta = Z_\beta \) and \( x \) is a condensation point of the set \( \{x_\beta : \beta \in B\} \), we have \( x \notin \cup \{Z_\beta : \beta \in B\} = Z_\omega_1 \). Hence, \( x \notin F \). To check that \( x \in \bar{F}^X_{\omega_1} \), it suffices to find a point of the type \( y_\beta \) in any neighbourhood \( O_\alpha \). We can assume that \( O_\alpha = \pi_\alpha^{-1}V \), where \( V \) is open in \( X_{\omega_1} \). The set \( B_1 = \{\beta \in B : x_\beta \in O_\alpha\} \) is uncountable. Consequently, there is \( \beta \in B_1 \) such that \( \alpha \leq \beta \). Then for this \( \beta \) and for every point \( z \in O_\alpha \) we have \( \pi_\beta^{-1}\pi_\beta(z) \subset O_r \), in particular \( \pi_\beta^{-1}\pi_\beta(x_\beta) \subset O_r \). But from (9) it follows that \( y_\beta \in \pi_\beta^{-1}\pi_\beta(x_\beta) \). Hence, \( y_\beta \in O_\alpha \). The proposition is proved. \( \square \)

**Lemma 2.10.** If \( F \) is an arbitrary closed subset of \( M_m^n \), then the set

\[
\Gamma = \left\{ \gamma : \bar{F} \cap Z_{\gamma} = \pi_\gamma \bar{F}^X_{\omega_1} \right\}
\]

is closed and unbounded in \( \omega_1 \).

**Proof.** Let \( \{\gamma_i : i \in \omega\} \) be an increasing sequence of elements of \( \Gamma \). We will show that \( \gamma = \sup \gamma_i \in \Gamma \). It suffices to check that \( \pi_\gamma \bar{F}^X_{\omega_1} \subset \bar{F} \cap Z_{\gamma} \). Take an arbitrary point \( x \in \bar{F}^X_{\omega_1} \) and a neighbourhood \( O_{\pi_\gamma}(x) \). Since \( S \) is continuous, there is a neighbourhood \( U \) of some \( \pi_\gamma \) such that \( (\pi_\gamma)^{-1}U \subset O_{\pi_\gamma}(x) \). By our assumption \( U \) meets the set \( F \cap Z_{\gamma} \). Therefore, the neighbourhood \( O_{\pi_\gamma}(x) \supset (\pi_\gamma)^{-1}U \) meets the set \( (\pi_\gamma)^{-1}(F \cap Z_{\gamma}) = F \cap Z_{\gamma} \).

Now we will show that \( \Gamma \cap \{\gamma_0, \omega_1\} \neq \emptyset \) for an arbitrary \( \gamma_0 \geq \beta \), where \( \beta \) satisfies Proposition 2.9. Let us note that by Proposition 2.7 for all \( \gamma \geq \beta \) we have

\[
\pi_\gamma^{-1}\pi_\gamma \bar{F}^X_{\omega_1} = \bar{F}^X_{\omega_1}.
\]

(10)

Let \( A_0 \) be a countable dense subset of \( \pi_{\gamma_0} \bar{F}^X_{\omega_1} \setminus F \cap Z_{\gamma_0} \). By the definition of the spectrum \( S \) there is \( \gamma_1 > \gamma_0 \) such that \( |(\pi_{\gamma_0})^{-1}a| \geq 2 \) for any point \( a \in A_0 \). Now
we inductively construct an increasing sequence of ordinals \( \gamma_i \) and countable sets \( A_i \) such that:

\[
A_i \text{ is dense in } \pi_{\gamma_i} F_{X_1} \setminus F \cap Z_{\gamma_i}, \tag{11}
\]

\[
\left( (\pi_{\gamma_i}^{-1}) a \right) \supset 2 \text{ for any } a \in A_i. \tag{12}
\]

Let us show that \( \gamma = \sup \gamma_i \in \Gamma \). From (11) it follows that \( A_i \cup (F \cap Z_{\gamma_i}) \) is dense in \( \pi_{\gamma_i} F_{X_{\omega_1}} \). In view of the continuity of \( S \) the set

\[
A = \bigcup \left\{ (\pi_{\gamma_i}^{-1})^{-1} \left( A_i \cup (F \cap Z_{\gamma_i}) \right) \colon i \in \omega \right\}
\]

is dense in \( \pi_{\gamma} F_{X_{\omega_1}} \). It remains to show that \( F \cap Z_{\gamma} \) is dense in \( A \). For that it suffices to verify that for an arbitrary \( a \in A \), we have

\[
\left( (\pi_{\gamma_i}^{-1})^{-1} a \right) = F \cap Z_{\gamma}. \tag{13}
\]

Since \( S \) has monotone projections and \( \left( \pi_{\gamma_i}^{-1} \right)^a \) consists of more than one point (according to (12)), the point \( a = \pi_{\gamma}(a) \) is a limit point of \( \left( \pi_{\gamma_i}^{-1} a \right) \). Hence, to check (13), it suffices to prove that

\[
\left( \pi_{\gamma_i}^{-1} a \right) \setminus \{ a \} \subset F \cap Z_{\gamma}. \tag{14}
\]

But \( \pi_{\gamma_i}^{-1} a \subset F_{X_{\omega_1}} \) according to (10). Therefore,

\[
\pi_{\gamma_i}^{-1} a \setminus \{ a \} = \left( \pi_{\gamma_i}^{-1} a \right) \cap Z_{\omega_1} \subset F_{X_{\omega_1}} \cap Z_{\omega_1} = F.
\]

Consequently, by (6) we have

\[
\left( \pi_{\gamma_i}^{-1} a \setminus \{ a \} \right) \cap Z_{\gamma} \subset F \cap Z_{\gamma}.
\]

The inclusion (14) is checked. The lemma is proved. \( \square \)

**Proposition 2.11.** \( M_m^n \) is countably compact, perfectly normal and hereditarily separable.

**Proof.** Countable compactness of \( M_m^n \) follows from Proposition 2.4. Proposition 2.9 implies that \( M_m^n \) is perfectly normal. Indeed, let \( F \) be closed in \( M_m^n \), and let \( \beta \) satisfy (7). There is a countable sequence \( U_i, i \in \omega \), of open subsets of the compactum \( X_{\beta} \) such that \( \pi_{\beta} F_{X_{\omega_1}} = \bigcap \{ U_i \colon i \in \omega \} \). Then setting \( V_i = (\pi_{\beta}^{-1} U_i) \cap M_m^n \), we have that \( F = \bigcap \{ V_i \colon i \in \omega \} \).

So the manifold \( M_m^n \) is separable and perfectly normal, thus it is hereditarily separable (see (13), 2.18). \( \square \)

**Proposition 2.12.** \( \dim M_m^n = m \) and \( \operatorname{Ind} M_m^n = m + n - 2. \)
A differentiable manifold with noncoinciding dimensions

**Proof.** We have \( \dim X_{\omega_1} = m \), because \( X_{\omega_1} \) is the limit of the spectrum \( S \), consisting of \( m \)-dimensional compacta \( X_n = Y_n \), and \( X_{\omega_1} \supset H_m^m \) with \( \dim H_m^m = m \). On the other hand for an arbitrary normal \( X \) we have (see [1])

\[
\dim X = \dim \beta X, \quad \text{Ind} \ X = \text{Ind} \ \beta X.
\]

Hence, \( \dim M_m^m = m \) by Proposition 2.3.

For any closed set \( F \) of \( X_{\omega_1} \) we put \( F^0 = F \setminus H_n^m X_{\omega_1} \).

**Lemma 2.13.** Let \( F \) be a closed set in \( X_{\omega_1} \), and let \( A \) and \( B \) be disjoint closed subsets of \( F \). Then in \( F \) there is a partition \( C \) between \( A \) and \( B \) such that:

1. if \( \dim F > n - 1 \) or \( F \cap H_m^m = \emptyset \), then \( \dim C \leq \dim F - 1 \);
2. if \( F \cap H_m^m \neq \emptyset \), then \( \dim C \cap H_m^m \leq \dim F \cap H_m^m - 1 \).

**Proof.** If \( F \cap H_m^m = \emptyset \), then our assertion follows from the equality \( \dim F = \text{Ind} F \). Let now \( F \cap H_m^m \neq \emptyset \). Assume that \( F = F^0 \). By Proposition 2.9 there is \( \beta < \omega_1 \) such that \( F = \pi_{\beta}^{-1} \pi_{\beta}(F) \). Take \( \gamma > \beta \) such that \( \pi_{\gamma}(A) \cap \pi_{\gamma}(B) = \emptyset \). All fibres of all neighbouring projections of \( S \) are contractible, so \( \pi_{\gamma} \) is an acyclic map. Hence, we have [14] \( \dim \pi_{\gamma}(F) \leq \dim F \). There is a partition \( D \) in \( \pi_{\gamma}(F) \) between \( \pi_{\gamma}(A) \) and \( \pi_{\gamma}(B) \) such that \( \dim D \leq \dim \pi_{\gamma}(F) - 1 \) and \( \dim(D \cap H_m^m) \leq \dim \pi_{\gamma}(F) \cap H_m^m - 1 \). The set \( C = \pi_{\gamma}^{-1}D \) is a partition in \( F \) between \( A \) and \( B \).

Recall that \( H_m^m \supset I^m \) and every point \( y \in I^m \) is a point of the second genus. Hence, if \( (\pi_n^{-1})^{-1}y \) consists of more than one point, then it is homeomorphic to the cone over \( M_1 \times I^{n-1} \), in particular \( \dim(\pi_n^{-1})^{-1}y = n - 2 \). On the other hand every map \( \pi_n \) is acyclic. Then it follows from (1), Proposition 2.6, Theorem C and [14] that

\[
\dim \pi_{\gamma}^{-1}y = n - 2 \quad \text{for any} \ y \in I^m \quad \text{and} \ \gamma < \omega_1.
\]

It follows from Theorem A that all neighbouring projections in the spectrum \( S \) are fully closed. Then by Theorem C we have \( \dim C \leq \max(\dim D, \dim \pi_{\gamma}(F)) \leq \max(\dim \pi_{\gamma}(F) + 1, n - 2) \). Since \( \dim \pi_{\gamma}(F) \leq \dim F \), we get \( \dim C \leq \dim F - 1 \) if \( \dim F > n - 1 \). On the other hand \( D \cap H_m^m \) is homeomorphic to \( C \cap H_m^m \), and \( \pi_{\gamma}(F) \cap H_m^m \) is homeomorphic to \( F \cap H_m^m \). Consequently, \( \dim(C \cap H_m^m) = \dim(D \cap H_m^m) \leq \dim(\pi_{\gamma}(F) \cap H_m^m) - 1 = \dim F \cap H_m^m - 1 \).

In the case \( F \neq F^0 \) we construct a partition \( C_0 \) in \( F^0 \) between \( A_0 = A \cap F^0 \) and \( B_0 = B \cap F^0 \) with the required property. After that, using a representation \( F = F^0 \cup (F \setminus F^0) \), we can extend the partition \( C_0 \) to a partition \( C \) in \( F \) between \( A \) and \( B \) such that \( \dim(C \setminus C_0) < \dim(F \setminus F^0) \). It is possible to do, because \( F \setminus F^0 \) is hereditarily normal and \( \dim(F \setminus F^0) = \text{ind}(F \setminus F^0) \). Since \( C \setminus C_0 \) is a countable sum of compacta, we have \( \dim C = \max(\dim C_0, \dim(C \setminus C_0)) \leq \max(\dim F^0 - 1, \dim(F \setminus F^0) - 1) = \dim F - 1 \). In the same way we get \( \dim C \cap H_m^m \leq \dim F \cap H_m^m - 1 \). The lemma is proved. \[ \square \]
Now we can prove that \( \text{Ind } X_{\omega_1} \leq m + n - 2 \). According to Lemma 2.13 it suffices to show that if \( F \) is closed in \( X_{\omega_1} \) and \( \dim F \leq n - 2 \), then \( \text{Ind } F \leq 2n - 4 \). Applying Lemma 2.13 \( n - 2 \) times, we reduce our problem to the following one:

If \( \dim F \leq n - 2 \) and \( \dim F \cap H^m_n = 0 \), then \( \text{Ind } F \leq n - 2 \).

Let \( A \) and \( B \) be disjoint closed subsets of \( F \). Since \( F \cap H^m_n \) is zero-dimensional, we can find neighbourhoods \( OA \) and \( OB \) in \( F \) such that \( \overline{OA} \cap \overline{OB} = \emptyset \) and \( F \cap H^m_n \subset OA \cup OB = U \). Let \( \Phi = F \setminus U \), \( A_0 = \overline{OA} \cap \Phi \), \( B_0 = \overline{OB} \cap \Phi \). Then \( \Phi \) is a compact subset of \( M^m_n \). Hence, \( \text{Ind } \Phi \leq n - 2 \) and there are disjoint neighbourhoods \( OA_0 \) and \( OB_0 \) in \( \Phi \) such that \( \text{Ind } C \leq n - 3 \), where \( C = \Phi \setminus OA_0 \cup OB_0 \). Now, setting \( V = OA_0 \cup OA \), \( W = OB_0 \cup OB \), we get the disjoint neighbourhoods \( V \) and \( W \) of the sets \( A \) and \( B \) in \( F \). In fact, \( C \cup W = (C \cup OB_0) \cup (B_0 \cup OB) \) is closed, because \( C \cup OB_0 \) is closed in \( \Phi \) and \( B_0 \cup OB = \overline{OB} \). Thus, \( V = F \setminus C \cup W \) is open in \( F \). By the same argument \( W \) is open in \( F \). Consequently, \( C = F \setminus V \cup W \) is a partition in \( F \) between \( A \) and \( B \) with \( \text{Ind } C \leq n - 3 \). The inequality \( \text{Ind } X_{\omega_1} \leq m + n - 2 \) is proved.

Let \( E = \pi^{-1}_0 I^m \). We will say that a closed set \( F \subset E \) is long if \( F^0 \cap I^m \neq \emptyset \). A long set \( F \) is said to be clean, if \( F = F^0 \). Recall that a subset \( U \) of a topological space \( X \) is said to be canonically open in \( X \) if \( U = \text{Int}_X \overline{U} \).

**Lemma 2.14.** Let \( F \) be clean and \( U \) canonically open in \( F \) with \( U \cap I^m \neq \emptyset \neq I^m \setminus \overline{U} \). Then \( \text{Bd } U \) is clean.

**Proof.** First of all we will prove that \( \overline{U} \) is clean. Let \( y \in U \cap I^m \), and let \( Oy \) be a neighbourhood of \( y \) in \( X_{\omega_1} \) such that \( Oy \cap F \subset U \). Since \( F \) is clean, we have \( Oy \cap (F \setminus I^m) \neq \emptyset \). But \( Oy \cap (F \setminus I^m) = Oy \cap (U \setminus I^m) \). Hence, \( y \) is in the closure of \( U \setminus I^m \). Therefore, \( \overline{U} \) is long. By the same argument \( \overline{U} \) is clean.

Now we set \( V = F \setminus \overline{U} \). Then \( V \) is clean. Because \( U \) is canonically open, we have \( \text{Bd } U = \overline{U} \cap V \). If \( y \in \text{Bd } U \cap I^m \subset \overline{U} \cap I^m \), then since \( \overline{U} \) is clean, it follows from Proposition 2.4 that \( \pi^{-1}_{\alpha} \pi_\alpha(y) \subset \overline{U} \) for some \( \alpha < \omega_1 \). By the same reason \( \pi^{-1}_{\beta} \pi_\beta(y) \subset \overline{U} \cap V = \text{Bd } U \) for some \( \beta > \alpha \). But \( \pi^{-1}_{\beta} \pi_\beta(y) \) is connected and consists of more than one point. Hence, \( y \) is in the closure of \( \pi^{-1}_{\alpha} \pi_\alpha(y) \setminus \{y\} \subset \text{Bd } U \setminus I^m \). The lemma is proved. \( \square \)

**Lemma 2.15.** If \( F \) is clean and \( F \cap I^m \neq \emptyset \), then

\[
\text{Ind } F \geq \text{Ind } F \cap I^m + n - 2.
\]

**Proof.** By induction: We start with \( \text{Ind } F \cap I^m = 0 \). Because \( F \) is clean, \( F \supset \pi^{-1}_\alpha \pi_\alpha^{-1}(y) \) for any \( y \in F \cap I^m \) and some \( \alpha < \omega_1 \). Then

\[
\text{Ind } F \geq \text{Ind } \pi^{-1}_\alpha \pi_\alpha^{-1}(y) \geq \dim \pi^{-1}_\alpha \pi_\alpha^{-1}(y) = n - 2.
\]

The last equality takes place according to (15). Suppose that we proved our assertion for all clean \( F \) with \( \text{Ind } F \cap I^m \leq k \), \( k > 0 \), and let \( \text{Ind } F \cap I^m = k + 1 \).
There are disjoint closed subsets $A$ and $B$ in $F \cap I^m$ such that $\text{Ind } C \geq k$ for any
partition $C$ in $F \cap I^m$ between $A$ and $B$. Since $k \geq 0$, the sets $A$ and $B$ are nonempty. Let $\Phi$ be an arbitrary partition in $F$ between $A$ and $B$. Reducing $\Phi$ if necessary, we can assume that $\Phi = \text{Bd } U$, where $U$ is a canonically open neigh-
bourhood of $A$ in $F$. Because $A \neq \emptyset \neq B$, we can apply Lemma 2.14. Hence $\Phi$ is
clean. Since $\Phi \cap I^m$ is a partition in $F \cap I^m$ between $A$ and $B$, we have
$\text{Ind } \Phi \cap I^m \geq k$. If $\text{Ind } \Phi \cap I^m = k$, then by the inductive assumption
$\text{Ind } \Phi \cap I^m + n - 2 = k + n - 2$. If $\text{Ind } \Phi \cap I^m = k + 1 = \text{Ind } F \cap I^m$ we
can find a clean set $D \subset \Phi$ such that $\text{Ind } D \cap I^m = k$. Indeed, $\Phi = \pi^{-1}_{\alpha}(\Phi)$ for
some $\alpha < \omega_1$ in view of Proposition 2.9. Then we take a closed subset $G \subset \Phi \cap I^m
with \text{Ind } G = k$ and set $D = \pi^{-1}_{\alpha}(G)$. Again by the inductive assumption we
have $\text{Ind } \Phi \geq \text{Ind } D \geq \text{Ind } D \cap I^m + n - 2 = k + n - 2$. Consequently, $\text{Ind } F \geq k + 1 + n - 2$. The lemma is proved. □

Applying Lemma 2.14 to the set $F = E$, we obtain $\text{Ind } E \geq m + n - 2$. So,
$\text{Ind } X_{as} = m + n - 2$ and Proposition 2.12 is proved.

Theorem 0.1 follows from Propositions 2.2, 2.11 and 2.12.

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