Steady fully developed natural convection flow in a vertical annular microchannel having temperature dependent viscosity: An exact solution

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Received 9 October 2015; revised 21 February 2016; accepted 25 February 2016
Available online 9 March 2016

KEYWORDS
Annular micro-channel; Natural convection; Temperature dependent viscosity; Velocity slip and temperature jump

Abstract This study is devoted to investigate the steady fully developed natural convection flow in a vertical annular micro-channel having temperature dependent viscosity in the presence of velocity slip and temperature jump at the annular micro-channel surfaces. The governing equations of the motion are a set of ordinary differential equations and their analytical solutions in dimensionless form have been obtained for the temperature field and velocity field. The effect of various flow parameters entering into the problem is discussed with the aid of line graphs. During the course of numerical investigation, it is found that increase in viscosity variation parameter enhances the fluid velocity and velocity slip. Furthermore, an increase in viscosity variation parameter leads to increase in the volume flow rate and skin friction.

1. Introduction

Fluid flow in micro-channel has continued to attract interest because of its practical applications in space systems, manufacturing and material processing operations, and in high-power-density chips in supercomputers and other electronics. Several investigations have been accomplished on forced convection fluid flow in micro-channels and micro-tube. However, only a few studies have been carried out on natural and mixed convection in vertical micro-channels and micro-tube. Chen and Weng [1] studied the flow mechanism in a vertical micro-channel and obtained an exact solution of the fully developed natural convection in an open-ended vertical parallel-plate micro-channel due to asymmetric heating of micro-channel walls. They found that the rarefaction and fluid–wall interaction have significant effects on the flow and thermal fields. Jha et al. [2] extended this work by taking into account suction/injection on the micro-channel walls. They concluded in their work that skin friction as well as rate of heat transfer is strongly dependent on suction/injection parameter. The transient hydrodynamics and thermal behaviors of fluid flow in an open-ended vertical parallel-plate micro-channel, under the effect of the hyperbolic-heat-conduction model, were investigated semi-analytically in [3]. They concluded that, as Knudsen number increases, velocity slip and temperature jump increase at the boundaries. Haddad et al. [4] numerically investigated the developing hydrodynamical behaviors of free convection gas flow in a vertical open-ended parallel-plate micro-channel and obtained an exact solution of the fully developed natural convection in an open-ended vertical parallel-plate micro-channel due to asymmetric heating of micro-channel walls. They found that the rarefaction and fluid–wall interaction have significant effects on the flow and thermal fields. Jha et al. [2] extended this work by taking into account suction/injection on the micro-channel walls. They concluded in their work that skin friction as well as rate of heat transfer is strongly dependent on suction/injection parameter.
micro-channel filled with porous media. Their results showed that the slip in velocity and jump in temperature decreased in the axial direction of the flow. Numerical solutions were obtained by Buonomo and Manca [5] for natural convection in parallel-plate vertical microchannels due to asymmetric heating by imposing constant heat flux on the boundaries. Buonomo and Manca [6] further performed a numerical study on transient mixed convection in a vertical micro-channel due to asymmetric as well as symmetric heat fluxes on the micro-channel surfaces. Chen and Weng [7] numerically studied the creep effect on the flow and heat transfer characteristics, for developing natural convective microflow in the same geometry.

In another related article, Weng and Chen [8] examined the influence of wall-surface curvature on the flow and thermal fields as well as the corresponding characteristics over the heated wall. The results of this work show the nonlinear behavior in temperature. It was also concluded that under certain rarefaction and wall interaction condition, by decreasing the curvature radius ratio, skin-friction decreases while rate of heat transfer increases. Jha et al. [9] extended the work of Weng and Chen [8] to the case in which the cylindrical surfaces forming the annulus are permeable, i.e. when there is a suction or injection through the annulus surfaces. They concluded that as suction/injection on the cylinder walls increases, the fluid velocity and temperature are enhanced. In another work, Jha and Aina [10] analyzed the work of Weng and Chen [8] by incorporating the pressure gradient in the vertical direction. It is observed that the probability of reverse flow formation increases at inner surface of outer cylinder of the micro-annulus by increasing curvature radius, while it decreases with increase in Knudsen number and fluid wall interaction parameter. Avci and Aydin [11,12] presented exact solutions for fully developed mixed convection in a vertical parallel-plate micro-channel with constant plate temperature and constant heat flux on the plates respectively. Also, Avci and Aydin [13] studied the fully developed mixed convective heat transfer of a Newtonian fluid in a vertical micro-annulus formed by two concentric micro-tubes. It is found that increasing mixed convection parameter enhances heat transfer while rarefaction effects considered by the velocity slip and the temperature jump in the slip flow regime decrease it. Jha and Aina [14] further extended the work of Avci and Aydin [13] to the case when suction/injection is imposed on the annulus surfaces. They concluded in their study that as suction/injection on the micro-annulus (MPA) increases, the fluid velocity and temperature increase. Recently, Jha and Aina [15] investigated steady fully developed mixed convection flow in a vertical micro-annulus in the presence of transverse magnetic field. Das et al. [16] presented a theoretical analysis to investigate the effect of buoyancy force on mixed convective Couette flow of a reactive viscous incompressible nanofluid between two concentric cylindrical pipes under bimolecular, Arrhenius and sensitized reaction rates. In another article, Das et al. [17] studied the fully developed mixed convection flow in a vertical channel filled with nanofluids in the presence of a uniform transverse magnetic field. They reported that the magnetic field tends to enhance the nanofluid velocity in the channel. Makinde [18] investigated the thermal analysis of a reactive generalized Couette flow of power law fluids between concentric cylindrical pipes.

On the other hand, most of the existing analytical studies for such problems are based on the constant physical properties of the fluid. However, accurate prediction for the flow formation and heat transfer can be achieved by considering variation of physical properties with temperature [19], especially for fluid viscosity. Makinde and Chinyoka [20] presented numerical solution of unsteady flow of a variable viscosity reactive fluid in a slit with wall suction/injection. Tshella et al. [21] investigated the entropy generation rate in a variable viscosity liquid flowing steadily through a cylindrical pipe with convective cooling at the pipe surface. They reported that a decrease in the fluid viscosity and an increase in viscous heating enhance total entropy generation in the flow fluid. Klemp
et al. [22] studied numerically the effect of temperature dependent viscosity on the entrance flow in a channel in the hydrodynamic case. The effects of variable viscosity on hydromagnetic flow and heat transfer have been studied by Seddeek [23]. Grasset and Parmentier [24] investigated thermal convection in a volumetrically heated fluid with strongly temperature dependent viscosity cooled from above. In another article, Saravanan and Kandaswamy [25] analyzed the hydromagnetic stability of convective flow of variable viscosity fluids generated by internal heat sources. The effect of variable temperature dependent viscosity on the mixed convection flow from vertical plate is investigated by many authors, e.g. Hady et al. [26] and Mahmud [27]. The free-or-mixed convection boundary layer flow from a horizontal surface in a saturated porous medium taking into account the effect of variable viscosity has been studied by Kumari [28]. Recently, Umavathi and Ojella [29] studied the effect of variable viscosity on free convection flow in a vertical rectangular duct. They observed that the negative values of viscosity variation parameter show intense velocity contours in the lower half region of the duct whereas positive values of viscosity variation parameter show the intense velocity contours in the upper half region of the duct. In another related article, Umavathi [30] studied analytically as well as numerically the combined effects of variable viscosity and variable thermal conductivity on double-diffusive convection flow of fluid in a vertical channel filled with porous material. However, derivation of any exact solution for steady fully developed natural convection flow in a vertical annular micro-channel with temperature dependent viscosity in the presence of velocity slip and temperature jump at the annular micro-channel surfaces with interfacial slip has not been attempted. It is well known that exact solutions have their own theoretical meaning, and many exact solutions played key roles in the early development of fluid mechanics and heat conduction [31,32]. Besides their theoretical importance, exact solutions can also be applied to checking the accuracy, convergence and effectiveness of various numerical computation methods and improving differencing schemes, grid generation ways and so on. Exact solutions are therefore very useful even for the newly rapidly developing computational fluid dynamics and heat transfer.

The objective of this work was to present exact solution of steady fully developed natural convection flow of fluid having temperature dependent viscosity in a vertical annular micro-channel in the presence of velocity slip and temperature jump at the annular micro-channel surfaces. The mathematical model employed herein represents a generalization of the work discussed in [8] to include temperature dependent viscosity.

2. Mathematical analysis

Consider a steady fully developed natural convection flow in a vertical annular micro-channel with temperature dependent viscosity in the presence of velocity slip and temperature jump at the annular micro-channel surfaces. A schematic geometry of the problem under investigation is shown in Fig. 1, where $X$-axis is parallel to the gravitational acceleration $g$ but in the opposite direction while the $r$-axis is in the radial direction. The radius of the inner and outer cylinder walls is $k_1$ and $k_2$, respectively. The outer surface of the inner cylinder is heated to a temperature ($T_1$) greater than that of the surrounding fluid having temperature ($T_0$) and the inner surface of the outer cylinder is maintained at temperature ($T_0$). Due to this temperature difference, natural convection occurs in the vertical annular micro-channel. Since the flow is fully developed and cylinders are of infinite length, the flow depends only on radial coordinate ($r$). Using Boussinesq’s approximation and considering temperature dependent viscosity, the governing momentum and energy equations, describing the present physical situation can be written in dimensional form as follows:

Conservation of momentum

$$\frac{1}{r} \frac{d}{dr} \left( r \mu \frac{du}{dr} \right) + g \rho_0 \beta_0 (T - T_0) = 0$$  \hspace{1cm} (1)

Conservation of energy

$$\frac{1}{r} \frac{d}{dr} \left( r \lambda \frac{dT}{dr} \right) = 0$$  \hspace{1cm} (2)

The boundary conditions for the velocity and temperature field in dimensional form are as follows:

$$u(r = k_1) = \frac{2 - \sigma_r}{\sigma_r} \lambda \frac{du}{dr} |_{r = k_1}$$  \hspace{1cm} (3)

$$u(r = k_2) = -\frac{2 - \sigma_r}{\sigma_r} \lambda \frac{du}{dr} |_{r = k_2}$$  \hspace{1cm} (4)

$$T(r = k_1) = T_1 + \frac{2 - \sigma_r}{\sigma_r} \frac{2 \nu}{\gamma + 1} \lambda \frac{dT}{dr} |_{r = k_1}$$  \hspace{1cm} (5)

$$T(r = k_2) = T_0 - \frac{2 - \sigma_r}{\sigma_r} \frac{2 \nu}{\gamma + 1} \lambda \frac{dT}{dr} |_{r = k_2}$$  \hspace{1cm} (6)

The mathematical model used in the present work to capture the viscosity variation with temperature is [33]

$$\mu = \mu_0 \exp[-b(T - T_0)]$$  \hspace{1cm} (7)
where $\mu_0$ is the viscosity when temperature is $T_0$ while the coefficient $b_0^c (C^{-1})$ determine the strength of dependency between $\mu$ and $T$.

Introducing the following dimensionless quantities in Eqs. (1)–(7),

\[
R = \frac{r - k_1}{w}, \quad w = k_2 - k_1, \quad U = \frac{u}{u_c}, \quad \theta = \frac{T - T_0}{T_1 - T_0}, \quad \eta = \frac{k_1}{k_2}, \quad \beta_i = \frac{2 - \sigma_i}{\sigma_i}, \quad B = b(T_1 - T_0),
\]

Eqs. (1) and (2) can be written in dimensionless form as

\[
\frac{1}{\eta + (1 - \eta)\bar{R}} \frac{d}{d\bar{R}} \left[ \eta + (1 - \eta)\bar{R} \exp (-B\theta) \frac{d\bar{U}}{d\bar{R}} \right] + \theta = 0
\]

subject to the following dimensionless boundary conditions [8]

\[
U(0) = \beta_i Kn \frac{d\bar{U}}{d\bar{R}} \big|_{\bar{R} = 0}, \quad U(1) = -\beta_i Kn \frac{d\bar{U}}{d\bar{R}} \big|_{\bar{R} = 1}
\]

\[
\theta(0) = 1 + \beta_i Kn \ln \frac{d\bar{\theta}}{d\bar{R}} \big|_{\bar{R} = 0}, \quad \theta(1) = -\beta_i Kn \ln \frac{d\bar{\theta}}{d\bar{R}} \big|_{\bar{R} = 1}
\]

By using the transformation $Z = \eta + (1 - \eta)\bar{R}$, the Eqs. (9)–(12) can be written as follows:

\[
\frac{1}{Z} \frac{d}{dZ} \left[ Z \exp (-B\theta) \frac{d\bar{U}}{dZ} \right] + \theta = 0
\]

subject to the boundary conditions

\[
U(\eta) = \beta_i Kn (1 - \eta) \frac{d\bar{U}}{dZ} \big|_{\eta = 0}, \quad U(1) = -\beta_i Kn (1 - \eta) \frac{d\bar{U}}{dZ} \big|_{\eta = 1}
\]

\[
\theta(\eta) = 1 + \beta_i Kn \ln (1 - \eta) \frac{d\bar{\theta}}{dZ} \big|_{\eta = 0}, \quad \theta(1) = -\beta_i Kn \ln (1 - \eta) \frac{d\bar{\theta}}{dZ} \big|_{\eta = 1}
\]

Integrating Eq. (14) and applying the boundary conditions (16) give the following:

\[
\theta(Z) = A_0 + A_1 \ln(\eta)
\]

where

\[
A_1 = \frac{1}{\ln \eta - \beta_i Kn F(1 - \eta) (1 + \frac{1}{\theta})}, \quad A_0 = -\beta_i Kn F(1 - \eta) A_1
\]

Substituting Eq. (17) into the momentum Eq. (13) and solving it using the boundary condition (15) give

\[
U(Z) = B_0 + \frac{B_i C_i(Z)^{\gamma_i}}{C_2} - \frac{1}{(1 - \eta)^2} \left[ \frac{A_0 C_i(Z)^{\gamma_i}}{2E_1} + A_1 C_i \left( \frac{(Z)^{\gamma_i}}{2E_i} (E_1 \ln(Z) - 1) - \frac{(Z)^{\gamma_i}}{4E_i} \right) \right]
\]

Two important parameters for convective micro-flow are the volume flow rate, and skin-friction. The temperature solution as well as rate of heat transfer is exactly the same as discussed by Chen and Weng [8].

The dimensionless volume flow rate is as follows:

\[
Q = \frac{q}{2\pi \rho_c u_c} = \frac{1}{(1 - \eta)^2} \int_1^1 Z U(Z) d\bar{Z}
\]

By substituting Eqs. (19) into (20) and integrating Eq. (20) give the following:

\[
Q = \frac{1}{(1 - \eta)^2} \left[ \frac{B_0 I_5 + B_i I_3 C_i}{C_2} \right]
\]

\[
- \frac{1}{(1 - \eta)^2} \left[ A_1 C_i I_2 + A_1 C_i \left( \frac{1}{2E_1} (E_1 I_3 - I_2) - \frac{I_2}{4E_1} \right) \right]
\]

The skin-frictions ($\tau$) at the cylinder walls are as follows:

\[
\tau_v = \mu(T) \frac{dU}{dR} \big|_{R = 0}
\]

\[
\tau_0 = e^{-b_0} (1 - \eta) \frac{dU}{dZ} \big|_{Z = \eta}
\]

\[
\tau_0 = \exp \left[ -B(A_0 + A_1 \ln(\eta)) \right] \left[ B_i C_i(\eta)^{\gamma_i - 1} - F_2 \right] (1 - \eta)
\]

\[
\tau_1 = e^{-b_0} (1 - \eta) \frac{dU}{dZ} \big|_{Z = 1}
\]

\[
\tau_1 = \exp \left[ -B A_0 \right] \left[ B_i C_i - F_3 \right] (1 - \eta)
\]

where $C_1$, $C_2$, $B_0$, $B_1$, $E_1$, $E_2$, $I_1$, $I_2$, $I_3$, $F_1$, $F_2$, $F_3$, $A_0$, $A_1$ are all constants given in the Appendix A.

In order to verify the accuracy of the present work, we have computed the numerical value for the velocity for small value of $B$. Table gives a comparison of the numerical values of the velocity obtained in the present work when $B \rightarrow 0$ with those obtained by Weng and Chen [8] for $\ln = 1.667$. As can be seen from Table 1, the solutions of the present work agree with those of Weng and Chen [8] for small values of $B$.

3. Results and discussion

MATLAB program is written to compute and generate line graphs for velocity, volume flow rate, and skin-friction at both cylinders for different values of the dimensionless parameters, such as rarefaction parameter ($\beta_i$), fluid–wall interaction parameter ($\ln$), radius ratio ($\eta$) and viscosity variation parameter ($B$) so as to comment on their relative significance in the flow formation. The present parametric study has been performed in the continuum and slip flow regimes ($Kn \ll 0.1$).
Also, for air and various surfaces, the values of $\beta$, and $\beta_i$ range from near 1 to 1.667 and from near 1.64 to more than 10, respectively. So, this study has been performed over the reasonable ranges of $0 \leq \beta_i Kn \leq 0.1$ and $0 \leq \ln \leq 10$. The selected reference values of $\beta_i Kn$, and $\ln$ for the present analysis are 0.05 and 1.64 respectively as given in Weng and Chen [8].

Temperature solution as well as rate of heat transfer is exactly the same as discussed by Weng and Chen [8]; therefore, we have only discussed the variation of velocity, volume flow rate and skin friction for different controlling parameters.

Fig. 2 shows the velocity distribution for different values of rarefaction parameter ($\beta_i Kn$) for fixed values of $B = 0.5$ and $\ln = 1.64$. It is observed that, as rarefaction parameter ($\beta_i Kn$) increases, the velocity slip at the cylindrical surfaces increases which reduces the retarding effect of the boundaries. This yields an observable increase in the fluid velocity. Furthermore, the slip induced by rarefaction effect increases as radius ratio ($\eta$) increases.

Table 1  Comparison of the values of velocity obtained in the present work with those obtained by Weng and Chen [8].

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$R$</th>
<th>Velocity ($\ln = 1.64$)</th>
<th>Weng and Chen [8]</th>
<th>Present work ($B \rightarrow 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.0713</td>
<td>0.0712</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.2</td>
<td>0.0838</td>
<td>0.0837</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.2</td>
<td>0.0741</td>
<td>0.0744</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.0505</td>
<td>0.0503</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
<td>0.0651</td>
<td>0.0650</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.2</td>
<td>0.0730</td>
<td>0.0725</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.2</td>
<td>0.0624</td>
<td>0.0621</td>
<td></td>
</tr>
<tr>
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<td>0.2</td>
<td>0.0409</td>
<td>0.0405</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.0527</td>
<td>0.0526</td>
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</tr>
<tr>
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<td>0.2</td>
<td>0.0538</td>
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</tr>
<tr>
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<td>0.0451</td>
<td>0.0450</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.0270</td>
<td>0.0269</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3 depicts the effect of fluid–wall interaction parameter ($\ln$) on the velocity distribution for fixed values of $\beta_i Kn = 0.05$ and $B = 0.5$. It is evident from Fig. 3 that, the increase in fluid–wall interaction parameter ($\ln$) leads to the decrease in fluid velocity and increase in slip velocity near the outer surface of the inner cylinder while reverse trend is observed at inner surface of the outer cylinder. In addition, the slip induced by fluid–wall interaction parameter ($\ln$) increases as radius ratio ($\eta$) decreases while the impact of fluid–wall interaction parameter ($\ln$) on the slip is more visible for smaller radius ratio ($\eta$).

Fig. 4 exhibits the effect of viscosity variation parameter ($B$) on velocity distribution for fixed values of $\beta_i Kn = 0.05$ and $\ln = 1.64$. It is observed that increase in viscosity variation parameter ($B$) enhances the fluid velocity and slip velocity while the profiles for constant viscosity ($B = 0$) lie above $B < 0$ and below $B > 0$ on velocity field. This is due to the fact that as viscosity variation parameter ($B > 0$) increases, the viscosity of the working fluid decreases due to exponentially decaying nature of viscosity which causes higher velocity and velocity slip while as the viscosity variation parameter ($B < 0$) increases, the viscosity of the working fluid increases which causes lower velocity and velocity slip. It is also evident in Fig. 4 that velocity slip increases for $B > 0$ and decreases for $B < 0$ with increase in radius ratio ($\eta$).

Fig. 5 illustrates the effects of rarefaction parameter ($\beta_i Kn$) and fluid–wall interaction parameter ($\ln$) on volume flow rate ($Q$). It is interesting to note that the volume flow rate ($Q$) is a decreasing function of fluid–wall interaction parameter ($\ln$). Furthermore, it is found that increase in radius ratio ($\eta$) and rarefaction parameter ($\beta_i Kn$) leads to increase in the volume flow rate.

Fig. 6 reveals the influences of rarefaction parameter ($\beta_i Kn$) and viscosity variation parameter ($B$) on volume flow rate ($Q$). It is observed from Fig. 6 that, the volume flow rate ($Q$) increases, as viscosity variation parameter ($B$) increases. In addition, it is interesting to note that increase in rarefaction parameter ($\beta_i Kn$) leads to increase in the volume flow rate.

Fig. 7 exhibits the effects of rarefaction parameter ($\beta_i Kn$) and fluid–wall interaction parameter ($\ln$) on skin-friction at
outer surface of inner cylinder ($R = 0$). It is clear that the skin-friction decreases with the increase in fluid–wall interaction parameter ($ln$) and rarefaction parameter ($\beta, Kn$). It is interesting to note that the impact of these parameters is significant for small value of radius ratio ($g$).

Fig. 8 shows the effects of rarefaction parameter ($\beta, Kn$) and fluid–wall interaction parameter ($ln$) on skin-friction at inner surface of outer cylinder ($R = 1$). It is evident from Fig. 8 that increases in radius ratio ($\eta$) and rarefaction parameter ($\beta, Kn$) lead to the increase in the skin-friction at inner surface of outer cylinder.

Figs. 9 and 10 illustrate the effects of rarefaction parameter ($\beta, Kn$) and viscosity variation parameter ($B$) on skin-friction at outer surface of inner cylinder ($R = 0$) and inner surface of outer cylinder ($R = 1$), respectively. It is observed from the these Figures that skin friction increases, as viscosity parameter ($B$) increases.
4. Conclusions

Steady fully developed natural convection flow formation in a vertical annular micro-channel having temperature dependent viscosity in the presence of velocity slip and temperature jump at the annular micro-channel surfaces is considered. The role of radius ratio ($\eta$), viscosity variation parameter ($B$), rarefaction parameter ($\beta, Kn$), and fluid-wall interaction parameter ($\ln$) on the fluid velocity, volume flow rate and skin-friction is investigated. This study agrees with the findings of Weng and Chen [8] for vanishing viscosity variation parameter ($B$). The main findings are as follows:

I. Increasing the values of viscosity variation parameter ($B$) enhances the velocity and velocity slip.

II. The volume flow rate increases with the increase in the values of both viscosity variation parameter ($B$) and Knudsen number ($Kn$).

III. Skin friction increases as viscosity variation parameter ($B$) increases.

Appendix A

Constants used in the present work.

\[ C_1 = \exp(BA_0), \quad C_2 = BA_1, \quad E_1 = C_2 + 2, \quad E_2 = E_1 + 1, \]
\[ B_0 = \frac{F_2F_5-F_2F_6}{F_2-F_3}, \quad B_1 = \frac{F_2-F_3}{F_2}. \]
\[ F_1 = \frac{1}{(1-\eta)^2} \left[ \frac{d_2C_4(\eta)^5}{2} + A_1C_4 \left( \frac{\eta(\eta)^5}{2} E_1 \ln(\eta) - \frac{(\eta)^5}{4} \right) \right], \]
\[ F_2 = \frac{1}{(1-\eta)^2} \left[ \frac{d_2C_4(\eta)^5}{2} + A_1C_4 \left( \frac{\eta(\eta)^5}{2} \ln(\eta) - \frac{(\eta)^5}{4} \right) \right], \]
\[ F_3 = \frac{C_4(\eta)^5}{C_4}, \]
\[ F_4 = C_1(\eta)^{C_1-1}, \quad F_5 = \frac{1}{(1-\eta)^{2}} \left[ \frac{d_2C_4}{2} - A_1C_4 \left( \frac{1}{(1-\eta)^{2}} + \frac{1}{\eta} \right) \right], \]
\[ F_6 = \frac{C_4(\eta)^5}{C_4}, \]
\[ F_7 = F_3 - \beta, Kn(1-\eta)F_4, \quad F_8 = F_1 - \beta, Kn(1-\eta)F_2, \]
\[ F_9 = \left( \frac{C_4}{C_4(\eta)^5} \right) + \beta, Kn(1-\eta)C_1, \]
\[ F_{10} = \beta, Kn(1-\eta)F_6 + F_3. \]
\[ I_1 = \frac{1-(\eta)^5}{\eta}, \quad I_2 = \frac{1-(\eta)^5}{\eta}, \quad I_3 = -\frac{(\eta)^5}{\eta} \ln(\eta) - \frac{1}{\eta} [1 - (\eta)^5], \]
\[ I_4 = \frac{1}{\eta} [1 - (\eta)^2], \]

References


