Enumeration of difference graphs

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Abstract

A difference graph is a bipartite graph \( G = (X, Y; E) \) such that all the neighborhoods of the vertices of \( X \) are comparable by inclusion. We enumerate labeled and unlabeled difference graphs with or without a bipartition of the vertices into two stable sets. The labeled enumerations are expressed in terms of combinatorial numbers related to the Stirling numbers of the second kind.

1. Introduction

We consider only finite, simple undirected graphs. The neighborhood of a vertex \( v \) of a graph \( G = (V, E) \) is the set \( \{ u \in V : uv \in E \} \). The degree of a vertex is the cardinality of its neighborhood. The degree partition of a graph is the partition of the vertices whose blocks are the sets of vertices with equal degrees. We consider the blocks as ordered by the degree, and include the block of vertices of degree zero even if this block is empty (the other blocks are non-empty by definition). Thus if \( \delta_1 < \cdots < \delta_m \) are the distinct positive degrees and \( \delta_0 = 0 \), the degree partition is \( (D_0, D_1, \ldots, D_m) \), where \( D_i \) is the set of all vertices of degree \( \delta_i \). Analogously the degree partition of a bipartite graph \( (X, Y; E) \) is defined as \( (X_0, \ldots, X_k; Y_0, \ldots, Y_t) \), where each \( X_i \) is the set of vertices of \( X \) having a given degree, and similarly for each \( Y_j \).

Again, only \( X_0 \) and \( Y_0 \) may be empty. A graph \( G = (V, E) \) is called a difference graph (and also a chain graph by some authors) when there exist vertex-weights \( a_v, v \in V \) and a positive number \( T \) such that \( |a_u - a_v| < T \) for all \( u, v \in V \) and for any \( u, v \in V, uv \in E \) if and only if \( |a_u - a_v| \geq T \). A difference graph must be bipartite with the bipartition \( (X, Y) \),

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Fig. 1. The structure of a difference graph. Only $X_0$ and $Y_0$ may be empty.

where $X = \{v: a_v \geq 0\}$ and $Y = \{v: a_v < 0\}$. Difference graphs play a prominent role in the proof by Yannakakis [10] of the NP-completeness of recognizing a partial order dimension $d$ for fixed $d \geq 3$, and in the recent polynomial-time recognition algorithm of threshold dimension 2 by Ma [7]. Various characterizations of difference graphs are given in [5]. In particular, their structure is given in the following result.

**Theorem 1** (Hammer et al. [5]). The following conditions are equivalent for a bipartite graph $G = (X, Y; E)$:

(i) $G$ is a difference graph;
(ii) all the neighborhoods of the vertices of $X$ are comparable by inclusion;
(iii) if $(X_0, \ldots, X_k; Y_0, \ldots, Y_l)$ is the degree partition of $G$, then $k = l$ and for each $x \in X_i$ and $y \in Y_j$ we have $xy \in E$ if and only if $i + j > k$. See Fig. 1.

Closely related to difference graphs are the threshold graphs. A graph $G = (V, E)$ is called a threshold graph when there exist vertex weights $a_v, v \in V$ and a number $T$ such that for any distinct vertices $u, v$ we have $uv \in E$ if and only if $a_u + a_v > T$. Many characterizations of threshold graphs are known. Among them are the following (see for example [3]):

**Theorem 2.** The following conditions are equivalent for a graph $G = (V, E)$:

(i) $G$ is a threshold graph;
(ii) $G$ can be constructed from the empty graph by repeatedly adding an isolated vertex or a vertex adjacent to all previous ones;
(iii) if $(D_0, \ldots, D_m)$ is the degree partition of $G$, then a vertex of $D_i$ is adjacent to a vertex of $D_j$ if and only if $i + j > m$.

The connection of difference and threshold graphs is expressed in the following theorem.

**Theorem 3** (Hammer et al. [5]). A graph $G = (V, E)$ is threshold if and only if there exist a partition $(X, Y)$ of $V$ and a difference graph $(X, Y; E')$ such that $E = E' \cup E_X$, where $E_X$ is the edge-set of the complete graph on $X$. 
Unlabeled threshold graphs have been enumerated by Peled [8]. He let $t_{nmk}$ denote the number of unlabeled threshold graphs with $n$ vertices, $m$ edges and largest clique (complete subgraph) of cardinality $k$, and found two expressions for the generating function $T_n(x, q) = \sum_{m,k} t_{nmk}q^mx^k$, namely

$$T_n(x, q) = x(1 + qx)(1 + q^2x)\cdots(1 + q^{n-1}x) = \sum_{j=1}^{n} \left[ \frac{n-1}{j-1} \right] q^{(j)} x^j,$$

where the square brackets denote the Gaussian binomial coefficient. As a consequence, the $q$-binomial theorem can be derived. An easy consequence of this or of Theorem 2 is the following result.

**Theorem 4.** The number of unlabeled threshold graphs on $n$ vertices is $2^{n-1}$.

Labeled threshold graphs have been enumerated by Beissinger and Pried [1]. We express their results in terms of the well-known numbers $g_n, n \geq 0$ defined by

$$\sum_{n=0}^{\infty} \frac{g_n}{n!} x^n = \frac{1}{2 - e^x}.$$

These numbers were studied by Cayley in the 19th century and several references to them can be found in [4, 6]. $g_n$ is the number of ordered partitions of an $n$-element set, that is to say, the number of ways to partition the set into non-empty blocks and to arrange the blocks in a sequence. Equivalently, $g_n$ is the number of $n$-letter words over the alphabet $\{1, 2, \ldots\}$ which for some $k$ contain precisely the letters $\{1, 2, \ldots, k\}$ with repetitions. To see this interpretation of $g_n$, we only have to write

$$\frac{1}{2 - e^x} = \frac{1}{1 - (e^x - 1)} = \sum_{k=0}^{\infty} (e^x - 1)^k$$

and note that $(e^x - 1)^k$ is the exponential generating function (egf) for the number of words over the alphabet $\{1, 2, \ldots, k\}$ with each letter appearing at least once. Because of this interpretation of $g_n$, we have

$$g_n = \sum_{k=0}^{n} k! \left\{ \begin{array}{c} n \\ k \end{array} \right\},$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ denotes the Stirling number of the second kind, the number of partitions of an $n$-element set into $k$ unordered non-empty blocks. This agrees with the well-known expansion (see [2] or [9])

$$(e^x - 1)^k = \sum_{k=0}^{\infty} \frac{x^n}{k!} \left\{ \begin{array}{c} n \\ k \end{array} \right\}.$$  

Knuth [6, Ex. 5.3.1.4] presents the following asymptotic for $g_n$:

$$\frac{g_n}{n!} = \frac{1}{2(ln 2)^{n+1}} + \sum_{k=1}^{\infty} \Re \left( \frac{1}{(ln 2 + 2\pi ik)^{n+1}} \right), \quad n \geq 1.$$
It is not hard to use this formula to obtain that

\[
\frac{g_n}{n!} = \frac{1}{2(\ln 2)^{n+1}} + O\left(\frac{1}{(2n)^n}\right), \quad n \to \infty.
\]

Beissinger and Peled [1] showed that the number \( t_n \) of labeled threshold graphs on \( n \) vertices satisfies

\[
t_n = 2(g_n - ng_{n-1}), \quad n \geq 2.
\]

They used this result to derive Frobenius's identity

\[
A_n(u) = u \sum_{k \geq 0} k! \binom{n}{k} (u-1)^{n-k}, \quad n \geq 1,
\]

where \( A_n(u) \) is the Eulerian polynomial, which can be defined (see [2] or [9]) by

\[
A_0(u) = 1 \quad \text{and} \quad 1 + \frac{1}{u} \sum_{n \geq 1} A_n(u) \frac{x^n}{n!} = \frac{u - 1}{u - e^{x(u-1)}}.
\]

Note that \( 2g_n = A_n(2) \) for \( n \geq 1 \).

Below we find the number of labeled and unlabeled difference graphs on \( n \) vertices. We do the same for bipartitioned difference graphs, that is to say difference graphs (which are bipartite graphs) together with a partition of their vertices into two stable sets.

### 2. Labeled difference graphs

**Theorem 5.** Let \( d_n \) be the number of labeled bipartitioned difference graphs on \( n \) vertices. Then

\[
d_0 = 1, \quad d_n = 2g_n, \quad n \geq 1.
\]

**Proof.** By Theorem 1, \( d_n \) is equal to the number of ordered partitions of \( \{1, \ldots, n\} \) of the form \((X_0, X_1, \ldots, X_k, Y_k, \ldots, Y_1, Y_0)\), where only the blocks \( X_0 \) and \( Y_0 \) may be empty. We define a mapping \( f \) from the set of all such partitions \( P \) into the set of ordered partitions of \( \{1, \ldots, n\} \) with non-empty blocks. We obtain \( f(P) \) by removing the empty blocks of \( P \), if any. Then for \( n \geq 1 \), \( f \) is precisely two-to-one. To see this, let \( Q = (B_1, \ldots, B_k) \) be a partition with non-empty blocks. If \( k \) is even, then

\[
f^{-1}(Q) = \{Q, (\emptyset, B_1, \ldots, B_k, \emptyset)\}
\]

and if \( k \) is odd, then

\[
f^{-1}(Q) = \{(\emptyset, B_1, \ldots, B_k), (B_1, \ldots, B_k, \emptyset)\}.
\]
The same result can also be obtained by writing the egf for the $d_n$. The egf for the number of ordered partitions of \{1, \ldots, n\} into $2k + 2$ ordered blocks, where only the first and last blocks may be empty, is $e^x(e^x - 1)^{2k}$. Therefore the egf for the $d_n$ is

$$
\sum_{k \geq 0} e^{2x}(e^x - 1)^{2k} = \frac{e^{2x}}{1 - (e^x - 1)^2} = \frac{e^x}{2 - e^x} = \frac{2}{2 - e^x} - 1,
$$

so $d_0 = 2g_0 - 1 = 1$ and $d_n = 2g_n$ for $n \geq 1$. 

One can introduce other parameters into the generating function. For example, suppose that we have a set of $p + q$ elements and want to partition the first $p$ elements into $k + 1$ ordered blocks where only the first block may be empty, and the last $q$ elements into $k + 1$ ordered blocks where only the last block may be empty. The number of possible ways to do this is the coefficient of $x^p y^q/(p!q!)$ in $e^x(e^x - 1)^k(e^y - 1)^k e^x e^y$. The coefficient of $x^p y^q/(p + q)!$, namely $(p^+ q^+)$ times the previous coefficient, is the number of ways to partition $p + q$ distinct elements into $2k + 2$ ordered blocks, where only the first and last blocks may be empty. By summing over all $k \geq 0$ we find that

$$
\sum_{p, q \geq 0} d_{pq} \frac{x^p y^q}{(p + q)!} = \frac{e^{x+y}}{1 - (e^x - 1)(e^y - 1)},
$$

where $d_{pq}$ is the number of labeled bipartitioned difference graphs with color classes of cardinality $p$ and $q$.

By manipulating the egf for the $d_n$ we can derive identities involving Stirling numbers of the second kind. For example, using the fact that

$$
e^x(e^x - 1)^k = \frac{d}{dx} (e^x - 1)^{k+1} = \frac{1}{k+1} \frac{d}{dx} \sum_{r \geq 0} \frac{x^r}{r!} (k+1)! \left\{ \begin{array}{c} r \\ k+1 \end{array} \right\}
$$

we find

$$
1 + \sum_{s \geq 1} \frac{x^s}{s!} 2 \sum_{j \geq 0} j! \left\{ \begin{array}{c} s \\ j \end{array} \right\} = \sum_{k \geq 0} \left( \sum_{r \geq 0} \frac{x^r}{r!} k! \left\{ \begin{array}{c} r+1 \\ k+1 \end{array} \right\} \right)^2 = \sum_{k \geq 0} \left( \sum_{r \geq 0} \frac{x^r}{r!} k! \left\{ \begin{array}{c} r+1 \\ k+1 \end{array} \right\} \right)^2,
$$

and therefore

$$
2 \sum_{j \geq 0} j! \left\{ \begin{array}{c} s \\ j \end{array} \right\} = \sum_{r \geq 0} \left( \begin{array}{c} s \\ r \end{array} \right) \sum_{k \geq 0} k! \sum_{r \geq 0} \frac{x^r}{r!} k! \left\{ \begin{array}{c} r+1 \\ k+1 \end{array} \right\} \left\{ \begin{array}{c} s-r+1 \\ k+1 \end{array} \right\}, \quad s \geq 1.
$$

**Theorem 6.** The number of labeled difference graphs on $n$ vertices is $\frac{1}{2}(1 + g_n)$.

**Proof.** By Theorem 1 any difference graph consists of zero or more isolated vertices and a connected difference graph, and conversely. As in Theorem 5, the egf for the
number of labeled non-empty bipartitioned connected difference graphs is 
\( \sum_{k \geq 1} (e^x - 1)^{2k} \). Hence the egf for the number of labeled non-empty connected difference graphs is \( \frac{1}{2} \sum_{k \geq 1} (e^x - 1)^{2k} \), where the factor \( \frac{1}{2} \) comes from exchanging the two color classes. By adding 1 we also include the empty graph, and by then multiplying by \( e^x \) we allow for the isolated vertices. Thus the egf for the number of labeled difference graphs is given by

\[
\frac{1}{2} \left( e^x + \frac{1}{2 - e^x} \right)
\]

and the number of labeled difference graphs on \( n \) vertices is the coefficient of \( x^n/n! \), namely \( \frac{1}{2} (1 + g_n) \).

3. Unlabeled difference graphs

We denote by \( \Theta_n \) the set of unlabeled threshold graphs on \( n \) vertices, and by \( A_n \) the set of unlabeled bipartitioned difference graphs on \( n \) vertices. By Theorem 4 we have \( |\Theta_n| = 2^{n-1} \). We shall use the following connection between \( |A_n| \) and \( |\Theta_n| \).

Lemma 7. \( |A_n| = |\Theta_n| + |A_{n-1}| \), \( n \geq 1 \).

Proof. Consider the mapping \( f: A_n \rightarrow \Theta_n \) given by \( f(D) = T \), where \( D = (X, Y; E) \in A_n \) is a bipartitioned difference graph with bipartition \( (X, Y) \), \( T = (X \cup Y, E \cup E_X) \), and \( E_X \) is the edge-set of the complete graph on \( X \). By Theorem 3 \( f \) is onto \( \Theta_n \), but it is not one-to-one. To find \( f^{-1}(f(D)) \), we note that \( T \) and \( D \) are determined by their degree partitions according to Theorem 2 and Theorem 1. In \( T \), the degree of every vertex of \( Y_i \) is \( |X_k \cup \cdots \cup X_{k-i+1}| \) and the degree of every vertex of \( X_i \) is \( |Y_k \cup \cdots \cup Y_{k-i+1}| + |X_0 \cup \cdots \cup X_k| - 1 \). Since \( X_i \) and \( Y_i \) are non-empty for \( i \geq 1 \), it follows that the degree partition of \( T \) is simply \( (Y_0, \ldots, Y_k, X_0, X_1, \ldots, X_k) \) unless one of the following conditions holds in \( D \):

(i) \( X_0 = \emptyset, |Y_k| = 1 \), in which case the degree partition of \( T \) is

\( (Y_0, \ldots, Y_k, X_k) \);

(ii) \( |X_0| = 1 \), in which case the degree partition of \( T \) is

\( (Y_0, \ldots, Y_k, X_k, X_0) \).

It follows that \( f^{-1}(f(D)) = \{D\} \) unless Condition (i) or (ii) holds for \( D \), in which case \( f^{-1}(f(D)) = \{D, D^*\} \), where \( D^* \) is obtained from \( D \) by moving the singleton \( Y_k \) to
X in Case (i) and the singleton \(X_0\) to \(Y\) in Case (ii). Note that if Case (i) holds for \(D\), Case (ii) holds for \(D^*\) and conversely. It follows that \(|\Theta_n| = |A_n| - |A'_n|\), where \(A'_n\) denotes the set of unlabeled bipartitioned difference graphs satisfying Condition (i) or (ii). We complete the proof by indicating a bijection from \(A'_n\) to \(A_{n-1}\). It is given by deleting the singleton \(Y_k\) in Case (i) and the singleton \(X_0\) in Case (ii). The resulting bipartitioned difference graph \(D'\) has isolated vertices in \(X\) in Case (i) and does not have them in Case (ii). Therefore we can uniquely reconstruct \(D\) from \(D'\) by adding a singleton \(Y_k\) to \(Y\) if \(D'\) has isolated vertices in \(X\) and adding a singleton \(X_0\) to \(X\) if \(D'\) has none. This shows that indeed we have a bijection.

**Theorem 8.** The number of unlabeled bipartitioned difference graphs on \(n\) vertices is \(2^n\).

**Proof.** By Lemma 7 and the fact that \(|\Theta_n| = 2^{n-1}\), we have

\[|A_n| = 2^{n-1} + 2^{n-2} + \cdots + 2^0 + |A_0| = 2^n - 1 + 1 = 2^n.\]

Another way to see this result is to modify the bijection between \(\Theta_n\) and the set of binary words of length \(n-1\) into a bijection between \(A_n\) and the set of binary words of length \(n\). We omit the details.

**Theorem 9.** The number of unlabeled difference graphs on \(n \geq 1\) vertices is \(2^{n-2} + 2^{[n/2]} - 1\).

**Proof.** Again, according to Theorem 1, a difference graph consists of zero or more isolated vertices and a connected difference graph. Hence the required number is \(\sum_{s=0}^n c_s\), where \(c_s\) is the number of unlabeled difference graphs with no isolated vertices on \(s\) vertices. According to Theorem 1, a bipartitioned difference graph with no isolated vertices is determined up to isomorphism by the cardinalities of the blocks \(X_1, \ldots, X_k\) and \(Y_1, \ldots, Y_k\), so the number of unlabeled bipartitioned difference graphs with no isolated vertices on \(s\) vertices is equal to the number of compositions of \(s\) into an even number of parts, i.e., sequences \((x_1, \ldots, x_k, y_1, \ldots, y_k)\) of positive integers whose sum is \(s\). For \(s \geq 2\), there are \(\binom{s-1}{k-1}\) compositions of \(s\) into \(2k\) parts, and therefore a total of \(\sum_{k=0}^s \binom{s-1}{k-1} = 2^{s-1}\) compositions of \(s\) into an even number of parts.

Since we are counting non-bipartitioned difference graphs with no isolated vertices, \(c_s\) equals the number of inequivalent compositions of \(s\) into an even number of parts, where the composition \((x_1, \ldots, x_k, y_1, \ldots, y_k)\) is considered equivalent to itself and to \((y_1, \ldots, x_k, x_1, \ldots, x_k)\). We find \(c_s\) by Burnside's Lemma. All \(2^{s-2}\) compositions into an even number of parts are invariant under the identity permutation. A composition into an even number of parts is invariant under the permutation \(\pi\) that switches color classes if and only if it has the form \((x_1, \ldots, x_k, x_1, \ldots, x_k)\), and \((x_1, \ldots, x_k)\) must then be a composition of \(s/2\). Hence there are \(\binom{s-1}{k-1}\) \((s\ even)\) compositions of \(s\) into \(2k\) parts that are invariant under \(\pi\), where the Iversonian \((s\ even)\) stands for \(1\ if\ s\ is\ even\ and\ for\ 0\ otherwise. By summing over all \(k \geq 1\), we find that there are \(2^{s-2} - 1\) \((s\ even)\) compositions of \(s\) into an even number of parts that are invariant under \(\pi\). Hence by
Burnside's lemma $c_s = \frac{1}{2}(2^{s-2} + 2^{s/2-1} \text{ (s even)})$ for $s \geq 2$. Clearly $c_0 = 1$ and $c_1 = 0$. Therefore the number of unlabeled difference graphs on $s \geq 2$ vertices is given by

$$
\sum_{s=0}^{n} c_s = 1 + 0 + \sum_{s=2}^{n} (2^{s-3} + 2^{s/2-2} \text{ (s even)})
$$

\begin{align*}
&= 2 + \sum_{s=3}^{n} (2^{s-3} + 2^{s/2-2} \text{ (s even)}) \\
&= 2 + 2^{n-2} - 1 + (1 + 2 + 4 + \cdots + 2^{\lfloor n/2 \rfloor - 2}) \\
&= 1 + 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1} - 1 = 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}.
\end{align*}

This value is also correct for $n = 1$. \[\square\]

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References


