Nash equilibrium and minimax theorem with $C$-concavity

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Received 21 January 2006
Available online 13 July 2006
Submitted by William F. Ames

Abstract

The purpose of this paper is to introduce a generalized $C$-concave condition, and by using Himmelberg’s fixed point theorem, to prove a new existence theorem of Nash equilibrium in non-compact generalized game with $C$-concavity. As applications, we shall prove a minimax theorem in non-compact settings and prove a minimax inequality in compact settings.

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Keywords: Nash equilibrium; $C$-concave; Minimax theorem; Minimax inequality

1. Introduction

In 1951, Nash established the well-known equilibrium existence theorem for $N$-person games. Since then, the classical results of Nash [18], Debreu [2,3] and Nikaido and Isoda [19] have served as basic references for the existence of Nash equilibrium for non-cooperative games. Next, in 1977, Friedman [9] established a generalization of the Nash theorem using the quasi-concavity assumption on every payoff function. In all of them, convexity of strategy spaces, continuity and concavity/quasi-concavity of the payoff functions were assumed. Till now there have been a number of generalizations, and also many applications of those theorems have been found in several areas, e.g., see [1,9] and references therein.

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0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.06.038
Two important concepts for removing the concavity/quasi-concavity assumptions of the pay-off functions are marked by the seminal papers of Fan [5,6] for 2-person zero-sum games, and the complete abandonment of concavity in Nishimura and Friedman [20]. In fact, the concept of concavelike payoffs due to Fan [6] does not require any linear structure on the strategy space. However, Joó [13] gave a general sum 2-person game where the payoff functions are continuous and concavelike, but the game has no Nash equilibrium. Horváth and Joó [11] also show that higher smoothness of the payoff functions does not change the situation. In [8], Forgó introduced the CF-concavity by adding continuity to Fan’s concavelike condition, and prove the existence theorem of Nash equilibrium and its applications using the CF-concavity without assuming the linear structure, and next, they proved an equilibrium existence theorem for a compact generalized N-person game.

In a recent paper [16], the authors introduced the C-concavity which generalizes both concave condition and CF-concavity without assuming the linear structure, and next, they proved an existence theorem of Nash equilibrium and its applications using the C-concavity. And, more recently, Kim and Kum [15] further generalize the C-convexity using constraint correspondences, and they prove an equilibrium existence theorem for a compact generalized N-person game.

In this paper, we will introduce a C-concave condition which generalizes both concave condition and CF-concavity without assuming the linear structure. Using this C-concavity and the partition of unity argument, we shall prove a new existence theorem of Nash equilibrium for non-compact generalized games. And we shall give a new minimax theorem and a minimax inequality as its applications. Those results generalize the existence theorems in [4,8,15,16,18,19] to non-compact generalized games with C-concavity. Finally we shall give an example of a game where C-concavity can be applied; but the concavity/quasi-concavity in [9,11–14,17,20] cannot be applied.

2. Preliminaries

We begin with some notations and definitions. Let $A$ be a subset of a topological space $X$. We shall denote by $2^A$ the family of all subsets of $A$. Let $I$ be a countable index set. For each $i \in I$, let $X_i$ be a non-empty topological space and denote $X := \prod_{i \in I} X_i$ and $X_i := \prod_{i \in I \setminus \{i\}} X_j$. If $x = (x_1, \ldots, x_n, \ldots) \in X$, we shall write $x_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \ldots) \in X_i$. If $x_i \in X_i$ and $x_j \in X_j$, we shall use the notation $(x_i, x_j) := (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n, \ldots) = x \in X$. Denote by $[0,1]^n$ the Cartesian product of $n$ unit intervals $[0,1] \times \cdots \times [0,1]$; and denote the unit simplex in $[0,1]^n$ by $\Delta_n$, i.e.,

$$\Delta_n := \left\{ (\lambda_1, \ldots, \lambda_n) \in [0,1]^n \left| \sum_{i=1}^n \lambda_i = 1 \right. \right\}.$$

Throughout this paper, all topological spaces are assumed to be Hausdorff.

Let $I = \{1, \ldots, n, \ldots\}$ be a countable set of players. A non-cooperative generalized game $\Gamma$ of normal form is an ordered tuple $(X_1, \ldots, X_n; f_1, \ldots, f_n, \ldots)$ where for each player $i \in I$, the non-empty set $X_i$ is the player’s pure strategy space, and $f_i : X = \prod_{i \in I} X_i \to \mathbb{R}$ is the player’s payoff function. The set $X$, joint strategy space, is the Cartesian product of the individual strategy sets, and an element of $X_i$ is called a strategy of the $i$th player. A strategy $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n, \ldots) \in X$ is called a Nash equilibrium for the game $\Gamma$ if the following system of inequalities holds: for each $i \in I$,

$$f_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \ldots, \bar{x}_n, \ldots) \geq f_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n, \ldots)$$

for all $x_i \in X_i$. When $I$ is an uncountable set of players, we can similarly define the non-cooperative game $\Gamma$ of normal form, and in this case, we also call $\Gamma$ the non-cooperative
generalized game. Here we remark that the model of a game in this paper is a non-cooperative game, i.e., there is no replay communicating between players, and so players act as free agents, and each player is trying to maximize his/her own payoff according to his/her strategy.

Now we recall some concepts which generalize the concavity. When $X$ and $Y$ are non-empty arbitrary sets, recall that $f : X \times Y \to \mathbb{R}$ is concavelike on $X$ with respect to $Y$ [6] if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, there exists $x_0 \in X$ such that

$$f(x_0, y) \geq \lambda f(x_1, y) + (1 - \lambda) f(x_2, y)$$

for all $y \in Y$.

Adding the continuity to concavelike functions, Forgó [8] introduced the CF-concavity as follows: Let $X$ be a non-empty topological space, $Y$ a non-empty arbitrary set. Then $f : X \times Y \to \mathbb{R}$ is said to be CF-concave on $X$ with respect to $Y$ if there exists a continuous function $\Psi : X \times X \times \mathbb{R} \to X$ such that for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $f(\Psi(x_1, x_2, \lambda), y) \geq \lambda f(x_1, y) + (1 - \lambda) f(x_2, y)$ for all $y \in Y$.

Also note that by using the induction, we can obtain the equivalent formulations to the concavelike and CF-concave conditions in general forms, respectively, e.g., see [16, Lemma 1] and [8, Lemma 1].

Next, we will introduce a concave condition which generalizes both CF-concavity and concavity as follows:

**Definition.** Let $X$ be a topological space, $Y$ an arbitrary set and $D$ be a non-empty subset of $X$. Then $f : X \times Y \to \mathbb{R}$ is called $C$-concave on $D$ if for every $n \geq 2$, whenever $n$ points $x_1, \ldots, x_n \in X$ are arbitrarily given, there exists a continuous function $\phi_n : \Delta_n \to D$ such that

$$f(\phi_n(\lambda_1, \ldots, \lambda_n), y) \geq \lambda_1 f(x_1, y) + \cdots + \lambda_n f(x_n, y)$$

for all $(\lambda_1, \ldots, \lambda_n) \in \Delta_n$ and for all $y \in Y$.

**Remarks.**

(a) When $X = D$ in Definition, the $C$-concavity is actually the same as the definition in [16]. In this case, the concavity clearly implies the $C$-concavity by letting $\phi_n(\lambda_1, \ldots, \lambda_n) := \lambda_1 x_1 + \cdots + \lambda_n x_n$ for each $\lambda_1, \ldots, \lambda_n \in \Delta_n$, whenever $x_1, \ldots, x_n \in X$ are given.

(b) Note that the continuous function $\phi_n$ need not be defined globally on $X \times \cdots \times X \times \mathbb{R}^n$ as in [8], but defined only on $\Delta_n$ in Definition. In fact, for any given $n$ points $x_1, \ldots, x_n \in X$, by defining

$$\phi_n(\lambda_1, \ldots, \lambda_n) := \Psi_n(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n)$$

for each $(\lambda_1, \ldots, \lambda_n) \in \Delta_n$, we can see that the CF-concavity implies the $C$-concavity.

(c) If $f$ is $C$-concave on $X$, then for any given points $x_1, x_2 \in X$ and for each $\lambda \in [0, 1]$, by defining $x_0 := \phi_2(\lambda, 1 - \lambda)$, we can see that $f$ is concavelike on $X$. Therefore, the following implication diagram holds:

$$\text{concave} \implies \text{CF-concave} \implies \text{$C$-concave} \implies \text{concavelike}.$$
Lemma 1. [10] Let $X$ be a convex subset of a locally convex Hausdorff topological vector space, $D$ a non-empty compact subset of $X$, and let $f : X \rightarrow D$ be a continuous mapping. Then there exists a point $\bar{x} \in D$ such that $f(\bar{x}) = \bar{x}$.

3. New existence theorem of Nash equilibrium

Let $\Gamma$ be a non-cooperative generalized game where $I$ is a countable (possibly uncountable) set of players and $X_i$ is the player’s pure strategy space. And let the strategy space $X := \prod_{i \in I} X_i$ be a non-empty subset of a locally convex Hausdorff topological vector space.

Now let us define the total sum of payoff functions $H : X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ associated with the non-cooperative generalized game $\Gamma$ as follows:

$$H(x, y) := \sum_{i \in I} f_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$$

for every $x = (x_1, \ldots, x_n, \ldots), y = (y_1, \ldots, y_n, \ldots) \in X = \prod_{i \in I} X_i$.

Then we shall need the following which is a general form of Lemma 3.1 in [19]:

Lemma 2. Let $\Gamma$ be a non-cooperative generalized game where $I$ is a countable (possibly uncountable) set of players. If there exists a point $\bar{x} \in X$ for which

$$H(\bar{x}, \bar{x}) \geq H(x, \bar{x})$$

for any $x \in X$, then $\bar{x}$ is a Nash equilibrium for $\Gamma$.

Proof. For each $i \in I$, we take any $x = (\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots) \in X$. Then, by substitution, we can see that

$$H(\bar{x}, \bar{x}) = \sum_{j \in I \setminus \{i\}} f_j(\bar{x}_1, \ldots, \bar{x}_i, \ldots) + f_i(\bar{x}_i, \bar{x}_i)$$

$$\geq H(x, \bar{x}) = \sum_{j \in I \setminus \{i\}} f_j(\bar{x}_1, \ldots, \bar{x}_i, \ldots) + f_i(x_i, \bar{x}_i)$$

for all $x_i \in X_i$. Therefore, we have

$$f_i(\bar{x}_i, \bar{x}_i) \geq f_i(x_i, \bar{x}_i)$$

for all $x_i \in X_i$;

hence $\bar{x}$ is a Nash equilibrium for $\Gamma$. \qed

Using the partition of unity argument, we now prove the following existence theorem of Nash equilibrium in non-compact generalized games:

Theorem 1. Let $I$ be a countable (possibly uncountable) set of index set, and let $\Gamma$ be a non-cooperative generalized game satisfying the following conditions:

(i) the strategy space $X := \prod_{i \in I} X_i$ is a paracompact convex subset of a locally convex Hausdorff topological vector space and $D$ be a non-empty compact subset of $X$;

(ii) the function $(x, y) \mapsto H(x, y)$ is continuous on $X \times X$;

(iii) the function $x \mapsto H(x, y)$ is $C$-concave on $D$;

(iv) for each $x \in D$, $H(x, x) \geq H(y, x)$ for all $y \in X \setminus D$. 

Then \( \Gamma \) has a Nash equilibrium \( \bar{x} \in D \), i.e., for each \( i \in I \),
\[
f_i(\bar{x}_i, \bar{x}_j) \geq f_i(x_i, x_j) \quad \text{for all} \quad x_i \in X_i.
\]

**Proof.** Suppose the contrary, i.e., assume that \( \Gamma \) has no Nash equilibrium. Then, by Lemma 2, for all \( x \in X \), there exists an \( y \in X \) such that \( H(x, x) < H(y, x) \).

For any \( z \in X \), we let
\[
U(z) := \{ x \in X \mid H(x, x) < H(z, x) \}.
\]

Then, since \( H \) is continuous, each \( U(z) \) is open (possibly empty) in \( X \); and also we have \( \bigcup_{z \in X} U(z) = X \). Here, without loss of generality, we may assume that \( X \setminus D \) is non-empty. By the assumption (iv), for each \( z \in X \setminus D \), we have that \( U(z) \subset X \setminus D \). Since
\[
X = \bigcup_{z \in X} U(z) = \left( \bigcup_{z \in D} U(z) \right) \cup \left( \bigcup_{z \in X \setminus D} U(z) \right),
\]
we obtain that \( D \subset \bigcup_{z \in D} U(z) \). Since \( D \) is compact and each \( U(z) \) is open, there exists a finite number of non-empty open sets \( U(z_1), \ldots, U(z_n) \) such that \( D \subset \bigcup_{i=1}^n U(z_i) \), where \( \{z_1, \ldots, z_n\} \subset D \). Since \( X \setminus D \) is non-empty, if possible, let \( z_{n+1} \in X \setminus D \) should be chosen satisfying that \( z_{n+1} \notin U(z_i) \) for each \( i \in \{1, \ldots, n\} \). And denote an open set \( U(z_{n+1}) := X \setminus D \). Then \( \{U(z_1), \ldots, U(z_{n+1})\} \) is a finite open covering of \( X \). Since \( X \) is paracompact, there exists a partition of unity \( \{\alpha_1, \ldots, \alpha_{n+1}\} \) subordinate to the open covering \( \{U(z_1), \ldots, U(z_{n+1})\} \), i.e.,
\[
0 \leq \alpha_i(x) \leq 1, \quad \sum_{i=1}^{n+1} \alpha_i(x) = 1 \quad \text{for all} \quad x \in X, \ i = 1, \ldots, n+1;
\]
and if \( x \notin U(z_j) \) for some \( j \), then \( \alpha_j(x) = 0 \).

For such \( \{z_1, \ldots, z_{n+1}\} \subset X \), since \( H \) is \( C \)-concave on \( D \), there exists a continuous mapping \( \phi_{n+1}: \Delta_{n+1} \to D \) satisfying the condition
\[
H(\phi_{n+1}(\lambda_1, \ldots, \lambda_{n+1}), x) \geq \lambda_1 H(z_1, x) + \cdots + \lambda_{n+1} H(z_{n+1}, x)
\]
for all \( (\lambda_1, \ldots, \lambda_{n+1}) \in \Delta_{n+1} \) and for all \( x \in X \).

Next we consider a continuous mapping \( \Psi : X \to D \), defined by
\[
\Psi(z) := \phi_{n+1}(\alpha_1(z), \ldots, \alpha_{n+1}(z)) \quad \text{for all} \quad z \in X.
\]

Since \( \phi_{n+1} \) and each \( \alpha_i \) are continuous, \( \Psi \) is continuous on \( X \). Moreover, \( \Psi \) maps a non-empty convex set \( X \) into a compact subset \( D \) in a locally convex Hausdorff topological vector space. Therefore, by Lemma 1, there exists a fixed point \( \bar{x} \in D \) such that \( \Psi(\bar{x}) = \bar{x} \). Since \( H \) is \( C \)-concave on \( D \), we have
\[
H(\Psi(\bar{x}), x) \geq \alpha_1(\bar{x}) H(z_1, x) + \cdots + \alpha_n(\bar{x}) H(z_n, x) + \alpha_{n+1}(\bar{x}) H(z_{n+1}, x)
\]
for all \( x \in X \); and so by putting \( x := \bar{x} \), we have
\[
H(\bar{x}, \bar{x}) \geq \alpha_1(\bar{x}) H(z_1, \bar{x}) + \cdots + \alpha_n(\bar{x}) H(z_n, \bar{x}) + \alpha_{n+1}(\bar{x}) H(z_{n+1}, \bar{x}).
\]

However, if \( \bar{x} \in U(z_j) \) for some \( 1 \leq j \leq n \), then \( H(\bar{x}, \bar{x}) < H(z_j, \bar{x}) \) and \( \alpha_j(\bar{x}) > 0 \); and if \( \bar{x} \notin U(z_k) \) for some \( 1 \leq k \leq n \), \( \alpha_k(\bar{x}) = 0 \). Also note that since \( \bar{x} \in D \), \( \bar{x} \notin X \setminus D = U(z_{n+1}) \); and so \( \alpha_{n+1}(\bar{x}) = 0 \). Therefore, we have
\[\sum_{i=1}^{n+1} \alpha_i(\bar{x}) H(z_i, \bar{x}) > \sum_{i=1}^{n+1} \alpha_i(\bar{x}) H(\bar{x}, \bar{x}) = H(\bar{x}, \bar{x}),\]

which contradicts (3). This completes the proof. \[\Box\]

Remarks.

(1) Theorem 1 generalizes the equilibrium existence theorems due to Nash [18] and Forgó [8] in the following aspects:
   (i) for each \(i \in I\), the strategy set \(X_i\) need not be compact; but the product space \(X = \prod_{i \in I} X_i\) must be a paracompact convex subset of a locally convex Hausdorff topological vector space;
   (ii) for each \(i \in I\), every payoff function \(f_i\) need not be concave nor continuous, and \(H\) need not be CF-concave;
   (iii) the set \(I\) of players need not be finite.

(2) Theorem 1 can be further generalized by using the constraint correspondences \(T_i\) as in Definition 1 in [15]. Also it should be noted that in our Theorem 1, the set of players \(I\) is a countable (possibly uncountable) set; however, in Theorem 1 in [15], the set of players \(I\) is a finite set.

When the strategy space \(X = D\) is compact in Theorem 1, the total sum of payoff functions \(H(x, y)\) must be bounded on \(X \times X\). In this case, the coercive condition (iv) is automatically satisfied, and so we have the following:

**Theorem 2.** Let \(I\) be a countable (possibly uncountable) set of players, and let \(\Gamma\) be a non-cooperative generalized game satisfying the following:

(i) the strategy space \(X := \prod_{i \in I} X_i\) is non-empty compact convex subset of locally convex Hausdorff topological vector space;
(ii) the function \((x, y) \mapsto H(x, y)\) is continuous on \(X \times X\);
(iii) the function \(x \mapsto H(x, y)\) is \(C\)-concave on \(X\).

Then \(\Gamma\) has at least one Nash equilibrium.

4. Some applications

As an application of Theorem 1, we shall prove the following minimax theorem in non-compact settings:

**Theorem 3.** Let \(X, Y, D, E\) be non-empty sets such that \(X \times Y\) is a paracompact convex in a locally convex Hausdorff topological vector space, \(D\) a non-empty compact subset of \(X\), and \(E\) a non-empty compact subset of \(Y\). Assume that

(a) the function \(f : X \times Y \to \mathbb{R}\) is continuous on \(X \times Y\);
(b) for each \(y \in Y\), the function \(x \mapsto -f(x, y)\) is \(C\)-concave on \(D\);
(c) for each \(x \in X\), the function \(y \mapsto f(x, y)\) is \(C\)-concave on \(E\);
(d) for each \((x, y) \in D \times E\), \(f(x, v) - f(u, y) \leq 0\) for all \((u, v) \in X \times Y \setminus D \times E\).
Then we have

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y).$$

**Proof.** Let \( f_1(x, y) := -f(x, y) \) and \( f_2(x, y) := f(x, y) \). In order to apply Theorem 1, we first note that the mapping \( H : (X \times Y) \times (X \times Y) \to \mathbb{R} \) is given by

$$H((x_1, y_1), (x_2, y_2)) := f_1(x_1, y_2) + f_2(x_2, y_1) \quad \text{for each} \ (x_1, y_1), (x_2, y_2) \in X \times Y.$$ 

Then \( H \) is clearly continuous, so it suffices to show that the assumptions (iii) and (iv) of Theorem 1 are satisfied. Let two points \((x_1, y_1), (x_2, y_2) \in X \times Y\) be given arbitrarily. Then for \([x_1, x_2]\), by the assumption (b), there exists a continuous function \( \Phi_1 : \Delta_2 \to D \) such that

$$f_1(\Phi_1(\lambda, 1 - \lambda), (x, v)) \geq \lambda f_1(x_1, v) + (1 - \lambda) f_1(x_2, v)$$

for every \( \lambda \in [0, 1] \) and every \( v \in Y \). Also, for \([y_1, y_2]\), by the assumption (c), there exists a continuous function \( \Phi_2 : \Delta_2 \to E \) such that

$$f_2(u, \Phi_2(\lambda, 1 - \lambda)) \geq \lambda f_2(u, y_1) + (1 - \lambda) f_2(u, y_2)$$

for every \( \lambda \in [0, 1] \) and every \( u \in X \).

Now we define a continuous function \( \Phi_2 : \Delta_2 \to D \times E \) by

$$\Phi_2(\lambda, 1 - \lambda) := (\Phi_1(\lambda, 1 - \lambda), \Phi_2(\lambda, 1 - \lambda)) \quad \text{for every} \ \lambda \in [0, 1].$$

Then it is easy to see that \( \Phi_2 \) is a continuous function on \( \Delta_2 \). Also, for every \( \lambda \in [0, 1] \), we have

$$\begin{align*}
\lambda H((x_1, y_1), (u, v)) + (1 - \lambda) H((x_2, y_2), (u, v)) & = \lambda (f_1(x_1, v) + f_2(u, y_1)) + (1 - \lambda) (f_1(x_2, v) + f_2(u, y_2)) \\
& = \left[ \lambda f_1(x_1, v) + (1 - \lambda) f_1(x_2, v) \right] + \left[ \lambda f_2(u, y_1) + (1 - \lambda) f_2(u, y_2) \right] \\
& \leq f_1(\Phi_1(\lambda, 1 - \lambda), (u, v)) + f_2(\Phi_2(\lambda, 1 - \lambda)) \\
& = H(\Phi_2(\lambda, 1 - \lambda), (u, v)) \quad \text{for all} \ (u, v) \in X \times Y.
\end{align*}$$

For arbitrarily given \( n \) points \((x_1, y_1), \ldots, (x_n, y_n) \in X \times Y\), we can similarly define a continuous function \( \Phi_n : \Delta_n \to D \times E \) by

$$\Phi_n(\lambda_1, \ldots, \lambda_n) := (\psi_1(\lambda_1, \ldots, \lambda_n), \psi_2(\lambda_1, \ldots, \lambda_n))$$

for every \((\lambda_1, \ldots, \lambda_n) \in \Delta_n\), where \( \psi_1 : \Delta_2 \to D \) is a continuous function suitable for \( f_1 \) with respect to \([x_1, \ldots, x_n]\), and \( \psi_2 : \Delta_2 \to E \) is a continuous function suitable for \( f_2 \) with respect to \([y_1, \ldots, y_n]\) in the \( C \)-concavity condition. Thus we can also show the condition (1) for the \( C \)-concavity of \( H \); and hence \( H \) is \( C \)-concave on \( D \times E \). It remains to show that \( H \) satisfies the coercive condition (iv) in Theorem 1. For each \((x, y) \in D \times E\),

$$H((x, y), (x, y)) = f_1(x, y) + f_2(x, y) = -f(x, y) + f(x, y) = 0.$$

And for each \((x, y) \in D \times E\),

$$H((u, v), (x, y)) = f_1(u, y) + f_2(x, v) = f(x, v) - f(u, y).$$

Therefore, by assumption (d), we have that for each \((x, y) \in D \times E\),

$$H((u, v), (x, y)) \geq H((u, v), (x, y)) \quad \text{for all} \ (u, v) \in X \times Y \setminus D \times E,$$

which implies the assumption (iv) of Theorem 1.

Therefore, by Theorem 1, there exists a Nash equilibrium \((x_0, y_0) \in D \times E\) such that

$$f_1(x_0, y_0) = \sup_{x \in X} f_1(x, y_0) \quad \text{and} \quad f_2(x_0, y_0) = \sup_{y \in Y} f_2(x_0, y).$$
Therefore, we have
\[-f(x_0, y_0) = f_1(x_0, y_0) \geq f_1(x, y_0) = -f(x, y_0) \quad \text{for all } x \in X,
\]
and
\[f(x_0, y_0) = f_2(x_0, y_0) \geq f_2(x_0, y) = f(x_0, y) \quad \text{for all } y \in Y.\]
Hence
\[\sup_{y \in Y} f(x_0, y) \leq f(x_0, y_0) \leq \inf_{x \in X} f(x, y_0),\]
which implies
\[\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq f(x_0, y_0) \leq \sup_{y \in Y} \inf_{x \in X} f(x, y).\]
And the reverse inequality
\[\sup_{y \in Y} f(x, y) \geq \sup_{y \in Y} \inf_{x \in X} f(x, y)\]
is trivial, and so we obtain the conclusion. 

As another application of Theorem 2, we shall prove the following which is comparable to the well-known minimax inequality due to Fan [7]:

**Theorem 4.** Let \(X\) be a non-empty compact convex set in a locally convex Hausdorff topological vector space \(E\) and let \(f : X \times X \to \mathbb{R}\) be a real-valued function on \(X \times X\) such that

(a) for each \(y \in X\), the function \(x \mapsto f(x, y)\) is lower semicontinuous on \(X\);
(b) for each \(x \in X\), the function \(y \mapsto f(x, y)\) is \(C\)-concave on \(X\).

Then the minimax inequality
\[\min_{x \in X} \sup_{y \in X} f(x, y) \leq \sup_{x \in X} f(x, x)\]
holds.

**Proof.** Let \(\mu := \sup_{x \in X} f(x, x)\). Clearly we may assume that \(\mu < \infty\). Suppose the contrary, i.e.,
\[\inf_{x \in X} \sup_{y \in X} f(x, y) > \sup_{x \in X} f(x, x) = \mu.\]
Then, for each \(x \in X\), there exists \(y \in X\) such that \(f(x, y) > \mu\). For any \(y \in X\), we let
\[U(y) := \{x \in X \mid f(x, y) > \mu\}.\]

Then, by the assumption (a), each \(U(y)\) is (possibly empty) open in \(X\) and also we have
\[\bigcup_{y \in X} U(y) = X.\]
Since \(X\) is compact, there exists a finite number of non-empty open sets \(U(y_1), \ldots, U(y_n)\) such that \(\bigcup_{i=1}^n U(y_i) = X\). Let \(\{\alpha_i \mid i = 1, \ldots, n\}\) be the partition of unity subordinate to the open covering \(\{U(y_i) \mid i = 1, \ldots, n\}\) of \(X\), i.e.,
\[0 \leq \alpha_i(x) \leq 1, \quad \sum_{i=1}^n \alpha_i(x) = 1 \quad \text{for all } x \in X, \ i = 1, \ldots, n;\]
and if \(x \notin U(y_j)\) for some \(j\), then \(\alpha_j(x) = 0\).
For such \( \{y_1, \ldots, y_n\} \subset X \), since \( y \mapsto f(x, y) \) is \( C \)-concave, there exists a continuous mapping \( \phi_n : \Delta_n \to X \) satisfying the condition
\[
f(x, \phi_n(\lambda_1, \ldots, \lambda_n)) \geq \lambda_1 f(x, y_1) + \cdots + \lambda_n f(x, y_n)
\]
for all \( (\lambda_1, \ldots, \lambda_n) \in \Delta_n \) and for all \( x \in X \).

Now consider a continuous mapping \( \Psi : X \to X \), defined by
\[
\Psi(x) \coloneqq \phi_n(\alpha_1(x), \ldots, \alpha_n(x)) \quad \text{for all } x \in X.
\]

Since \( \phi_n \) and each \( \alpha_i \) are continuous, \( \Psi \) is continuous on \( X \). Moreover, \( \Psi \) maps \( X \), which is a compact convex subset of a locally convex Hausdorff topological vector space, into itself. Therefore, by Lemma 1, there exists a fixed point \( \bar{x} \in X \) such that \( \Psi(\bar{x}) = \bar{x} \).

On the while, by the \( C \)-concavity of \( f \), we have
\[
f(x, \Psi(\bar{x})) \geq \alpha_1(\bar{x}) f(x, y_1) + \cdots + \alpha_n(\bar{x}) f(x, y_n) \quad \text{for all } x \in X;
\]
and so we have
\[
f(\bar{x}, \bar{x}) \geq \sum_{i=1}^{n} \alpha_i(\bar{x}) f(\bar{x}, y_i).
\]
(4)

However, if \( \bar{x} \in U(y_j) \) for some \( 1 \leq j \leq n \), then we have \( f(\bar{x}, y_j) > \mu \) and \( \alpha_j(\bar{x}) > 0 \); and if \( \bar{x} \notin U(y_k) \) for some \( 1 \leq k \leq n \), then \( \alpha_k(\bar{x}) = 0 \). Thus we have
\[
\mu = \sup_{x \in X} f(x, x) \geq f(\bar{x}, \bar{x}) \geq \sum_{i=1}^{n} \alpha_i(\bar{x}) f(\bar{x}, y_i) > \mu,
\]
which is a contradiction. This completes the proof. \( \square \)

As we mentioned before, the generalized game described in [8,19] has an equilibrium if the payoff function \( f_i \) satisfies either CF-concavity or concavity. Indeed, many of the assumptions made in the preceding theorems in [8,19] have been weakened and the existence of equilibrium has been proved; however, it is hard to improve the equilibrium theorem by relaxing quasi-concavity assumption of the payoff functions and the convexity assumption on the strategy space. On the other hand, in this paper, we introduce a meaningful \( C \)-concavity, and prove a new Nash equilibrium existence theorem. Since the Nash equilibrium is an useful tool in many areas of mathematical economics including oligopoly theory, general equilibrium and social choice theory, the \( C \)-concavity should be helpful in developing the theory of Nash equilibrium. Also note that Theorem 1 can be improved to more general spaces by using Eilenberg–Montgomery’s fixed point theorem without assuming the linear structure on \( X \).

Finally, we shall give an example where Theorem 1 can be applied but previous results due to Nash [18], Nikaido and Isoda [19], and Friedman [9] can not be applied.

**Example.** Let \( \Gamma = \{X_1, X_2; f_1, f_2\} \) be a 2-person game where \( X_1 = (-1, 1) \), \( X_2 = [0, 1] \), \( D = [0, 1] \subset X_1 \), \( E = [0, 1] = X_2 \), and payoff functions be given as follows:
\[
f_1(x_1, x_2) := x_1^2 x_2 \quad \text{for every } (x_1, x_2) \in X = X_1 \times X_2,
\]
\[
f_2(y_1, y_2) := y_1 \sqrt{y_2} \quad \text{for every } (y_1, y_2) \in X = X_1 \times X_2.
\]
Clearly, \( f_1(\cdot, x_2) \) is not quasi-concave for any \( x_2 \in [0, 1] \), and thus theorems of Nash [18], Nikaido and Isoda [19], and Friedman [9] cannot be applied. For this game, the related total sum of payoff functions \( H : X \times X \to \mathbb{R} \) is given by

\[
H((x_1, x_2), (y_1, y_2)) = f_1(x_1, y_2) + f_2(y_1, x_2) = x_1^2 y_2 + y_1 \sqrt{x_2},
\]

for every \(((x_1, x_2), (y_1, y_2)) \in X \times X\). Then \( H(x, y) \) is continuous on \( X \times X \). For arbitrarily given two points \((x_1, x_2), (x_3, x_4) \in X\), we now define a continuous function \( \phi_2 : \Delta_2 \to D \times E \) by

\[
\phi_2(\lambda, 1 - \lambda) := \left( \sqrt{\lambda x_1^2 + (1 - \lambda)x_3^2}, \left[ \lambda \sqrt{x_2} + (1 - \lambda) \sqrt{x_4} \right]^2 \right) \quad \text{for all } \lambda \in [0, 1].
\]

Then it is easy to see that \( \phi_2 \) is a continuous function on \( \Delta_2 \). Also, for every \( \lambda \in [0, 1] \) and \((y_1, y_2) \in X\), we have

\[
H(\phi_2(\lambda, 1 - \lambda), (y_1, y_2))
\]

\[
= H\left(\left( \sqrt{\lambda x_1^2 + (1 - \lambda)x_3^2}, \left[ \lambda \sqrt{x_2} + (1 - \lambda) \sqrt{x_4} \right]^2 \right), (y_1, y_2)\right)
\]

\[
= (\lambda x_1^2 + (1 - \lambda)x_3^2)y_2 + \left( \lambda \sqrt{x_2} + (1 - \lambda) \sqrt{x_4} \right)y_1
\]

\[
\geq \lambda (x_1^2 y_2 + y_1 \sqrt{x_2}) + (1 - \lambda)(x_3^2 y_2 + y_1 \sqrt{x_4})
\]

\[
= \lambda H((x_1, x_2), (y_1, y_2)) + (1 - \lambda) H((x_3, x_4), (y_1, y_2)).
\]

For arbitrarily given \( n \) points \((x_1, x_2), \ldots, (z_1, z_2) \in X\), we can similarly define a continuous function \( \phi_n : \Delta_n \to D \times E \) by

\[
\phi_n(\lambda_1, \ldots, \lambda_n) := \left( \sqrt{\lambda_1 x_1^2 + \cdots + \lambda_n z_1^2}, \left[ \lambda_1 \sqrt{x_2} + \cdots + \lambda_n \sqrt{z_2} \right]^2 \right)
\]

for all \( (\lambda_1, \ldots, \lambda_n) \in \Delta_n \); then we can show the \( C \)-concave condition (1); and hence \( H \) is \( C \)-concave on \( D \times E \). Therefore, we can apply the Theorem 1 to the game \( \Gamma \); and clearly, (1, 1) is a Nash equilibrium for \( \Gamma \). In fact,

\[
1 = f_1(1, 1) \geq f_1(x_1, 1) = x_1^2 \quad \text{for every } x_1 \in X_1,
\]

\[
1 = f_2(1, 1) \geq f_2(y_2, 1) = \sqrt{y_2} \quad \text{for every } y_2 \in X_2.
\]

References


