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Nonlinear mobility continuity equations and generalized displacement convexity

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Abstract

We consider the geometry of the space of Borel measures endowed with a distance that is defined by generalizing the dynamical formulation of the Wasserstein distance to concave, nonlinear mobilities. We investigate the energy landscape of internal, potential, and interaction energies. For the internal energy, we give an explicit sufficient condition for geodesic convexity which generalizes the condition of McCann. We take an eulerian approach that does not require global information on the geodesics. As by-product, we obtain existence, stability, and contraction results for the semigroup obtained by solving the homogeneous Neumann boundary value problem for a nonlinear diffusion equation in a convex bounded domain. For the potential energy and the interaction energy, we present a nonrigorous argument indicating that they are not displacement semiconvex.

Keywords: Gradient flows; Displacement convexity; Nonlinear diffusion equations; Parabolic equations; Wasserstein distance; Nonlinear mobility

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1. Introduction

1.1. Displacement convexity and Wasserstein distance

In [33], McCann introduced the notion of *displacement convexity* for integral functionals of the form: given $U : [0, +\infty) \to \mathbb{R}$,

$$\mathscr{U}(\mu) := \int_\Omega U\big(\rho(x)\big)\,\mathrm{d}x \quad \text{for } \mu = \rho\mathscr{L}^d,$$

defined on the set $\mathscr{P}_{ac}(\Omega)$ of the Borel probability measures in a convex open domain $\Omega \subset \mathbb{R}^d$, which are absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^d$. Displacement convexity of $\mathscr{U}$ means convexity along a particular class of curves, given by *displacement interpolation* between two given measures. These curves turned out to be the geodesics of the space $\mathscr{P}_{ac}(\Omega)$ endowed with the euclidean Wasserstein distance.

We recall that the Wasserstein distance W between two Borel probability measures μ_0 and μ_1 on Ω is defined by the following optimal transportation problem (Kantorovich relaxed version) (see [42,43])

$$W^2(\mu_0, \mu_1) := \min\left\{ \int_{\Omega\times\Omega} |x-y|^2\,\mathrm{d}\gamma(x,y)\colon\ \gamma \in \Gamma(\mu_0, \mu_1)\right\},$$

where $\Gamma(\mu_0, \mu_1)$ is the set of admissible plans/couplings between μ_0 and μ_1, that is the set of all Borel probability measures on $\Omega \times \Omega$ with first marginal μ_0 and second marginal μ_1.

We introduce the “pressure” function P, defined by

$$P(r) := rU'(r) - \big(U(r) - U(0)\big) = \int_0^r sU''(s)\,\mathrm{d}s \quad \text{so that}$$

$$P'(r) = rU''(r), \qquad P(0) = 0. \tag{1.1}$$

The main result of [33] states that under the assumption

$$P'(r)r \geqslant (1 - 1/d)P(r) \geqslant 0 \quad \forall r \in (0, +\infty), \tag{1.2a}$$

or, equivalently,

$$r \mapsto \frac{P(r)}{r^{1-1/d}} \quad \text{is nonnegative and nondecreasing on } (0, +\infty), \tag{1.2b}$$

the functional $\mathscr{U}$ is convex along the constant speed geodesics induced by W, i.e. for every curve $(\mu_s)_{s\in[0,1]} \subset \mathscr{P}_{ac}(\Omega)$ satisfying

$$W(\mu_{s_1}, \mu_{s_2}) = |s_1 - s_2|W(\mu_0, \mu_1) \quad \forall s_1, s_2 \in [0, 1], \tag{1.3}$$

the map $s \mapsto \mathscr{U}(\mu_s)$ is convex in $[0,1]$. This class of curves can be, equivalently, defined by displacement interpolation, using the Brenier's optimal transportation map pushing μ_0 onto μ_1 (see [42], for example). Note that (1.2a) and (1.1) imply the convexity of U.

For power-like functions U, P

$$U(\rho)=\begin{cases}\frac{1}{\beta-1}\rho^\beta & \text{if } \beta\neq 1,\\ \rho\log\rho & \text{if } \beta=1,\end{cases}\qquad P(\rho)=\rho^\beta,\quad \text{(1.2a) is equivalent to } \beta\geqslant 1-1/d. \tag{1.4}$$

1.2. The link with a nonlinear diffusion equation

Among the various applications of this property, a remarkable one concerns a wide class of nonlinear diffusion equations. The seminal work of Otto [34] contributed the key idea that a solution of the nonlinear diffusion equation

$$\partial_t\rho-\nabla\cdot\big(\rho\nabla U'(\rho)\big)=0\quad \text{in } (0,+\infty)\times\Omega, \tag{1.5}$$

with homogeneous Neumann boundary condition on $\partial\Omega$ can be interpreted as the trajectory of the *gradient flow* of $\mathscr{U}$ with respect to the Wasserstein distance. This means that the equation is formally the gradient flow of $\mathscr{U}$ with respect to the local metric which for a tangent vector s has the form

$$\langle s,s\rangle_\rho=\int_\Omega \rho|\nabla p|^2\,\mathrm{d}x\quad \text{where } \begin{cases}-\nabla\cdot(\rho\nabla p)=s & \text{in } \Omega,\\ \nabla p\cdot \boldsymbol{n}=0 & \text{on } \partial\Omega\end{cases}$$

where $\boldsymbol{n}$ is a unit normal vector to $\partial\Omega$. Let us note here that Eq. (1.5) corresponds via (1.1) to

$$\partial_t\rho-\Delta P(\rho)=0. \tag{1.6}$$

In particular, the heat equation, for $P(\rho)=\rho$, is the gradient flow of the logarithmic entropy $\mathscr{U}(\rho)=\int_\Omega\rho\log\rho\,\mathrm{d}x$. Let us also note that the metric above satisfies

$$\langle s,s\rangle_\rho=\inf\left\{\int_\Omega \rho|\boldsymbol{v}|^2\,\mathrm{d}x\colon\ s+\nabla\cdot(\rho\boldsymbol{v})=0 \text{ in } \Omega \text{ and } \boldsymbol{v}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\right\}.$$

The key property of this metric is that the length of the minimal geodesic between two given measures is nothing but the Wasserstein distance. More precisely

$$W^2(\mu_0,\mu_1)=\inf\Bigg\{\int_0^1\int_{\mathbb{R}^d}\big|\boldsymbol{v}_s(x)\big|^2\rho_s(x)\,\mathrm{d}x\,\mathrm{d}s\colon\ \partial_s\rho+\nabla\cdot(\rho\boldsymbol{v})=0 \text{ in } (0,1)\times\mathbb{R}^d,$$
$$\operatorname{supp}(\rho_s)\subset\bar\Omega,\ \rho_0\mathscr{L}^d=\mu_0,\ \rho_1\mathscr{L}^d=\mu_1\Bigg\}.$$

This *dynamical formulation* of the Wasserstein distance was rigorously established by Benamou and Brenier in [5] and extended to more general situations in [2] and [30].

As for the classical gradient flows of convex functions in euclidean spaces, the flow associated with (1.5) is a contraction with respect to the Wasserstein distance. In [2] the authors showed that one of the possible ways to rigorously express the link between the functional $\mathscr{U}$, the distance W, and the solution of the diffusion equation (1.5) is given by the *evolution variational inequality* satisfied by the measures $\mu_t = \rho(t,\cdot)\mathscr{L}^d$ associated with (1.5):

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W^2(\mu_t,\nu) \leqslant \mathscr{U}(\nu) - \mathscr{U}(\mu_t) \quad \forall \nu \in \mathscr{P}_{ac}(\Omega). \tag{1.7}$$

1.3. A new class of "dynamical" distances

In a number of problems from mathematical biology [8,9,14,17,18,26,27,36] mainly in chemotaxis with prevention of overcrowding, mathematical physics [10,13,21,22,28,29,40], studies of phase segregation [23,39], and studies of thin liquid films [6], the mobility of "particles" depends on the density ρ itself. For instance, a typical choice of the mobility to avoid overcrowding in chemotaxis is to assume a saturation of the density, see [26], after normalization this leads to $\mathsf{m}(\rho) = \rho(1-\rho)$. The equation for the population density is

$$\partial_t\rho = \nabla\cdot\big[\mathsf{m}(\rho)\nabla(\rho * W) + \nabla\rho\big] = \nabla\cdot\big[\mathsf{m}(\rho)\nabla\big(\rho * W + U'(\rho)\big)\big]$$

with $U(\rho) = \rho\log\rho + (1-\rho)\log(1-\rho)$ and W the fundamental solution of $-\Delta S + \delta S$ with $\delta \geqslant 0$. Another source of models comes from mathematical physics. The same equation has been derived as a hydrodynamical limit of interacting particles system with Kawasaki exchange dynamics in studies of phase segregation [23]. Moreover, these type of equations appear in the study of relaxation towards equilibrium of fermionic or bosonic particles based on kinetic models [28,29]. In their diffusive approximation, they lead to equations of the form

$$\partial_t\rho = \nabla\cdot\big[\mathsf{m}(\rho)x + \nabla\rho\big] = \nabla\cdot\big[\mathsf{m}(\rho)\nabla\big(V + U'(\rho)\big)\big]$$

with $\mathsf{m}(\rho) = \rho(1\pm\rho)$, $V(x) = \frac{|x|^2}{2}$ and $U(\rho) = \rho\log\rho \pm (1\mp\rho)\log(1\mp\rho)$, with $+$ corresponding to bosons and $-$ to fermions.

More precisely the local metric in the configuration space is formally given as follows: For a tangent vector s (euclidean variation)

$$\langle s,s\rangle_\rho = \int_\Omega \mathsf{m}(\rho)|\nabla p|^2\,\mathrm{d}x \quad \text{where } \begin{cases} -\nabla\cdot(\mathsf{m}(\rho)\nabla p) = s & \text{in } \Omega,\\ \nabla p\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\mathsf{m} : [0,+\infty) \to [0,+\infty)$ is the *mobility function*. The global distance generated by the local metric is given by

$$\mathcal{W}^2_{\mathsf{m},\Omega}(\mu_0,\mu_1) := \inf\bigg\{\int_0^1\int_{\mathbb{R}^d}\big|\boldsymbol{v}_s(x)\big|^2\mathsf{m}\big(\rho_s(x)\big)\,\mathrm{d}x\,\mathrm{d}s\colon\ \partial_s\rho + \nabla\cdot\big(\mathsf{m}(\rho)\boldsymbol{v}\big) = 0$$

$$\text{in } (0,1)\times\mathbb{R}^d,\ \operatorname{supp}(\rho_s)\subset\bar\Omega,\ \rho_0\mathscr{L}^d = \mu_0,\ \rho_1\mathscr{L}^d = \mu_1\bigg\}. \tag{1.8}$$

This distance was recently introduced and studied in [19] in the case when m is concave and nondecreasing. Similarly to the case $\mathsf{m}(r) = r$, it is easy to check formally that the trajectory of the *gradient flow* of $\mathscr{U}$ with respect to the modified distance $\mathcal{W}_{\mathsf{m},\Omega}$ solves

$$\partial_t \rho - \nabla \cdot \big(\mathsf{m}(\rho)\nabla U'(\rho)\big) = 0 \quad \text{in } (0, +\infty) \times \Omega \tag{1.9}$$

with homogeneous Neumann boundary conditions on $\partial\Omega$. Assuming that $U''\mathsf{m}$ and $U''\mathsf{m}\mathsf{m}'$ are locally integrable, we can define in this case the function P and the auxiliary function H by

$$P(r) := \int_0^r U''(z)\mathsf{m}(z)\,\mathrm{d}z, \qquad H(r) := \int_0^r U''(z)\mathsf{m}(z)\mathsf{m}'(z)\,\mathrm{d}z = \int_0^r P'(z)\mathsf{m}'(z)\,\mathrm{d}z,$$

so that

$$P' = \mathsf{m}U'', \qquad H' = \mathsf{m}'P', \qquad P(0) = H(0) = 0,$$

and, at least for smooth solutions, the problem (1.9) is equivalent to (1.6).

By means of a formal computation, detailed in Section 2, the second derivative of the internal energy functional $\mathscr{U}$ along a geodesic curve $(\mu_s)_{s\in[0,1]}$ satisfying as in (1.3)

$$\mathcal{W}_{\mathsf{m},\Omega}(\mu_{s_1}, \mu_{s_2}) = |s_1 - s_2| \mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) \quad \forall s_1, s_2 \in [0, 1],$$

is nonnegative, i.e. $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathscr{U}(\mu_s) \geqslant 0$, if the following generalization of McCann condition (1.2a), (1.2b) holds

$$P'(r)\mathsf{m}(r) \geqslant (1 - 1/d)H(r) \geqslant 0 \quad \forall r \in (0, +\infty). \tag{1.10a}$$

It can also be expressed by requiring that

$$r \mapsto \frac{H(r)}{\mathsf{m}^{1-1/d}(r)} \quad \text{is nonnegative and nondecreasing in } (0, +\infty). \tag{1.10b}$$

Note that $P'(r) \geqslant 0$ by (1.10a) implies the convexity of U.

As in the case of the Wasserstein distance, in dimension $d = 1$ the condition (1.10a) reduces to the usual convexity of U. In dimension $d \geqslant 2$, still considering the relevant example of power-like functions U, P, m as in (1.4), we get

$$U(\rho) = \begin{cases} \frac{1}{\beta-1}\rho^\beta & \text{if } \beta \neq 1, \\ \rho\log\rho & \text{if } \beta = 1 \end{cases}, \qquad \mathsf{m}(\rho) = \rho^\alpha, \qquad P(\rho) = \frac{\beta}{\gamma}\rho^\gamma, \qquad \gamma := \alpha + \beta - 1$$

and condition (1.10a) is equivalent to

$$\alpha \in (0, 1], \quad \gamma \geqslant 1 - \alpha/d.$$

In this case the heat equation corresponds to $\gamma = \alpha + \beta - 1 = 1$ and it is therefore the gradient flow of the functional

$$\mathscr{U}(\rho) = \frac{1}{(2-\alpha)(1-\alpha)} \int_\Omega \rho^{2-\alpha}\,\mathrm{d}x \tag{1.11}$$

with respect to the distance $\mathcal{W}_{\mathsf{m},\Omega}$ induced by the mobility function $\mathsf{m}(\rho) = \rho^\alpha$.

Understanding diffusion equations as gradient flows with respect to these new metrics has a functional analysis interest, since it naturally leads to new functional inequalities and to a new interpretation of classical ones. As in the Wasserstein framework [1,15], geodesic/displacement convexity of integral functionals usually plays a crucial role; in particular, in the case of Fokker–Planck equations with a log-concave invariant measure $\gamma = e^{-V}\mathscr{L}^d$, one can recover the family of the classical Beckner inequalities (interpolating between Log–Sobolev and Poincaré inequalities, see [4]) by exploiting the geodesic convexity of the functional obtained integrating $\rho^{2-\alpha}$ with respect to γ as in (1.11). In this case (considered in [20]) also the definition of the transport distance involves the reference measure γ [19]. Analogous inequalities can be expected in the nonlinear diffusion case.

Convexity and dissipation inequalities for second order diffusion equations are also a crucial tool (see the "metric" techniques developed by [25,32] in the linear mobility case $\mathsf{m}(\rho) = \rho$) for the study of nonnegative solutions to a certain class of fourth-order nonlinear diffusion equations

$$\partial_t \rho + \nabla \cdot \big(\mathsf{m}(\rho)\nabla\big(\rho^{\beta-1}\Delta\rho^\beta\big)\big) = 0. \tag{1.12}$$

Particularly interesting cases correspond to the values $\beta = 1$ (equation of thin-film type for power-like mobilities or Cahn–Hiliard when $\mathsf{m}(\rho) = \rho(1-\rho)$) and $\beta = 1/2$ (the so called Derrida–Lebowitz–Speer–Spohn equation also arising in quantum drift-diffusion models). They are formally the gradient flows of the first-order functionals

$$\mathscr{U}(\rho) := \frac{1}{2\beta} \int_\Omega \big|\nabla\big(\rho^\beta\big)\big|^2\,\mathrm{d}x \tag{1.13}$$

with respect to $\mathcal{W}_{\mathsf{m},\Omega}$.

In view of applications to Cahn–Hiliard models, another interesting example, still leading to the heat equation, is represented by the functional

$$\mathscr{U}(\rho) = \int_\Omega \big(\rho\log\rho + (1-\rho)\log(1-\rho)\big)\,\mathrm{d}x, \quad 0 \leqslant \rho \leqslant 1, \quad \mathscr{L}^d\text{-a.e. in } \Omega,$$

and the distance induced by $\mathsf{m}(\rho) = \rho(1-\rho)$, $\rho \in [0,1]$. Notice that in this case the positivity domain of the mobility m is the finite interval $[0,1]$, a case that has not been explicitly considered in [19], but that can be still covered by a careful analysis (see [31]).

1.4. Geodesic convexity and contraction properties

Our aim is to prove rigorously the geodesic convexity of the integral functional $\mathscr{U}$ under conditions (1.10a), (1.10b) and the metric characterization of the nonlinear diffusion equation

(1.9) as the gradient flow of $\mathscr{U}$ with respect to the distance $\mathcal{W}_{\mathsf{m},\Omega}$ (1.8). If one tries to follow the same strategy which has been developed in the more familiar Wasserstein framework, one immediately finds a serious technical difficulty, due to the lackness of an "explicit" representation of the geodesics for $\mathcal{W}_{\mathsf{m},\Omega}$. In fact, the McCann's proof of the displacement convexity of the functionals $\mathscr{U}$ is strictly related to the canonical representation of the Wasserstein geodesics in terms of optimal transport maps.

Existence of a minimal geodesic connecting two measures at a finite $\mathcal{W}_{\mathsf{m},\Omega}$ distance has been proved by [19]. However, an explicit representation is no longer available. On the other hand in [16], following the eulerian approach introduced in [35], the authors presented a new proof of McCann's convexity result for integral functionals defined on a compact manifold without the use of the representation of geodesics. Here, following the same approach of [16], we reverse the usual strategy which derives the existence and the contraction property of the gradient flow of a functional from its geodesic convexity. On the contrary, we show that under the assumption (1.10a) smooth solutions of (1.9) satisfy the following Evolution Variational Inequality analogous to (1.7),

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\mathcal{W}^2_{\mathsf{m},\Omega}(\mu_t,\nu) \leqslant \mathscr{U}(\nu) - \mathscr{U}(\mu_t) \\ &\quad \forall t \in [0,+\infty),\ \forall \nu \in \mathscr{P}(\Omega){:}\ \mathcal{W}_{\mathsf{m}}(\nu,\mu_0) < +\infty. \end{aligned} \tag{1.14}$$

This is sufficient to construct a nice gradient flow generated by $\mathscr{U}$ and metrically characterized by (1.14), as showed in [2] and [3]. The remarkable fact proved by [16] is that whenever a functional $\mathscr{U}$ admits a flow, defined at least in a dense subset of $D(\mathscr{U})$, satisfying (1.14), the functional itself is convex along the geodesics induced by the distance $\mathcal{W}_{\mathsf{m},\Omega}$. As a by-product we obtain stability, uniqueness, and regularization results for the solutions of the problem (1.9) in a suitable subspace of $\mathscr{P}(\Omega)$ metrized by $\mathcal{W}_{\mathsf{m},\Omega}$.

Concerning the assumptions on m, its *concavity* is a necessary and sufficient condition to write the definition of $\mathcal{W}_{\mathsf{m},\Omega}$ with a jointly convex integrand [19], which is crucial in many properties of the distance, in particular for its lower semicontinuity with respect to the usual weak convergence of measures. Since $\mathsf{m} \geqslant 0$ on $[0,\infty)$ the concavity implies that the mobility must be nondecreasing. This is the case considered in [19]. However we are also able to treat the case when the mobility is defined on an interval $[0, M)$ where it is nonnegative and concave. It that case the configuration space is restricted to absolutely continuous measures with densities bounded from above by M. Such mobilities are of particular interest in applications as mentioned before.

1.5. Plan of the paper

In next section, we show the heuristic computations for the convexity of functionals with respect to $\mathcal{W}_{\mathsf{m},\Omega}$. Section 3 is devoted to introduce the notation and to review the needed concepts on $\mathcal{W}_{\mathsf{m},\Omega}$ from [19]. Moreover, we prove a key technical regularization lemma: Lemma 3.6. Subsection 3.4 addresses the question of finiteness of $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0,\mu_1)$, providing new sufficient conditions on m and μ_0,μ_1 in order to ensure that $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0,\mu_1) < +\infty$. After a brief review of some basic properties of the diffusion equation (1.6), in Section 4 we try to get some insight on the features of the generalized McCann condition (1.10a), (1.10b), we recall some basic facts on the metric characterization of contracting gradient flows and their relationships with geodesic convexity borrowed from [2,16], and we state our main results Theorems 4.10 and 4.12. The core of our argument in smooth settings is collected in Section 5, whereas the last Section concludes the proofs of the main results. At the end of the paper we collect some final remarks and open problems.

2. Heuristics

We first discuss, in a formal way, the conditions for the displacement convexity of the internal, the potential and the interaction energy, with respect to the geodesics corresponding to the distance (1.8). For simplicity, we assume that $\Omega = \mathbb{R}^d$ and that densities are smooth and decaying fast enough at infinity so that all computations are justified.

2.1. Geodesics

We first obtain the optimality condition for the geodesic equations in the fluid dynamical formulation of the new distance (1.8). As in [7], we insert the nonlinear mobility continuity equation (1.8)

$$\partial_s \rho + \nabla \cdot \big(\mathsf{m}(\rho)\boldsymbol{v}\big) = 0 \quad \text{in } (0,1) \times \mathbb{R}^d. \tag{2.1}$$

inside the minimization problem as a Lagrange multiplier. As a result, we get the unconstrained minimization problem

$$\begin{aligned} \mathcal{W}_{\mathsf{m}}^2(\mu_0,\mu_1) = \inf_{(\rho,\boldsymbol{v})} \sup_{\psi} \Bigg\{ & \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} \big|\boldsymbol{v}_s(x)\big|^2 \mathsf{m}\big(\rho_s(x)\big)\,\mathrm{d}x\,\mathrm{d}s \\ & - \int_0^1 \int_{\mathbb{R}^d} \big[\rho_s(x)\partial_s\psi(s,x) + \mathsf{m}\big(\rho_s(x)\big)\big(\boldsymbol{v}_s(x)\cdot\nabla\psi(s,x)\big)\big]\,\mathrm{d}x\,\mathrm{d}s \\ & + \int_{\mathbb{R}^d} \rho_1(x)\psi(1,x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_0(x)\psi(0,x)\,\mathrm{d}x \Bigg\}. \end{aligned}$$

Applying a formal minimax principle and thus taking first an infimum with respect to $\boldsymbol{v}$ we obtain the optimality condition $\boldsymbol{v} = \nabla\psi$, and the following formal characterization of the distance

$$\begin{aligned} \mathcal{W}_{\mathsf{m}}^2(\mu_0,\mu_1) = \sup_{\psi} \inf_{\rho} \Bigg\{ & -\frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla\psi|^2 \mathsf{m}(\rho)\,\mathrm{d}x\,\mathrm{d}s - \int_0^1 \int_{\mathbb{R}^d} \rho\,\partial_s\psi\,\mathrm{d}x\,\mathrm{d}s \\ & + \int_{\mathbb{R}^d} \rho_1(x)\psi(1,x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_0(x)\psi(0,x)\,\mathrm{d}x \Bigg\}, \end{aligned}$$

which provides the further optimality condition

$$\partial_s\psi + \frac{1}{2}\mathsf{m}'\big(\rho_s(x)\big)|\nabla\psi|^2 = 0. \tag{2.2}$$

We thus end up with a coupled system of differential equations in $(0,1)\times\mathbb{R}^d$ [19, Remark 5.19],

$$\begin{cases} \partial_s\rho+\nabla\cdot\big(\mathsf{m}(\rho)\nabla\psi\big)=0, \\ \partial_s\psi+\dfrac{1}{2}\mathsf{m}'(\rho)|\nabla\psi|^2=0. \end{cases} \tag{2.3}$$

2.2. Internal energy

We use the formal equations (2.3) for the geodesics associated to the distance (1.8) to compute the conditions under which the internal energy functional is displacement convex. If therefore (ρ_s,ψ_s) is a smooth solution of (2.3), which decays sufficiently at infinity, we proceed as usual [11,35,42] to obtain the following formulas:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{U}(\mu)=-\int_{\mathbb{R}^d}P(\rho)\Delta\psi\,\mathrm{d}x,$$

where ρ denotes, as usual in this paper, the density of μ with respect to the Lebesgue measure, and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathscr{U}(\mu)=\int_{\mathbb{R}^d}\big(P'(\rho)\mathsf{m}(\rho)-H(\rho)\big)(\Delta\psi)^2\,\mathrm{d}x+\int_{\mathbb{R}^d}H(\rho)\bigg(-\nabla\psi\cdot\nabla\Delta\psi+\frac{1}{2}\Delta|\nabla\psi|^2\bigg)\,\mathrm{d}x$$
$$-\frac{1}{2}\int_{\mathbb{R}^d}P'(\rho)\mathsf{m}''(\rho)|\nabla\rho|^2|\nabla\psi|^2\,\mathrm{d}x.$$

As usual, the Bochner formula

$$-\nabla\psi\cdot\nabla\Delta\psi+\frac{1}{2}\Delta|\nabla\psi|^2=|\operatorname{Hess}\psi|^2\geqslant\frac{1}{d}(\Delta\psi)^2,$$

and the fact that $H(\rho)\geqslant 0$, allow us to estimate it as

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathscr{U}(\mu)\geqslant\int_{\mathbb{R}^d}\big(P'(\rho)\mathsf{m}(\rho)-(1-1/d)H(\rho)\big)(\Delta\psi)^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^d}P'(\rho)\mathsf{m}''(\rho)|\nabla\rho|^2|\nabla\psi|^2\,\mathrm{d}x.$$

Therefore, under conditions of concavity of the mobility $\mathsf{m}(\rho)$ and the generalized displacement McCann's condition (1.10a), the functional $\mathscr{U}$ is convex along the geodesics of the distance $\mathcal{W}_{\mathsf{m}}$.

2.3. Potential energy

Similar heuristic formulas can be obtained for the potential and the interaction energy, as in [11,42]. We consider the potential energy functional

$$\mathscr{V}(\mu):=\int_{\mathbb{R}^d}V(x)\,\mathrm{d}\mu,$$

with V a given smooth potential. The equation that we formally obtain as the gradient flow of $\mathscr{V}$ with respect to the distance $\mathcal{W}_{\mathsf{m}}$ is the conservation law

$$\partial_t \rho = \nabla \cdot \big(\mathsf{m}(\rho)\nabla V\big), \tag{2.4}$$

which is an equation of hyperbolic type. As before, it is easy to check that the second derivative of $\mathscr{V}$ along a geodesic satisfying (2.3) is

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathscr{V}(\mu) = \int_{\mathbb{R}^d} \mathsf{m}(\rho)\mathsf{m}'(\rho)\,(\mathrm{Hess}\, V\,\nabla\psi)\cdot\nabla\psi\,\mathrm{d}x + \int_{\mathbb{R}^d} \mathsf{m}(\rho)\mathsf{m}''(\rho)\bigg((\nabla\rho\cdot\nabla\psi)(\nabla V\cdot\nabla\psi) - \frac{1}{2}(\nabla\rho\cdot\nabla V)|\nabla\psi|^2\bigg)\,\mathrm{d}x.$$

This formula allows us to show that this functional cannot be convex along geodesics if m is not linear. Technically, the reason is the presence of the terms linearly depending on $\nabla\rho$. We present a simple example:

Example. Let us first construct the example in one dimension. The expression for the second derivative of the functional above reduces to

$$\frac{d^2}{\mathrm{d}s^2}\mathscr{V}(\mu) = \int_{\mathbb{R}} \mathsf{m}(\rho)\mathsf{m}'(\rho)V_{xx}\psi_x^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \mathsf{m}(\rho)\mathsf{m}''(\rho)\rho_x V_x\psi_x^2\,\mathrm{d}x =: I + II.$$

Consider the case that V is nontrivial. Then $V_x \neq 0$ on some interval. For notational simplicity, we assume that

$$V_x > 0 \quad \text{on } [-2, 2].$$

Since the mobility m we are considering is not a linear function of ρ there exists $z > 0$ such that $\mathsf{m}''(z) \neq 0$. Again for notational simplicity, let us assume that

$$\mathsf{m}''(z) < 0 \quad \text{on } \left[\frac{1}{2}, \frac{3}{2}\right].$$

The fact that we chose V_x to be positive and m'' negative is irrelevant because the sign of term II can be controlled by the sign of ρ_x. Let η be a piecewise linear function on $\mathbb{R}$:

$$\eta(x) = \begin{cases} \frac{3}{2} & \text{if } x < -\frac{1}{2}, \\ 1 - x & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ \frac{1}{2} & \text{if } x > \frac{1}{2}. \end{cases}$$

The fact that the function is Lipschitz, but not smooth is irrelevant; smooth approximations of the given η, can also be used in the construction. Let $\eta_\varepsilon(x) = \eta(\frac{x}{\varepsilon})$. Let $\sigma \in C_0^\infty(\mathbb{R}, [0, 1])$,

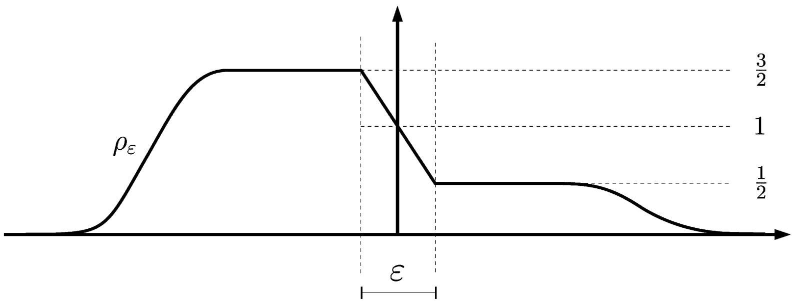

Fig. 1. A profile at which the potential energy is not convex.

supported in $[-1,1]$, such that $\sigma = 1$ on $[-\frac{1}{4},\frac{1}{4}]$ and $\int_{\mathbb{R}} \sigma(x)\mathrm{d}x = 1$. Let $\rho_\varepsilon = \sigma\eta_\varepsilon$. Note that $\int_{\mathbb{R}} \rho_\varepsilon \,\mathrm{d}x = 1$. A typical profile of ρ_ε is given in Fig. 1.

The test velocity (tangent vector at $s = 0$) we consider also needs to be localized near zero. A simple choice is $\psi_\varepsilon(0) = \eta_\varepsilon$. Let $\rho_\varepsilon(s)$ be the corresponding geodesics given by (2.1) and (2.2). Let us observe how, at $s = 0$, the terms I and II scale with ε:

$$I_\varepsilon \leqslant \max_{z\in[0,2]} \mathsf{m}(z)\mathsf{m}'(z) \max_{x\in[-1,1]} V_{xx}(x)\frac{1}{\varepsilon^2}\varepsilon \sim \frac{1}{\varepsilon},$$

$$II_\varepsilon \leqslant -\frac{1}{2} \min_{z\in[\frac{1}{2},\frac{3}{2}]} \mathsf{m}(z)\big|\mathsf{m}''(z)\big|\frac{1}{\varepsilon} \min_{x\in[-1,1]} V_x \frac{1}{\varepsilon^2}\varepsilon \sim -\frac{1}{\varepsilon^2}.$$

Thus, for ε small enough, $\frac{d^2}{\mathrm{d}s^2}\big|_{s=0}\mathscr{V}(\rho_\varepsilon(s)) < 0$. Furthermore note that the square of the length of the tangent vector $\frac{d}{\mathrm{d}t}\rho_\varepsilon(0)$ is

$$\int_{\mathbb{R}} \mathsf{m}\big(\rho_\varepsilon(0)\big)(\partial_x\psi_\varepsilon)^2\,\mathrm{d}x \sim \frac{1}{\varepsilon}.$$

Thus for any $\lambda \in \mathbb{R}$ there exists $\varepsilon > 0$ such that

$$\frac{d^2}{\mathrm{d}s^2}\bigg|_{s=0}\mathscr{V}\big(\rho_\varepsilon(s)\big) + \lambda\int_{\mathbb{R}} \mathsf{m}\big(\rho_\varepsilon(0)\big)(\partial_x\psi_\varepsilon)^2\,\mathrm{d}x < 0$$

which implies that $\mathscr{V}$ is not λ-convex for any $\lambda \in \mathbb{R}$.

Let us conclude the example by remarking that it can be extended to multidimensional domains. In particular it suffices to extend the 1-D profile to d-D to be constant in every other direction and then use a cut-off. We only sketch the elements of the construction.

We can assume that $\nabla V(0) = e_d$. Let $\tilde{\rho}_\varepsilon(x) = \rho_\varepsilon(x_d)$. Let $\hat{x} = (x_1,\ldots,x_{d-1})$. To cut-off in the directions perpendicular to e_d we use the length scales $1 \gg l \gg \delta \gg \varepsilon$. Let $\theta_{l,\delta}$ be smooth cut-off function equal to 1 on $[-l,l]$ and equal to 0 outside of $[-l-\delta, l+\delta]$; with $|\nabla\theta_{l,\delta}| < \frac{C}{\delta}$ and $|D^2\theta_{l,\delta}| < \frac{C}{\delta^2}$. Let $\rho_{l,\delta,\varepsilon}(x) = \tilde{\rho}_\varepsilon(x_d)\theta_{l,\delta}(|\hat{x}|)$. Let $\psi_{l,\delta,\varepsilon}(x) = \eta_\varepsilon(x_d)\theta_{l,\delta}(|\hat{x}|)$. Checking the scaling of appropriate terms is straightforward.

2.4. Interaction energy

Consider the interaction energy functional

$$\mathscr{W}(\mu) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y,$$

with W a given smooth potential. The equation that we formally obtain as the gradient flow of $\mathscr{W}$ with respect to the distance $\mathcal{W}_{\mathsf{m}}$ is the interaction equation

$$\partial_t\rho = \nabla\cdot\big(\mathsf{m}(\rho)(\rho*\nabla W)\big). \tag{2.5}$$

As before, it is easy to check that

$$\begin{aligned}
\frac{d^2}{ds^2}\mathscr{W}(\mu) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathsf{m}\big(\rho(x)\big)\mathsf{m}'\big(\rho(x)\big)\rho(y)\nabla\psi(x)\cdot\big(\mathrm{Hess}\, W(x-y)\nabla\psi(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathsf{m}\big(\rho(x)\big)\mathsf{m}\big(\rho(y)\big)\nabla\psi(y)\cdot\big(\mathrm{Hess}\, W(x-y)\nabla\psi(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathsf{m}\big(\rho(x)\big)\mathsf{m}''\big(\rho(x)\big)\rho(y)\big(\nabla\rho(x)\cdot\nabla\psi(x)\big)\big(\nabla W(x-y)\cdot\nabla\psi(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathsf{m}\big(\rho(x)\big)\mathsf{m}''\big(\rho(x)\big)\rho(y)\big(\nabla\rho(x)\cdot\nabla W(x-y)\big)\big|\nabla\psi(x)\big|^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

It can be demonstrated that if m is nonlinear then the interaction energy is not geodesically convex. As for the potential energy, the reason lies in the presence of derivatives of ρ in the expression above. More precisely, in one dimension the second derivative of $\mathscr{W}(\rho)$ reduces to

$$\begin{aligned}
\frac{d^2}{ds^2}\mathscr{W}(\rho) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathsf{m}\big(\rho(x)\big)\mathsf{m}'\big(\rho(x)\big)\rho(y)\psi_x^2(x)W_{xx}(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}}\int_{\mathbb{R}} \mathsf{m}\big(\rho(x)\big)\mathsf{m}\big(\rho(y)\big)\psi_y(y)W_{xx}(x-y)\psi_x(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathsf{m}\big(\rho(x)\big)\mathsf{m}''\big(\rho(x)\big)\rho(y)\rho_x(x)\psi_x^2(x)\,W_x(x-y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

It turns out that the example for the lack of (semi)convexity provided for the potential energy is also an example (with V replaced by W) for the interaction energy. The estimates of the terms are similar, so we leave the details to the reader.

3. Notation and preliminaries

In this section, following [19], we shall recall the main properties of the distance $\mathcal{W}_{\mathsf{m},\Omega}$ introduced in (1.8). For the sake of simplicity, we only consider here the case of a bounded open domain Ω, so that it is not restrictive to assume that all the measures (Radon, i.e. locally finite, in the general approach of [19]) involved in the various definitions have finite total variation. Since we deal with arbitrary mobility functions m, these distances do not exhibit nice homogeneity properties as in the Wasserstein case; therefore we deal with finite Borel measures without assuming that their total mass is 1.

3.1. Measures and continuity equation

We denote by $\mathcal{M}^+(\mathbb{R}^d)$ (resp. $\mathcal{M}^+_c(\mathbb{R}^d)$) the space of finite positive Borel measures on $\mathbb{R}^d$ (resp. with compact support) and by $\mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued Borel measures on $\mathbb{R}^d$ with finite total variation. By Riesz representation theorem, the space $\mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ can be identified with the dual space of $C^0_c(\mathbb{R}^d;\mathbb{R}^d)$ and it is endowed with the corresponding weak* topology. We denote by $|\boldsymbol{\nu}| \in \mathcal{M}^+(\mathbb{R}^d)$ the total variation of the vector measure $\boldsymbol{\nu} \in \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$. $\boldsymbol{\nu}$ admits the polar decomposition $\boldsymbol{\nu} = \boldsymbol{w}|\boldsymbol{\nu}|$ with $\boldsymbol{w} \in L^1(|\boldsymbol{\nu}|;\mathbb{R}^d)$. If B is a Borel subset of $\mathbb{R}^d$ (typically an open or closed set) we denote by $\mathcal{M}^+(B)$ (resp. $\mathcal{M}^+(B;\mathbb{R}^d)$) the subset of $\mathcal{M}^+(\mathbb{R}^d)$ (resp. $\mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$) whose measure μ are concentrated on B, i.e. $\mu(\mathbb{R}^d \setminus B) = 0$ (resp. $|\mu|(\mathbb{R}^d \setminus B) = 0$). Notice that if B is a compact subset of $\mathbb{R}^d$ then the convex set in $\mathcal{M}^+(B)$ of measures with a fixed total mass $\mathfrak{m}$ is compact with respect to the weak* topology. If $\mathfrak{m} > 0$, $\mathcal{M}^+(B,\mathfrak{m})$ is the convex subset of $\mathcal{M}^+(B)$ whose measures have fixed total mass $\mu(B) = \mathfrak{m}$.

Let Ω be a bounded open subset of $\mathbb{R}^d$. Given $\mu_0, \mu_1 \in \mathcal{M}^+(\bar{\Omega})$ we denote by $\mathcal{CE}_\Omega(\mu_0 \to \mu_1)$ the collection of time dependent measures $(\mu_s)_{s\in[0,1]} \subset \mathcal{M}^+(\bar{\Omega})$ and $(\boldsymbol{\nu}_s)_{s\in(0,1)} \in \mathcal{M}(\bar{\Omega};\mathbb{R}^d)$ such that

1. $s \mapsto \mu_s$ is weakly* continuous in $\mathcal{M}^+(\mathbb{R}^d)$ with $\mu|_{s=0} = \mu_0$ and $\mu|_{s=1} = \mu_1$;
2. $(\boldsymbol{\nu}_s)_{s\in(0,1)}$ is a Borel family with $\int_0^1 |\boldsymbol{\nu}_s|(\bar{\Omega})\,\mathrm{d}s < +\infty$;
3. $(\mu, \boldsymbol{\nu})$ is a distributional solution of

$$\partial_s \mu_s + \nabla \cdot \boldsymbol{\nu}_s = 0 \quad \text{in } (0,1) \times \mathbb{R}^d.$$

If $(\mu, \boldsymbol{\nu}) \in \mathcal{CE}_\Omega(\mu_0 \to \mu_1)$ then it is immediate to check that the total mass $\mu_s(\mathbb{R}^d) = \mu_s(\bar{\Omega}) = \mathfrak{m}$ is a constant, independent of s. In particular, $\mu_0(\mathbb{R}^d) = \mu_1(\mathbb{R}^d)$.

3.2. Mobility and action functional

We fix a right threshold $M \in (0, +\infty]$ and a concave *mobility function* $\mathsf{m} \in C^0[0, M)$ strictly positive in $(0, M)$. We denote by $\mathsf{m}(M)$ the left limit of $\mathsf{m}(r)$ as $r \uparrow M$. We can also introduce the maximal left interval of monotonicity of m whose right extreme is

$$M_\uparrow := \sup\bigl\{m \in [0, M)\colon\ \mathsf{m}|_{[0,m]} \text{ is nondecreasing}\bigr\}.$$

We distinguish two situation:

Case A. $M = +\infty$, so that m is nondecreasing and $M_\uparrow = M = +\infty$; typically $\mathsf{m}(0) = 0$ and the main example is provided by $\mathsf{m}(r) = r^\alpha$, $\alpha \in [0, 1]$. This is the case considered in [19]. When $\mathsf{m}'(+\infty) := \lim_{r\uparrow+\infty} r^{-1}\mathsf{m}(r) = \lim_{r\uparrow+\infty} \mathsf{m}'(r) = 0$ we are in the *sublinear growth* case. A *linear growth* of m corresponds to $\mathsf{m}'(+\infty) > 0$.

Case B. $M < +\infty$, so that $0 \leqslant M_\uparrow \leqslant M$ and m is nonincreasing in the right interval $[M_\uparrow, M]$ (but we also allow m to be constant or even decreasing in $[0, M)$ with $M_\uparrow = 0$). Typically $\mathsf{m}(0) = \mathsf{m}(M) = 0$ (in this case $0 < M_\uparrow < M$) and the main example is $\mathsf{m}(r) = r(M - r)$, or, more generally, $\mathsf{m}(r) = r^{\alpha_0}(M - r)^{\alpha_1}$, $\alpha_0, \alpha_1 \in (0, 1]$.

Remark 3.1. Many properties proved in the Case A can be extended to the Case B, but there are important exceptions. The most important one concerns the upper bound on the two measures μ^0, μ^1 in order to satisfy $\mathcal{W}_{\mathsf{m},\Omega}(\mu^0, \mu^1) < +\infty$ in Case B: they should be absolutely continuous with respect to $\mathscr{L}^d$ with essentially bounded densities $\rho^i \leqslant M$, $i = 0, 1$.

Another important difference concerns the *subadditivity property* [19, Theorem 5.12],

$$\mathcal{W}_{\mathsf{m},\Omega}\big(\mu^0 + \sigma^0, \mu^1 + \sigma^1\big) \leqslant \mathcal{W}_{\mathsf{m},\Omega}\big(\mu^0, \mu^1\big) + \mathcal{W}_{\mathsf{m},\Omega}\big(\sigma^0, \sigma^1\big)$$

which does not hold in Case B. We refer to [31] for further technical details.

Using the conventions

$$\begin{aligned} a/b &= 0 && \text{if } a = b = 0,\\ a/b &= +\infty && \text{if } a > 0 = b, \end{aligned} \tag{3.1}$$

the corresponding *action density function* $\phi_{\mathsf{m}} : \mathbb{R} \times \mathbb{R}^d \to [0, +\infty]$ is defined by

$$\phi_{\mathsf{m}}(\rho, \boldsymbol{w}) = \begin{cases} \frac{|\boldsymbol{w}|^2}{\mathsf{m}(\rho)} & \text{if } \rho \in [0, M],\\ +\infty & \text{if } \rho < 0 \text{ or } \rho > M. \end{cases}$$

It is not difficult to check that, under the convention (3.1), the function ϕ_{m} is (jointly) convex and lower semicontinuous.

Given that m is concave and ϕ_{m} is convex, when $M = +\infty$ we can define the recession function $\varphi^\infty_{\mathsf{m}} : \mathbb{R}^d \mapsto [0, +\infty]$ (recall (3.1)),

$$\varphi^\infty_{\mathsf{m}}(\boldsymbol{w}) := \lim_{r\uparrow+\infty} r\phi_{\mathsf{m}}(1, \boldsymbol{w}/r) = \frac{|\boldsymbol{w}|^2}{\mathsf{m}'(\infty)}, \qquad \mathsf{m}'(\infty) := \lim_{r\to+\infty} \mathsf{m}'(r) = \lim_{r\to+\infty} \frac{\mathsf{m}(r)}{r} \geqslant 0.$$

We introduce now the *action functional*

$$\Phi_{\mathsf{m},\Omega} : \mathcal{M}^+\big(\mathbb{R}^d\big) \times \mathcal{M}\big(\mathbb{R}^d; \mathbb{R}^d\big) \to [0, +\infty],$$

defined on couples of measures $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $\boldsymbol{\nu} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$. In order to define it we consider the usual Lebesgue decomposition $\mu = \rho\mathscr{L}^d + \mu^\perp$, $\boldsymbol{\nu} = \boldsymbol{w}\mathscr{L}^d + \boldsymbol{\nu}^\perp$ and distinguish the following cases:

1. If the support of μ or $\boldsymbol{\nu}$ is not contained in $\bar{\Omega}$ then $\Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) = +\infty$;
2. When $M < +\infty$ (Case B), we set

$$\Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) := \begin{cases} \int_\Omega \phi_{\mathsf{m}}(\rho, \boldsymbol{w})\,\mathrm{d}x & \text{if } \mu^\perp = 0,\ \boldsymbol{\nu}^\perp = 0, \\ +\infty & \text{otherwise;} \end{cases}$$

   notice that if $\Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) < +\infty$ then $\rho \in L^\infty(\Omega)$ with $0 \leqslant \rho \leqslant M$, $\mathscr{L}^d$-a.e. in Ω and $\boldsymbol{w} \in L^2(\Omega; \mathbb{R}^d)$.
3. When $M = +\infty$ and $\mathsf{m}'(\infty) = 0$ (Case A, sublinear growth) then

$$\Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) := \begin{cases} \int_\Omega \phi_{\mathsf{m}}(\rho, \boldsymbol{w})\,\mathrm{d}x & \text{if } \boldsymbol{\nu}^\perp = 0, \\ +\infty & \text{otherwise;} \end{cases}$$

4. Finally, when $M = +\infty$ and $\mathsf{m}'(\infty) > 0$ (Case A, linear growth) then we set

$$\Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) := \begin{cases} \int_\Omega \phi_{\mathsf{m}}(\rho, \boldsymbol{w})\,\mathrm{d}x + \int_{\bar{\Omega}} \varphi^\infty_{\mathsf{m}}(\boldsymbol{w}^\perp)\,\mathrm{d}\mu^\perp & \text{if } \boldsymbol{\nu}^\perp = \boldsymbol{w}^\perp\mu^\perp \ll \mu^\perp, \\ +\infty & \text{otherwise.} \end{cases}$$

3.3. The modified Wasserstein distance

Let Ω be a bounded open set. Given $\mu^0, \mu^1 \in \mathcal{M}^+(\bar{\Omega})$ we define

$$\mathcal{W}_{\mathsf{m},\Omega}(\mu^0, \mu^1) := \inf\left\{ \left(\int_0^1 \Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s)\,\mathrm{d}s \right)^{1/2} :\ (\mu, \boldsymbol{\nu}) \in \mathcal{CE}_\Omega(\mu^0 \to \mu^1) \right\} \tag{3.2}$$

$$= \inf\left\{ \int_0^1 \left(\Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s)\right)^{1/2}\,\mathrm{d}s:\ (\mu, \boldsymbol{\nu}) \in \mathcal{CE}_\Omega(\mu^0 \to \mu^1) \right\}. \tag{3.3}$$

We refer to [19, Theorem 5.4] for the equivalence between (3.2) and (3.3). $\mathcal{W}_{\mathsf{m},\Omega}(\mu^0, \mu^1) = +\infty$ if the set of connecting curves $\mathcal{CE}_\Omega(\mu^0 \to \mu^1)$ is empty. The following three propositions are proved in [19], see Theorems 5.5–5.7, 5.15, and Proposition 5.14.

Proposition 3.2. *The space $\mathcal{M}^+(\bar{\Omega})$ endowed with the distance $\mathcal{W}_{\mathsf{m},\Omega}$ is a complete pseudo-metric space* (*the distance can assume the value $+\infty$*)*, inducing as strong as, or stronger topology than the weak* one.*

Given a measure $\sigma \in \mathcal{M}^+(\bar{\Omega})$, the space $\mathcal{M}^+_{\mathsf{m},\Omega}[\sigma] := \{\mu \in \mathcal{M}^+(\bar{\Omega}):\ \mathcal{W}_{\mathsf{m},\Omega}(\mu, \sigma) < +\infty\}$ is a complete metric space whose measures have the same total mass of σ.

Moreover, for every $\mu_0,\ \mu_1 \in \mathcal{M}^+(\bar{\Omega})$ such that $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) < +\infty$ there exists a minimizing couple $(\mu, \boldsymbol{\nu})$ in (3.2) (*unique, if* m *is strictly concave and sublinear*) *and the curve $(\mu_s)_{s\in[0,1]}$ is a constant speed geodesic for $\mathcal{W}_{\mathsf{m},\Omega}$, thus satisfying*

$$\mathcal{W}_{\mathsf{m},\Omega}(\mu_t, \mu_s) = |t - s|\mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) \quad \forall s, t \in [0, 1].$$

Proposition 3.3 *(Lower semicontinuity). If Ω_n, Ω are bounded open sets such that $\mathscr{L}^d|_{\Omega_n}$ weakly* converges to $\mathscr{L}^d|_{\Omega}$, $M_n \in (0, +\infty]$ is a nonincreasing sequence converging to M, m_n is a sequence of nonnegative concave functions in the intervals $(0, M_n)$ such that*

$$\mathsf{m}_{n'}(r) \geqslant \mathsf{m}_{n''}(r) \quad \forall r \in (0, M_{n''}) \quad \text{if } n' \leqslant n'', \qquad \lim_{n\to\infty} \mathsf{m}_n(r) = \mathsf{m}(r) \quad \forall r \in (0, M),$$

and μ_0^n, μ_1^n are sequences of measures weakly convergent to μ_0 and μ_1 respectively, then*

$$\liminf_{n\to+\infty} \mathcal{W}_{\mathsf{m}_n,\Omega_n}\big(\mu_0^n, \mu_1^n\big) \geqslant \mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1). \tag{3.4}$$

Proposition 3.4 *(Monotonicity). Let $\tilde{\Omega} \supset \Omega$, $\tilde{\mathsf{m}} \geqslant \mathsf{m}$, $\mu_0, \mu_1 \in \mathcal{M}^+(\bar{\Omega})$. Then the following inequality holds*

$$\mathcal{W}_{\tilde{\mathsf{m}},\tilde{\Omega}}(\mu_0, \mu_1) \leqslant \mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1).$$

Proposition 3.5. *Let $k \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative convolution kernel, with $\int_{\mathbb{R}^d} k(x)\,\mathrm{d}x = 1$ and $\operatorname{supp}(k) = \bar{B}_1(0)$, and let $k_\varepsilon(x) := \varepsilon^{-d} k(x/\varepsilon)$. For every $\mu, \mu_0, \mu_1 \in \mathcal{M}^+(\bar{\Omega})$ and $\boldsymbol{\nu} \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^d)$ we have*

$$\Phi_{\mathsf{m},\Omega_\varepsilon}(\mu * k_\varepsilon, \boldsymbol{\nu} * k_\varepsilon) \leqslant \Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) \quad \forall \varepsilon > 0,$$

$$\mathcal{W}_{\mathsf{m},\Omega_\varepsilon}(\mu_0 * k_\varepsilon, \mu_1 * k_\varepsilon) \leqslant \mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) \quad \forall \varepsilon > 0, \tag{3.5}$$

$$\lim_{\varepsilon\to 0} \mathcal{W}_{\mathsf{m},\Omega_\varepsilon}(\mu_0 * k_\varepsilon, \mu_1 * k_\varepsilon) = \mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1), \tag{3.6}$$

where $\Omega_\varepsilon := \Omega + B_\varepsilon(0)$.

Proof. If $\Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) < +\infty$ then μ, $\boldsymbol{\nu}$ are supported in $\bar{\Omega}$ and [19, Theorem 2.3] yields

$$\Phi_{\mathsf{m},\Omega}(\mu, \boldsymbol{\nu}) = \Phi_{\mathsf{m},\mathbb{R}^d}(\mu, \boldsymbol{\nu}) \geqslant \Phi_{\mathsf{m},\mathbb{R}^d}(\mu * k_\varepsilon, \boldsymbol{\nu} * k_\varepsilon) = \Phi_{\mathsf{m},\Omega_\varepsilon}(\mu * k_\varepsilon, \boldsymbol{\nu} * k_\varepsilon),$$

being $\mu * k_\varepsilon$, $\boldsymbol{\nu} * k_\varepsilon$ supported in $\bar{\Omega}_\varepsilon$. Notice that only the concavity of m (and not its monotonicity) plays a role here. A similar argument and [19, Theorem 5.15] yields (3.5). The limit (3.6) is an immediate consequence of (3.4) and (3.5). □

The next technical lemma provides a crucial approximation result for curves with finite $\Phi_{\mathsf{m},\Omega}$ energy. It allows for measures to be approximated by ones with smooth, positive densities.

Lemma 3.6. *Let Ω be an open bounded convex set and let $(\mu, \boldsymbol{\nu}) \in \mathcal{CE}_\Omega(\mu_0 \to \mu_1)$ with given constant mass $\mathfrak{m}$ and finite energy $\int_0^1 \Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s)\,\mathrm{d}s < +\infty$. For every $\varepsilon > 0$, $\delta \in [0, 1]$ there exist a decreasing family of smooth convex sets $\Omega^\varepsilon \downarrow \Omega$ and a family of curves $(\mu^{\varepsilon,\delta}, \boldsymbol{\nu}^{\varepsilon,\delta}) \in \mathcal{CE}_{\Omega^\varepsilon}(\mu_0^{\varepsilon,\delta} \to \mu_1^{\varepsilon,\delta})$ with the following properties*

$$\mu_i^{\varepsilon,\delta} = (1-\delta)\mu_i * k_\varepsilon + \delta\lambda^\varepsilon, \qquad \lambda^\varepsilon := \frac{\mathfrak{m}}{\mathscr{L}^d(\Omega^\varepsilon)}\mathscr{L}^d|_{\Omega^\varepsilon}, \qquad \mu_s^{\varepsilon,\delta}(\Omega^\varepsilon) = \mathfrak{m}, \tag{3.7}$$

$$\mu_s^{\varepsilon,\delta} = \rho_s^{\varepsilon,\delta}\mathscr{L}^d|_{\Omega^\varepsilon}, \qquad \boldsymbol{\nu}_s^{\varepsilon,\delta} = \boldsymbol{w}_s^{\varepsilon,\delta}\mathscr{L}^d|_{\Omega^\varepsilon}, \qquad \rho^{\varepsilon,\delta}, \boldsymbol{w}^{\varepsilon,\delta} \in C^\infty\big([0,1]\times\bar{\Omega}^\varepsilon\big), \tag{3.8}$$

$$\partial_s\rho_s^{\varepsilon,\delta} + \nabla\cdot\boldsymbol{w}_s^{\varepsilon,\delta} = 0 \quad \textit{in } (0,1)\times\Omega^\varepsilon, \qquad \rho^{\varepsilon,\delta} \geqslant \delta\frac{\mathfrak{m}}{\mathscr{L}^d(\Omega)} > 0,$$

$$\frac{1}{c_\varepsilon^2}\int_0^1 \Phi_{\mathsf{m},\Omega^\varepsilon}\big(\mu_s^{\varepsilon,\delta}, \boldsymbol{\nu}_s^{\varepsilon,\delta}\big)\,\mathrm{d}s \leqslant \int_0^1 \Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s)\,\mathrm{d}s = \lim_{\varepsilon,\delta\downarrow 0}\int_0^1 \Phi_{\mathsf{m},\Omega^\varepsilon}\big(\mu_s^{\varepsilon,\delta}, \boldsymbol{\nu}_s^{\varepsilon,\delta}\big)\,\mathrm{d}s,$$

where $c_\varepsilon := 1 + 2\varepsilon$.

Proof. Let us extend $(\mu_s, \boldsymbol{\nu}_s)$ outside the unit interval by setting $\boldsymbol{\nu}_s \equiv 0$ and $\mu_s \equiv \mu_0$ if $s < 0$, $\mu_s \equiv \mu_1$ if $s > 1$; it is immediate to check that $(\mu, \boldsymbol{\nu})$ still satisfy the continuity equation. We then consider a family of smooth and convex open sets Ω^ε satisfying $\Omega + B_{2\varepsilon}(0) \subset \Omega^\varepsilon \subset \Omega + B_{3\varepsilon}(0)$ and define $\tilde{\mu}_s^\varepsilon := \mu * k_\varepsilon$, $\tilde{\boldsymbol{\nu}}_s^\varepsilon := \boldsymbol{\nu} * k_\varepsilon$ which have smooth densities $\tilde{\rho}_s^\varepsilon$, $\tilde{\boldsymbol{w}}_s^\varepsilon$ and are concentrated in $\bar{\Omega} + B_\varepsilon(0)$. We perform a further time convolution with respect to a 1-dimensional family of nonnegative smooth mollifiers $h_\varepsilon(z) := \varepsilon^{-1}h(z/\varepsilon)$ with support in $[-\varepsilon, \varepsilon]$ and integral 1,

$$\bar{\mu}_s^\varepsilon := \int_{\mathbb{R}} \tilde{\mu}_z^\varepsilon h_\varepsilon(s-z)\,\mathrm{d}z, \qquad \bar{\boldsymbol{\nu}}_s^\varepsilon := \int_{\mathbb{R}} \tilde{\boldsymbol{\nu}}_z^\varepsilon h_\varepsilon(s-z)\,\mathrm{d}z,$$

with corresponding densities ρ_s^ε, $\boldsymbol{w}_s^\varepsilon$. Notice that $\bar{\mu}_{-\varepsilon}^\varepsilon = \mu_0^{\varepsilon,0}$, $\bar{\mu}_{1+\varepsilon}^\varepsilon = \mu_1^{\varepsilon,0}$ and, by the convexity of ϕ_{m} and Jensen's inequality, we have

$$\phi_{\mathsf{m}}\big(\rho_s^\varepsilon, \boldsymbol{w}_s^\varepsilon\big) \leqslant \int_{\mathbb{R}} \phi_{\mathsf{m}}\big(\tilde{\rho}_z^\varepsilon, \tilde{\boldsymbol{w}}_z^\varepsilon\big)h_\varepsilon(s-z)\,\mathrm{d}z, \qquad \Phi_{\mathsf{m},\Omega^\varepsilon}\big(\bar{\mu}_s^\varepsilon, \bar{\boldsymbol{\nu}}_s^\varepsilon\big) \leqslant \int_{\mathbb{R}} \Phi_{\mathsf{m},\Omega^\varepsilon}\big(\tilde{\mu}_s^\varepsilon, \tilde{\boldsymbol{\nu}}_s^\varepsilon\big)h_\varepsilon(s-z)\,\mathrm{d}z$$

so that, being $\bar{\boldsymbol{\nu}}_s^\varepsilon = 0$ if $s < -\varepsilon$ or $s > 1+\varepsilon$,

$$\int_{-\varepsilon}^{1+\varepsilon} \Phi_{\mathsf{m},\Omega^\varepsilon}\big(\bar{\mu}_s^\varepsilon, \bar{\boldsymbol{\nu}}_s^\varepsilon\big)\,\mathrm{d}s = \int_{\mathbb{R}} \Phi_{\mathsf{m},\Omega^\varepsilon}\big(\bar{\mu}_s^\varepsilon, \bar{\boldsymbol{\nu}}_s^\varepsilon\big)\,\mathrm{d}s \leqslant \int_{\mathbb{R}} \Phi_{\mathsf{m},\Omega^\varepsilon}\big(\tilde{\mu}_s^\varepsilon, \tilde{\boldsymbol{\nu}}_s^\varepsilon\big)\,\mathrm{d}s \leqslant \int_0^1 \Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s)\,\mathrm{d}s.$$

We eventually set

$$\mu_s^\varepsilon := \bar{\mu}_{c_\varepsilon s-\varepsilon}^\varepsilon, \qquad \boldsymbol{\nu}_s^\varepsilon := c_\varepsilon\bar{\boldsymbol{\nu}}_{c_\varepsilon s-\varepsilon}^\varepsilon, \qquad c_\varepsilon := 1+2\varepsilon$$

and

$$\mu_s^{\varepsilon,\delta} := (1-\delta)\mu_s^\varepsilon + \delta\lambda^\varepsilon, \qquad \boldsymbol{\nu}_s^{\varepsilon,\delta} := \boldsymbol{\nu}_s^\varepsilon.$$

It is then easy to check that all the requirements are satisfied. $\square$

3.4. Couple of measures at finite $\mathcal{W}_{\mathsf{m},\Omega}$ distance

We discuss now some cases when it is possible to prove that the distance between two measures is finite. We already know [19, Cor. 5.25] (in the Case A, but the same argument can be easily adapted to cover the case $M < +\infty$) that when Ω is convex and bounded

$$\text{if } \mu_i = \rho_i \mathscr{L}^d \text{ with } \|\rho_i\|_{L^\infty(\mathbb{R}^d)} < M \quad \text{then } \mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) < \infty. \tag{3.9}$$

We focus on the Case A, $M = +\infty$, and exploit some ideas of [37]. In order to refine the condition (3.9), we first introduce the functions

$$k_{\mathsf{m},d}(r) := \left(r^{1+2/d}\mathsf{m}(r)\right)^{-1/2}, \qquad K_{\mathsf{m},d}(r) := \frac{1}{d}\int_r^{+\infty} k_{\mathsf{m},d}(z)\,\mathrm{d}z, \quad r > 0.$$

Observe that $K_{\mathsf{m},d}$ is either everywhere finite or identically $+\infty$. In particular, in the case $\mathsf{m}(r) = r^\alpha$, $K_{\mathsf{m},d}$ is finite if and only if $\alpha > 1 - 2/d$.

Theorem 3.7. *Let Ω be a bounded, open convex set of $\mathbb{R}^d$. Suppose that $M = +\infty$, $\mathfrak{m} > 0$, and that $K_{\mathsf{m},d}$ is finite (in particular $\alpha > 1 - 1/2d$ in the homogeneous case $\mathsf{m}(r) = r^\alpha$). Then any two measures $\mu_0, \mu_1 \in \mathcal{M}^+(\bar{\Omega}, \mathfrak{m})$ have finite distance $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) < +\infty$ and the topology induced by $\mathcal{W}_{\mathsf{m},\Omega}$ on the space $\mathcal{M}^+(\bar{\Omega}, \mathfrak{m})$ coincides with the usual weak* topology. In particular, the metric space $(\mathcal{M}^+(\bar{\Omega}, \mathfrak{m}), \mathcal{W}_{\mathsf{m},\Omega})$ is compact and separable.*

Proof. We fix an open set B with compact closure in Ω and a reference measure $\lambda = \bar{\rho}\mathscr{L}^d|_B$ with $\lambda(\Omega) = \mathfrak{m}$ and $0 < \bar{\rho}(x) \leqslant b$ for $\mathscr{L}^d$-a.e. x in B. Since $\mathcal{W}_{\mathsf{m},\Omega}$ satisfies the triangular inequality, the first part of the theorem follows if we show that $\mathcal{W}_{\mathsf{m},\Omega}(\lambda, \mu) < +\infty$ for every $\mu \in \mathcal{M}^+(\bar{\Omega}, \mathfrak{m})$.

Let $\boldsymbol{r} : B \to \bar{\Omega}$ be the Brenier map pushing λ onto μ: we know that $\boldsymbol{r}$ is cyclically monotone. We set $\boldsymbol{r}_s := (1-s)\boldsymbol{i} + s\boldsymbol{r}$ with image $B_s \subset (1-s)\bar{B} + s\bar{\Omega} \subset \bar{\Omega}$ and inverse $\boldsymbol{s}_s = \boldsymbol{r}_s^{-1} : B_s \to B$, and $\boldsymbol{v}_s := (\boldsymbol{r} - \boldsymbol{i}) \circ \boldsymbol{r}_s^{-1} = \boldsymbol{i} - \boldsymbol{s}_s$. It is well known that $\boldsymbol{s}_s$ is a Lipschitz map with Lipschitz constant bounded by $(1-s)^{-1}$ and that the curve $\mu_s := (\boldsymbol{r}_s)_\#\lambda$ belongs to $\mathcal{CE}_\Omega(\lambda \to \mu)$ with

$$\mu_s = \bar{\rho}_s \mathscr{L}^d|_{B_s}, \qquad \bar{\rho}_s = \chi_{B_s}\bar{\rho}(\boldsymbol{s}_s)J_s, \qquad J_s := \det \mathrm{D}\boldsymbol{s}_s, \qquad \boldsymbol{\nu}_s = \bar{\rho}_s \boldsymbol{v}_s \mathscr{L}^d.$$

Since the map $r \mapsto r/\mathsf{m}(r)$ is nondecreasing and $J_s \leqslant (1-s)^{-d}$, it follows that

$$\Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s) = \int_{B_s} \frac{\bar{\rho}_s^2}{\mathsf{m}(\bar{\rho}_s)}|\boldsymbol{v}_s|^2\,\mathrm{d}x = \int_B \frac{\bar{\rho}(y)J_s(\boldsymbol{r}_s(y))}{\mathsf{m}(\bar{\rho}(y)J_s(\boldsymbol{r}_s(y)))}\left|\boldsymbol{r}(y) - y\right|^2\bar{\rho}(y)\,\mathrm{d}y \tag{3.10}$$

$$\leqslant \frac{b(1-s)^{-d}}{\mathsf{m}(b(1-s)^{-d})}\int_B \left|\boldsymbol{r}(y) - y\right|^2\bar{\rho}(y)\,\mathrm{d}y = \frac{b(1-s)^{-d}}{\mathsf{m}(b(1-s)^{-d})}W_2^2(\lambda, \mu). \tag{3.11}$$

Taking the square root and applying (3.3), since

$$\int_0^1 \left(\frac{b(1-s)^{-d}}{\mathsf{m}(b(1-s)^{-d})}\right)^{1/2} \mathrm{d}s = \frac{b^{1/d}}{d} \int_b^{+\infty} \left(\frac{z}{\mathsf{m}(z)}\right)^{1/2} z^{-1-1/d} \, \mathrm{d}z = b^{1/d} \, K_{\mathsf{m},d}(b)$$

we get the estimate

$$\mathcal{W}_{\mathsf{m},\Omega}(\lambda,\mu) \leqslant b^{1/d} \, K_{\mathsf{m},d}(b) \, W_2(\lambda,\mu). \tag{3.12}$$

A completely analogous calculation with $\mu := \mu_0$ (resp. $\mu := \mu_1$) and $\mu_s = \mu_{0,s}$ (resp. $\mu_s = \mu_{1,s}$) shows that

$$\mathcal{W}_{\mathsf{m},\Omega}(\mu_{i,1-\varepsilon},\mu_i) \leqslant b^{1/d} K_{\mathsf{m},d}\big(b\varepsilon^{-d}\big) \, W_2(\lambda,\mu_i) \quad \forall \varepsilon > 0, \ i = 0, 1.$$

On the other hand, taking into account that the density of $\mu_{i,1-\varepsilon}$ is bounded by $b\varepsilon^{-d}$, we can apply (3.12) with $\mu_{0,1-\varepsilon}$ instead of λ, obtaining

$$\mathcal{W}_{\mathsf{m},\Omega}(\mu_{0,1-\varepsilon},\mu_{1,1-\varepsilon}) \leqslant b^{1/d} \varepsilon^{-1} K_{\mathsf{m},d}\big(b\varepsilon^{-d}\big) \, W_2(\mu_{0,1-\varepsilon},\mu_{1,1-\varepsilon}).$$

Therefore, the triangular inequality yields

$$\mathcal{W}_{\mathsf{m},\Omega}(\mu_0,\mu_1) \leqslant b^{1/d} K_{\mathsf{m},d}\big(b\varepsilon^{-d}\big)\big(W_2(\mu_0,\lambda) + W_2(\mu_1,\lambda) + \varepsilon^{-1} W_2(\mu_{0,1-\varepsilon},\mu_{1,1-\varepsilon})\big).$$

Applying this estimate to a sequence μ_n weakly* converging to μ (and therefore converging also with respect to W_2), since the corresponding geodesic interpolants with λ $\mu_{n,1-\varepsilon}$ converge to $\mu_{1-\varepsilon}$ as $n \to \infty$ with respect to W_2, we easily obtain

$$\limsup_{n\to\infty} \mathcal{W}_{\mathsf{m},\Omega}(\mu_n,\mu) \leqslant 2b^{1/d} \, K_{\mathsf{m},d}\big(b\varepsilon^{-d}\big) W_2(\mu,\lambda).$$

Since $\lim_{\varepsilon\downarrow 0} K_{\mathsf{m},b}(b\varepsilon^{-d}) = 0$, taking ε arbitrarily small, we conclude. □

In the next result we do not assume any particular condition on m, but we ask that $\mu_i \ll \mathscr{L}^d$ with densities satisfying some extra integrability assumptions.

Theorem 3.8. *Let Ω be a bounded, open convex set of $\mathbb{R}^d$ and assume that $M = +\infty$, $\mathfrak{m} > 0$. If the measures $\mu_i = \rho_i \mathscr{L}^d|_\Omega \in \mathcal{M}^+(\Omega,\mathfrak{m})$, $i = 0, 1$, satisfy*

$$\int_\Omega \frac{\rho_i(x)^2}{\mathsf{m}(\rho_i(x))} \, \mathrm{d}x < +\infty, \quad i = 0, 1, \tag{3.13}$$

then $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0,\mu_1) < +\infty$.

Proof. We argue as in the previous proof, keeping the same notation and observing that for $0 \leqslant s \leqslant 1/2$, (3.11) yields

$$\Phi_{\mathsf{m},\Omega}(\mu_s,\boldsymbol{\nu}_s) \leqslant \frac{b\,2^d}{\mathsf{m}(b\,2^d)} W_2^2(\lambda,\mu). \tag{3.14}$$

When $1/2 \leqslant s \leqslant 1$ we invert the role of λ and $\mu = \rho\mathscr{L}^d$ in (3.10) obtaining

$$\Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{v}_s) = \int_\Omega \frac{\rho(y)\tilde{J}_s(\tilde{\boldsymbol{s}}_s(y))}{\mathsf{m}(\rho(y)\tilde{J}_s(\tilde{\boldsymbol{s}}_s(y)))}\big|\tilde{\boldsymbol{s}}(y) - y\big|^2\rho(y)\,\mathrm{d}y \tag{3.15}$$

where $\tilde{\boldsymbol{s}}_s = (1-s)\boldsymbol{s} + s\boldsymbol{i}$ is the optimal map pushing μ onto μ_s and $\tilde{J}_s = \det \mathrm{D}\tilde{\boldsymbol{s}}_s^{-1}$ satisfies $\tilde{J}_s \leqslant s^{-d}$. (3.15) then yields for $1/2 \leqslant s \leqslant 1$,

$$\Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{v}_s) \leqslant 2^{d+1}\int_\Omega \frac{\rho(y)^2}{\mathsf{m}(\rho(y))}\big(\big|\tilde{\boldsymbol{s}}(y)\big|^2 + |y|^2\big)\,\mathrm{d}y. \tag{3.16}$$

Since the range of $\tilde{\boldsymbol{s}}(y)$ is μ-essentially bounded, the integral in (3.16) is finite thanks to (3.13). Integrating (3.14) in $(0, 1/2)$ and (3.16) in $(1/2, 1)$ we conclude that $\mathcal{W}_{\mathsf{m},\Omega}(\lambda, \mu)$ is finite. □

4. Geodesic convexity of integral functionals and their gradient flows

4.1. Nonlinear diffusion equations: weak and limit solutions

We consider a

$$\text{convex } \textit{density function } U \in W^{2,1}_{\mathrm{loc}}(0, M) \text{ with } \mathsf{m}U'' \in L^1_{\mathrm{loc}}\big([0, M)\big) \tag{4.1a}$$

and a *pressure function* $P : [0, M) \to \mathbb{R}$ defined by

$$P(r) := \int_0^r \mathsf{m}(z)U''(z)\,\mathrm{d}z. \tag{4.1b}$$

Let us observe that $P \in W^{1,1}_{\mathrm{loc}}([0, M))$ is nondecreasing, continuous, and $P(0) = 0$. When U has a superlinear growth at $+\infty$ the corresponding *internal energy functional* $\mathscr{U} : D(\mathscr{U}) \subset \mathcal{M}^+_{\mathrm{c}}(\mathbb{R}^d) \to (-\infty, +\infty]$ is defined as

$$\mathscr{U}(\mu) := \int_{\mathbb{R}^d} U\big(\rho(x)\big)\,\mathrm{d}x, \qquad D(\mathscr{U}) := \big\{\mu = \rho\mathscr{L}^d \in \mathcal{M}^+_{\mathrm{c}}\big(\mathbb{R}^d\big)\colon\ U(\rho) \in L^1\big(\mathbb{R}^d\big)\big\}. \tag{4.2}$$

Since U is bounded from below by a linear function and μ has compact support, the integral in (4.2) is always well defined. $\mathscr{U}$ is lower semicontinuous with respect to weak convergence in $\mathcal{M}^+_{\mathrm{c}}(\mathbb{R}^d)$ if and only if

$$U'(+\infty) := \lim_{r\uparrow+\infty}\frac{U(r)}{r} = \lim_{r\uparrow+\infty}U'(r) = +\infty.$$

When $U'(+\infty) < +\infty$ we define the functional $\mathscr{U}$ as

$$\mathscr{U}(\mu) := \int_{\mathbb{R}^d} U(\rho)\,\mathrm{d}x + U'(+\infty)\mu^\perp\big(\mathbb{R}^d\big), \quad \mu = \rho\mathscr{L}^d + \mu^\perp,$$

where $\mu^\perp$ is the singular part of μ in the usual Lebesgue decomposition.

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, and connected set with Lipschitz boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}$. We will often suppose that Ω is convex in the sequel. We consider the homogeneous Neumann boundary value problem for the nonlinear diffusion equation

$$\partial_t \rho - \Delta P(\rho) = 0 \quad \text{in } (0,+\infty)\times\Omega, \qquad \partial_{\boldsymbol{n}} P(\rho) = 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{4.3}$$

with *nonnegative* initial condition $\rho(0,\cdot) = \rho_0$. We also introduce the dissipation rate of $\mathscr{U}$ along the flow by

$$\mathscr{D}(\rho) = \int_\Omega \frac{|\nabla P(\rho)|^2}{\mathsf{m}(\rho)}\,\mathrm{d}x = \int_\Omega \phi_{\mathsf{m}}\big(\rho, \nabla P(\rho)\big)\,\mathrm{d}x \quad \forall 0 \leqslant \rho \in L^1(\Omega),\ P(\rho) \in W^{1,1}(\Omega). \tag{4.4}$$

We collect in the following result some well established facts [41] on weak and classical solutions to (4.3).

Theorem 4.1 *(Very weak and classical solutions). Let us suppose that Ω is bounded and $\rho_0 \in L^\infty(\Omega)$. There exists a unique solution $\rho \in L^\infty((0,+\infty)\times\Omega) \cap C^0([0,+\infty); L^1(\Omega))$ with $P(\rho) \in L^\infty((0,+\infty)\times\Omega) \cap L^2((0,+\infty); W^{1,2}(\Omega))$ to* (4.3) *satisfying the following weak formulation*

$$\int_0^{+\infty}\int_\Omega \big(\rho\partial_t\zeta - \nabla P(\rho)\cdot\nabla\zeta\big)\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \forall \zeta \in C^\infty_{\mathrm{c}}\big((0,+\infty)\times\mathbb{R}^d\big), \tag{4.5}$$

and the initial condition $\rho(0,\cdot) = \rho_0$. The energy $\mathscr{U}$ is decreasing along the flow and satisfies the identity

$$\int_\Omega U\big(\rho(T,x)\big)\,\mathrm{d}x + \int_0^T\int_\Omega \frac{|\nabla P(\rho)|^2}{\mathsf{m}(\rho)}\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega U\big(\rho_0(x)\big)\,\mathrm{d}x \quad \forall T > 0. \tag{4.6}$$

The map $\rho_0 \mapsto S_t\rho_0 := \rho(t,\cdot)$ can be extended to a C^0 contraction semigroup $S = S(P,\Omega)$ in the positive cone of $L^1(\Omega)$, whose curves $S_t\rho_0$ are also called "limit L^1-solutions" of (4.3)*, and it satisfies*

$$\operatorname{ess\,inf}_\Omega \rho_0 \leqslant S_t\rho_0 \leqslant \operatorname{ess\,sup}_\Omega \rho_0.$$

If moreover $U, \mathsf{m} \in C^\infty(0,M)$, U is uniformly convex, Ω is smooth and $\inf_\Omega \rho_0 > 0$*, then $\rho \in C^\infty((0,+\infty)\times\bar{\Omega})$ and is a classical solution to* (4.3)*.*

Let us briefly discuss here two useful lemma, whose proof follows from a standard variational argument.

Lemma 4.2. *If ρ_0, $U(\rho_0) \in L^1(\Omega)$ then the limit L^1-solution $\rho = S(\rho_0)$ satisfies $P(\rho) \in L^1_{\mathrm{loc}}([0,+\infty); W^{1,1}(\Omega))$, the weak formulation* (4.5)*, and the energy inequality*

$$\int_\Omega U\big(\rho(T,x)\big)\,\mathrm{d}x + \int_0^T\int_\Omega \frac{|\nabla P(\rho)|^2}{\mathsf{m}(\rho)}\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_\Omega U\big(\rho_0(x)\big)\,\mathrm{d}x. \tag{4.7}$$

Proof. Let us first show that we can find a constant C depending only on P, $\omega := \mathscr{L}^d(\Omega)$, $\mathfrak{m} = \int_\Omega \rho\,\mathrm{d}x$, and the constant c_p in the Poincaré inequality for Ω such that

$$\begin{gathered}\big\|P(\rho)\big\|_{L^1(\Omega)} \leqslant C\big(1 + \big\|\nabla P(\rho)\big\|_{L^1(\Omega)}\big) \quad \forall\rho \in L^1(\Omega),\\ \int_\Omega \rho\,\mathrm{d}x = \mathfrak{m}, \qquad P(\rho) \in W^{1,1}(\Omega).\end{gathered} \tag{4.8}$$

In fact, setting $\mathsf{p} := \int_\Omega P(\rho)\,\mathrm{d}x$ and $\ell := \mathscr{L}^d(\{x \in \Omega\colon P(\rho) \geqslant \mathsf{p}/2\})$ Poincaré and Chebyshev inequality yield

$$\frac{1}{2}\mathsf{p}(\omega - \ell) \leqslant \int_\Omega \big|P(\rho) - \mathsf{p}\big|\,\mathrm{d}x \leqslant c_p \int_\Omega \big|\nabla P(\rho)\big|\,\mathrm{d}x, \quad \frac{1}{2}\mathsf{p} \leqslant P(\mathfrak{m}/\ell),$$

so that if $\ell \geqslant \omega/2$ we get $\mathsf{p} \leqslant 2P(2\mathfrak{m}/\omega)$, whereas if $\ell \leqslant \omega/2$ we obtain

$$\mathsf{p} \leqslant 4\omega^{-1} c_p \int_\Omega \big|\nabla P(\rho)\big|\,\mathrm{d}x.$$

If now $\rho_t = S_t\rho_0$ is the $L^1(\Omega)$-limit of a sequence $\rho_{n,t} = S_t\rho_{n,0}$ of bounded solutions with $U(\rho_{n,0}) \to U(\rho_0)$ in $L^1(\Omega)$ as $n \uparrow +\infty$, from the uniform bound (4.6) we obtain for every bounded Borel set $\mathcal{T} \subset (0,+\infty)$, every $B \subset \Omega$, and every nonnegative constants a, b such that $\mathsf{m}(r) \leqslant a + br$,

$$\begin{aligned}\int_{\mathcal{T}}\int_B \big|\nabla P(\rho_n)\big|\,\mathrm{d}x\,\mathrm{d}t &\leqslant \big\|\mathsf{m}(\rho_n)\big\|^{1/2}_{L^1(\mathcal{T}\times B)}\bigg(\int_{\mathcal{T}\times B} \frac{|\nabla P(\rho_n)|^2}{\mathsf{m}(\rho_n)}\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/2}\\ &\leqslant C\|a + b\rho_n\|^{1/2}_{L^1(\mathcal{T}\times B)}.\end{aligned}$$

Taking $\mathcal{T} = (0,T)$, $B = \Omega$ and applying (4.8), we obtain a uniform bound of the sequence $P(\rho_n)$ in $L^1(0,T; W^{1,1}(\Omega))$; since ρ_n converges to ρ in $L^1((0,T)\times\Omega)$, we obtain that $\nabla P(\rho_n)$ is uniformly integrable and therefore it converges weakly to $\nabla P(\rho)$ in $L^1((0,T)\times\Omega)$. It follows that $P(\rho) \in L^1(0,T; W^{1,1}(\Omega))$ and we can then pass to the limit in the weak formulation (4.5) written for ρ_n, obtaining the same identity for ρ. The inequality (4.7) eventually follows by the same limit procedure, recalling that the dissipation functional (4.4) is lower semicontinuous with respect to weak convergence in $L^1(\Omega)$. $\square$

The following stability result is used in the sequel; its proof is an easy adaption of [41, Prop. 6.10].

Proposition 4.3. *Let $\Omega^n \subset \mathbb{R}^d$ be a decreasing sequence of open, bounded, convex sets converging to Ω and let $S^n = S(P, \Omega^n)$, $S(P, \Omega)$ be the associated semigroups provided by Theorem* 4.1. *If (after a trivial extension to* 0 *outside Ω^n) $\rho_0^n \in L^1(\Omega^n)$ is converging strongly in $L^1(\mathbb{R}^d)$ to $\rho_0 \in L^1(\Omega)$, then $S_t^n(\rho_0^n) \to S_t(\rho_0)$ in the same L^1 sense, as $n \uparrow +\infty$ for every $t > 0$.*

4.2. The generalized McCann condition

Let us assume that $P'\mathsf{m}' \in L^1_{\mathrm{loc}}([0, M))$ and let us introduce a primitive function H of $h := P'\mathsf{m}' = U''\mathsf{m}\mathsf{m}'$,

$$H(r) := H_0 + \int_0^r P'(z)\mathsf{m}'(z)\,\mathrm{d}z \quad \text{for some } H_0 \geqslant 0. \tag{4.9}$$

Notice that in the most common case when $\mathsf{m}'(0_+) = \lim_{r\downarrow 0} r^{-1}\mathsf{m}(r) > 0$, the local integrability of h in a right neighborhood of 0 implies the local integrability of $\mathsf{m}\,U''$ (which we already required in (4.1a)) and the fact that P is bounded from below. These restrictions can be removed in the 1-dimensional case, see Remark 4.15.

Definition 4.4 *(Generalized McCann condition).* Let U, P, H and m be defined in the interval $(0, M)$ according to (4.1a), (4.1b) and (4.9). We say that the energy density U and the corresponding pressure function P satisfy the *d-dimensional generalized McCann condition for the mobility* m, denoted by *GMC*(m, d), if for a suitable choice of H_0

$$U''(r)\mathsf{m}^2(r) = P'(r)\mathsf{m}(r) \geqslant (1 - 1/d)H(r) \geqslant 0 \quad \forall r \in (0, M), \tag{4.10a}$$

or, equivalently,

$$r \mapsto \frac{H(r)}{\mathsf{m}^{1-1/d}(r)} \quad \text{is nonnegative and nondecreasing in } (0, +\infty). \tag{4.10b}$$

Before analyzing some properties related to *GMC*(m, d) let us consider in more detail the nonegativity condition of (4.10a) in dimension $d = 1$ and in the two distinct Cases A–B we introduced in Section 3.2.

Dimension $d = 1$. In the 1-dimensional the generalized McCann condition *GMC*$(\mathsf{m}, 1)$ reduces to the usual convexity of U: we will also comment on this issue in the next Remark 4.15.

Case A. $M = +\infty$ and $d > 1$. The minimal admissible choice for H corresponds to $H_0 = 0$ in (4.9). Notice that the existence of a nonnegative primitive of $h = P'\mathsf{m}'$ in $(0, +\infty)$ is in fact equivalent to its local integrability in a right neighborhood of 0.

Case B. $M < +\infty$ and $d > 1$. In this case we have to assume $h = P'\mathsf{m}' \in L^1(0, M)$ and we can choose

$$H_0 := \left(\int_0^M P'\mathsf{m}'\,\mathrm{d}x\right)^- < +\infty. \tag{4.11}$$

If moreover P is locally Lipschitz near 0 and M, and $\mathsf{m}(0) = \mathsf{m}(M) = 0$, then imposing the first inequality of (4.10a) at $r = 0_+$ yields $H_0 = 0$ and at $r = M_-$ yields the compatibility condition $\int_0^M P'\mathsf{m}'\,\mathrm{d}r = 0$.

We collect in the following remarks some simple properties related to this definition.

Remark 4.5 *(Elementary properties).*

1. (Linear mobility) (4.10a) is consistent with the usual McCann condition (1.2a) in the linear case of $\mathsf{m}(r) = r$.
2. (Dimension $d = 1$) As in the case of McCann condition, in space dimension $d = 1$ (4.10a) is equivalent to the convexity of U or to the monotonicity of P.
3. (Local boundedness of U when $d > 1$) In dimension $d > 1$ the energy density function U is bounded in a right neighborhood of 0 (and in a left neighborhood of M, in the case $M < +\infty$). Since $U'' = P'/\mathsf{m}$ the property is immediate if $\mathsf{m}(0) > 0$. If $\mathsf{m}(0) = 0$ then $\mathsf{m}'(0) > 0$ and therefore P is bounded around 0 and the formula

$$U(r) = U(r_0) + U'(r_0)(r - r_0) + \int_0^{r_0} \frac{(z-r)^+}{\mathsf{m}(z)} P'(z)\,\mathrm{d}z, \qquad r \in (0, r_0],$$

   shows that $\lim_{r\downarrow 0} U(r) < +\infty$.
4. (Constant mobility) When $\mathsf{m}(r) \equiv c > 0$ (4.10a) is still equivalent to the convexity of U.
5. (The power-like case) In the case of $P(r) = r^\gamma$ ($\gamma = \alpha + \beta - 1$ if $U(r) = r^\beta$) and $\mathsf{m}(r) = r^\alpha$, (4.10a) is satisfied if and only if

$$\gamma \geqslant 1 - \frac{\alpha}{d}. \tag{4.12}$$

6. (The case $P(r) = r$) It is immediate to check that the couple (r, m) always satisfies (4.10a): it corresponds to the entropy function U_m whose second derivative is m^{-1}. After fixing some $r_0 \in (0, M)$ (the choice $r_0 = 0$ is admissible if m^{-1} is integrable in a right neighborhood of 0), we obtain

$$U_\mathsf{m}(r) := \int_{r_0}^{r} \frac{r - z}{\mathsf{m}(z)}\,\mathrm{d}z, \qquad P_\mathsf{m}(r) = r - r_0.$$

7. (The case of the logarithmic entropy) $U(r) = r\log r$ satisfies $GMC(r^\alpha, d)$ if and only if $\gamma = \alpha \geqslant d/(d+1)$.
8. (Linearity) If P_1 and P_2 satisfy $GMC(\mathsf{m}, d)$ then also $\alpha_1 P_1 + \alpha_2 P_2$ satisfies $GMC(\mathsf{m}, d)$, for every $\alpha_1, \alpha_2 \geqslant 0$. Analogously, if P satisfies $GMC(\mathsf{m}_1, d)$ and $GMC(\mathsf{m}_2, d)$ then P satisfies $GMC(\alpha_1\mathsf{m}_1 + \alpha_2\mathsf{m}_2, d)$. In particular, if P satisfies $GMC(\mathsf{m}, d)$ then $P(r) + \alpha r$ satisfies $GMC(\mathsf{m} + \beta, d)$ for every $\alpha, \beta \geqslant 0$.

9. (Shift) If $M = +\infty$ and P satisfies $GMC(\mathsf{m}, d)$ then P satisfies $GMC(\mathsf{m}(\cdot + \alpha), d)$ and $P(\cdot - \alpha)$ satisfies $GMC(\mathsf{m}, d)$, for every $\alpha \geqslant 0$.

The next two properties are more technical and require a detailed proof.

Lemma 4.6 *(Smoothing). Let us assume that P satisfies $GMC(\mathsf{m}, d)$ and let us fix two constants $0 < M' < M'' < M$. Then there exists a family P_η, m_η, $\eta > 0$, with smooth restriction to $[M', M'']$ such that $P_\eta \geqslant P$ is strictly increasing, $\mathsf{m}_\eta \geqslant \mathsf{m}$ is concave, P_η satisfies $GMC(\mathsf{m}_\eta, d)$ (in $[0, M'']$), and P_η, m_η converge uniformly to P, m in $[M', M'']$ as $\eta \downarrow 0$. Moreover, if P' is locally integrable in a right neighborhood of 0, then we can choose $M' = 0$.*

Proof. When $M' > 0$ it is not restrictive (up to choosing a smaller M') to assume that M' is a Lebesgue point of the derivative of P. Let H be as in (4.9) and let us set $\tilde{\mathsf{m}}_\eta(r) := \mathsf{m}(r) + \eta$, $\tilde{P}_\eta(r) = P(r) + \eta r$,

$$\tilde{H}_\eta(r) = H_0 + \eta\mathsf{m}(0) + \eta^2 + \int_0^r \tilde{P}_\eta'(r)\tilde{\mathsf{m}}_\eta'(r)\,\mathrm{d}r = H(r) + \eta\mathsf{m}(r) + \eta^2 \geqslant \eta^2 > 0.$$

By the previous remark (points 6 and 8) $\tilde{P}_\eta$ satisfies $GMC(\tilde{\mathsf{m}}_\eta, d)$ and moreover

$$\tilde{P}_\eta'\tilde{\mathsf{m}}_\eta - (1 - 1/d)\tilde{H}_\eta = P'\mathsf{m} - (1 - 1/d)H + \frac{\eta}{d}(\eta + \mathsf{m}) \geqslant \frac{\eta}{d}(\mathsf{m} + \eta) \geqslant \eta^2/d. \tag{4.13}$$

By choosing a family of mollifiers h_δ, $\delta > 0$, with support in $[0, \delta]$, we introduce the functions

$$\tilde{P}_{\eta,\delta}(r) := \begin{cases} \tilde{P}_\eta(M') + \int_{M'}^r \tilde{P}_\eta' * h_\delta\,\mathrm{d}s & \text{if } r \geqslant M', \\ \tilde{P}_\eta(r) & \text{if } r < M', \end{cases}$$

$$\tilde{\mathsf{m}}_{\eta,\delta}(r) := \begin{cases} \tilde{\mathsf{m}}_\eta(M') + \int_{M'}^r \tilde{\mathsf{m}}_\eta' * h_\delta\,\mathrm{d}s & \text{if } r \geqslant M', \\ \tilde{\mathsf{m}}_\eta(r) & \text{if } r < M', \end{cases}$$

which are smooth in $[M', M'']$ and satisfy the requested monotonicity/concavity conditions. Since $\tilde{P}_{\eta,\delta}'$ converges to $\tilde{P}_\eta'$ in $L^1_{\mathrm{loc}}(0, M']$ and $\tilde{\mathsf{m}}_{\eta,\delta}$ is uniformly bounded and converges pointwise a.e. to $\tilde{\mathsf{m}}_\eta'$ as $\delta \to 0$, we conclude that the corresponding continuous functions $\tilde{H}_{\eta,\delta}$ converge uniformly to $\tilde{H}_\eta$ as $\delta \downarrow 0$. By (4.13), we can find a sufficiently small $\delta = \delta_\eta$ depending on η such that

$$\tilde{P}_{\eta,\delta_\eta}'\tilde{\mathsf{m}}_{\eta,\delta_\eta}' \geqslant (1 - 1/d)\tilde{H}_{\eta,\delta_\eta} \geqslant 0.$$

A standard diagonal argument concludes the proof. □

Lemma 4.7 *(Minimal asymptotic behaviour). When $d > 1$ and $M = +\infty$, the function $P_{\min}(r) := \int_0^r \mathsf{m}(z)^{-1/d}\,\mathrm{d}z$ satisfies $GMC(\mathsf{m}, d)$ and provides an (asymptotic) lower bound for every any other P, since for every $r_0 > 0$ there exists a constant $c_0 > 0$ such that*

$$P'(r) \geqslant c_0 P_{\min}'(r) = c_0\mathsf{m}(r)^{-1/d}, \qquad U''(r) \geqslant c_0\mathsf{m}(r)^{-1-1/d} \quad \textit{for a.e. } r \geqslant r_0. \tag{4.14}$$

Proof. In fact $f(r) := P'(r)\mathsf{m}(r)$ satisfies

$$f(r) \geqslant (1-1/d)\left(H(r_0) + \int_{r_0}^{r} f(r)\mathsf{m}'(r)/\mathsf{m}(r)\,\mathrm{d}r\right).$$

Gronwall Lemma then yields (4.14) with $c_0 := (1-1/d)H(r_0)\mathsf{m}(r_0)^{1/d-1}$. $\square$

Notice that in the case $\mathsf{m}(r) = r^\alpha$ we obtain the functions $P_{\min}(r) = c\,r^{\gamma_0}$ with exponent $\gamma_0 = 1-\alpha/d$, which is consistent with (4.12). The corresponding energy density functions are then $U_{\min}(r) = cr^{2-\alpha(1+1/d)}$: in particular, when $\alpha < d/(d+1)$, all the energy functions have a superlinear growth as $r \uparrow \infty$.

Remark 4.8 *(A sufficient condition).* It is possible to give a simpler sufficient condition than (4.10a), at least when $\mathsf{m}U''$ is integrable in a right neighborhood of 0 and $M = +\infty$: if

$$\text{the map } r \mapsto \mathsf{m}^{1/d}(r)P'(r) = \mathsf{m}^{1+1/d}(r)U''(r) \text{ is nondecreasing in } (0,+\infty) \tag{4.15}$$

then (4.10a) is satisfied. In fact, assuming U smooth for simplicity, (4.15) is equivalent to

$$0 \leqslant \mathsf{m}^{1/d}P'' + 1/d\,\mathsf{m}^{1/d-1}\mathsf{m}'P'.$$

Multiplying the inequality by $\mathsf{m}^{1-1/d}$ and integrating from 0 to r we get (4.10a). Condition (4.15) gives the same sharp bound (4.12) in the power case.

4.3. The metric approach to gradient flows

We recall here some basic facts about the metric notion of gradient flows, referring to [2] for further details. Let $(D, \mathcal{W})$ be a metric space, not assumed to be complete, and let $\mathscr{V} : D(\mathscr{V}) \to (-\infty, +\infty]$ be a lower semicontinuous functional. A family of continuous maps $\mathsf{S}_t : D \to D$, $t \geqslant 0$, is a C^0-*(metric) contraction gradient flow of* $\mathscr{V}$ with respect to $\mathcal{W}$ if

$$\mathsf{S}_{t+h}(u) = \mathsf{S}_h\big(\mathsf{S}_t(u)\big), \qquad \lim_{t\downarrow 0}\mathsf{S}_t(u) = \mathsf{S}_0(u) = u \quad \forall u \in D,\ t,h \geqslant 0, \tag{4.16a}$$

$$\frac{1}{2}\mathcal{W}^2\big(\mathsf{S}_t(u), v\big) - \frac{1}{2}\mathcal{W}^2(u,v) \leqslant t\big(\mathscr{V}(v) - \mathscr{V}\big(\mathsf{S}_t(u)\big)\big) \quad \forall t > 0,\ u \in D,\ v \in D(\mathscr{V}). \tag{4.16b}$$

Thanks to [16, Prop. 3.1], conditions (4.16a), (4.16b) imply

$$\mathsf{S}_t(D) \subset D(\mathscr{V})\ \forall t > 0 \text{ and the map } t \mapsto \mathscr{V}\big(\mathsf{S}_t(u)\big) \text{ is not increasing in } (0,+\infty),$$

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\mathcal{W}^2\big(\mathsf{S}_t(u), v\big) + \mathscr{V}\big(\mathsf{S}_t(u)\big) \leqslant \mathscr{V}(v) \quad \forall u \in D,\ v \in D(\mathscr{V}),\ t \geqslant 0, \tag{4.17}$$

$$\mathscr{V}\big(\mathsf{S}_t(u)\big) \leqslant \mathscr{V}(v) + \frac{1}{2t}\mathcal{W}^2(u,v) \quad \forall u \in D,\ v \in D(\mathscr{V}),\ t > 0,$$

$$\mathcal{W}^2\big(\mathsf{S}_{t_1}(u), \mathsf{S}_{t_0}(u)\big) \leqslant 2(t_1 - t_0)\big(\mathscr{V}(\mathsf{S}_{t_0}u) - \mathscr{V}_{\inf}\big) \quad \forall u \in D(\mathscr{V}),\ 0 \leqslant t_0 \leqslant t_1, \tag{4.18}$$

$$\mathcal{W}\big(\mathsf{S}_t(u), \mathsf{S}_t(v)\big) \leqslant \mathcal{W}(u,v) \quad \forall u, v \in D,\ t \geqslant 0.$$

In (4.17) we used the usual notation

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\zeta(t) = \limsup_{h\to 0^+} \frac{\zeta(t+h)-\zeta(t)}{h}$$

for every real function $\zeta : [0,+\infty) \to \mathbb{R}$.
The following approximated convexity estimate [16, Theorem 3.2] plays an important role in the sequel.

Theorem 4.9 *(Approximated convexity). Let us suppose that* S *is metric contraction gradient flow of* $\mathscr{V}$ *with respect to* $\mathcal{W}$ *according to* (4.16a), (4.16b) *and let* $s \mapsto u_s \in D$, $s \in [0,1]$, *be a Lipschitz* (*"almost" geodesic*) *curve such that* $u_0, u_1 \in D(\mathscr{V})$ *and*

$$\mathcal{W}(u_r,u_s) \leqslant L|r-s| \quad \forall r,s \in [0,1], \qquad L^2 \leqslant \mathcal{W}^2(u_0,u_1)+\delta^2. \tag{4.19}$$

Then for every $s \in [0,1]$ *and* $t > 0$, *we have*

$$\mathscr{V}\big(\mathsf{S}_t(u_s)\big) \leqslant (1-s)\mathscr{V}(u_0) + s\mathscr{V}(u_1) + \frac{s(1-s)}{2t}\delta^2.$$

In particular, if u_s *is a minimal geodesic, i.e.* (4.19) *holds with* $\delta = 0$, *then*

$$\mathscr{V}(u_s) \leqslant (1-s)\mathscr{V}(u_0) + s\mathscr{V}(u_1) \quad \forall s \in [0,1].$$

4.4. Main results

We state our main result about the generation of a contractive gradient flow of $\mathscr{U}$ with respect to $\mathcal{W}_{\mathsf{m},\Omega}$.

Theorem 4.10 *(Contractive gradient flow). Let us assume that* Ω *is a bounded, convex open set, and the functions* U, P, H *satisfy the generalized McCann condition* $GMC(\mathsf{m},d)$. *For every reference measure* $\sigma \in \mathcal{M}^+(\Omega)$ *with finite energy* $\mathscr{U}(\sigma) < +\infty$ *the functional* $\mathscr{U}$ *generates a unique metric contraction gradient flow* $\mathcal{S} = \mathcal{S}(\mathscr{U},\mathsf{m},\Omega)$ *in the space*

$$D := \big\{\mu \in \mathcal{M}^+(\Omega):\ \mu \ll \mathscr{L}^d|_\Omega,\ \mathcal{W}_{\mathsf{m},\Omega}(\mu,\sigma) < +\infty,\ \mathscr{U}(\mu) < +\infty\big\}$$

endowed with the distance $\mathcal{W}_{\mathsf{m},\Omega}$. *Moreover* $\mathcal{S}$ *is characterized by the formula* $\mathcal{S}_t\mu_0 = \rho_t\mathscr{L}^d|_\Omega$, *where* $\rho_t = S_t\rho_0$ *is a limit* L^1*-solution of* (4.3).

When m satisfies the finiteness condition of Theorem 3.7 (in particular $\mathsf{m}(r) = r^\alpha$ with $\alpha > 1 - 2/d$) we obtain a much more refined result, which in particular shows the continuous dependence of $\mathcal{S}$ on the weak* topology.

Corollary 4.11. *Under the same assumptions on* Ω, U, P *of the previous theorem, if moreover* $M = +\infty$ *and* m *satisfies the finiteness condition of Theorem* 3.7, *then the semigroup* $\mathcal{S}$ *can be uniquely extended to a contraction semigroup on every convex set* $\mathcal{M}^+(\bar{\Omega},\mathfrak{m})$, *which is continuous with respect to the weak* convergence of the initial data. If* U *has a superlinear growth, then* $\mathcal{S}_t(\mu_0) = \rho_t\mathscr{L}^d \ll \mathscr{L}^d|_\Omega$ *for every* $t > 0$ *and* ρ_t *is a weak solution of* (4.3) *according to* (4.5).

We conclude this section with our main convexity result.

Theorem 4.12 *(Convexity). Let us assume that Ω is a bounded convex open set, and the functions U, P, H satisfy the generalized McCann condition $GMC(\mathsf{m}, d)$. For every $\mu_0, \mu_1 \in \mathcal{M}^+(\Omega) \cap D(\mathscr{U})$ with finite distance $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) < +\infty$ there exists a constant speed minimizing geodesic for $\mathcal{W}_{\mathsf{m},\Omega}$, $\mu : [0, 1] \to \mathcal{M}^+(\Omega)$ connecting μ_0 to μ_1 such that*

$$\mathscr{U}(\mu_s) \leqslant s\mathscr{U}(\mu_1) + (1-s)\mathscr{U}(\mu_0) \quad \forall s \in [0, 1]. \tag{4.20}$$

The proof of Theorems 4.10 and 4.12 will be developed in the next two sections.

Remark 4.13 *(Weak and strong convexity).* When (4.20) holds for all the (constant speed, minimizing) geodesics, the functional $\mathscr{U}$ is called *strongly* geodesically convex. When m is strictly concave and has a sublinear growth (or $M < +\infty$) then every two measures with finite $\mathcal{W}_{\mathsf{m},\Omega}$-distance can be connected by a *unique* geodesic [19, Theorem 5.11], so that there is no difference between strong or weak convexity and (4.20) yields that the map $s \mapsto \mathscr{U}(\mu_s)$ is convex in [0, 1].

Remark 4.14 *(Absolutely continuous measures).* Even when geodesics are not unique, the proof of Theorem 4.12 shows in fact that (4.20) is satisfied by *any* geodesic μ_s with $\mu_s \ll \mathscr{L}^d$ for every $s \in \mathbb{R}^d$, which surely exist if U has a superlinear growth. Along this class of geodesics we still obtain that the map $s \mapsto \mathscr{U}(\mu_s)$ is convex in [0, 1].

Remark 4.15 *(The one-dimensional case).* When the space dimension $d = 1$, then the generalized McCann condition $GMC(\mathsf{m}, 1)$ reduces to the usual convexity of U. In this case, a simple approximation argument shows that we can cover also the case of functions U which are not bounded in a right neighborhood of 0 (and in a left neighborhood of M, if $M < +\infty$) and the integrability assumptions on $U''\mathsf{m}$ of (4.1a) and on $U''\mathsf{m}\mathsf{m}'$ of (4.11) can be dropped.

5. Action inequalities in the smooth case

This Section contains the proof of the action inequality in the smooth case. This is a core of the proof of the main result. Indeed, in the next Section, the proof of Theorem 4.10 will be obtained by using the approximation results of Lemma 3.6 and 4.6, and passing to the limit on the inequality (5.13).

In this section we assume that Ω is a *smooth and bounded open set*. We consider a smooth curve

$$\mu_s := \rho_s \mathscr{L}^d|_\Omega, \quad \rho \in C^\infty\big([0, 1] \times \bar{\Omega}\big),\ 0 < m_0 \leqslant \rho \leqslant m_1 < M, \\ \mu_s(\Omega) \equiv \mathfrak{m}, \quad s \in [0, 1]. \tag{5.1}$$

We also assume that P and m are of class C^∞ in $[m_0, m_1]$. We consider the semigroup $S = S(P, \Omega)$ defined by Theorem 4.1 and we set

$$\mu_{s,t} := \rho_{s,t}\mathscr{L}^d|_\Omega, \qquad \rho_{s,t}(\cdot) = \rho(s, t, \cdot) := S_{st}\rho_s, \quad s \in [0, 1],\ t \geqslant 0. \tag{5.2}$$

Classical theory of quasilinear parabolic equation shows that $\rho \in C^\infty([0, 1] \times [0, \infty) \times \Omega) \cap C^\infty([0, 1] \times (0, +\infty) \times \bar{\Omega})$.

Since the semigroup S_t preserve the lower and upper bounds on ρ and $\int_\Omega \partial_s\rho\,\mathrm{d}x = 0$, for every $(s,t) \in [0,1] \times [0,+\infty)$ we can introduce the unique solution $\zeta_{s,t} = \zeta(s,t,\cdot) \in C^\infty(\bar{\Omega})$, of the uniformly elliptic Neumann boundary value problem

$$\begin{cases} -\nabla\cdot\big(\mathsf{m}(\rho)\nabla\zeta\big) = \partial_s\rho & \text{in } \Omega,\\ \nabla\zeta\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \zeta(x)\,\mathrm{d}x = 0. \end{cases} \tag{5.3}$$

It is easy to check that ζ depends smoothly on s and t. Notice that (5.3) is equivalent to

$$\int_\Omega \mathsf{m}(\rho)\,\nabla\zeta\cdot\nabla\eta\,\mathrm{d}x = \int_\Omega \partial_s\rho\,\eta\,\mathrm{d}x \quad \forall\eta \in C^1(\bar{\Omega}). \tag{5.4}$$

By construction, for every $t \geqslant 0$ the curve $s \mapsto (\mu_{s,t}, \boldsymbol{\nu}_{s,t})$ with $\boldsymbol{\nu}_{s,t} := \mathsf{m}(\rho_{s,t})\nabla\zeta_{s,t}\mathscr{L}^d|_\Omega$ belongs to $\mathcal{CE}_\Omega(\mu_0 \to \mu_{1,t})$ and its energy can be evaluated by integrating the action

$$A_{s,t} := \Phi_{\mathsf{m},\Omega}(\mu_{s,t}, \boldsymbol{\nu}_{s,t}) = \int_\Omega \mathsf{m}(\rho_{s,t})|\nabla\zeta_{s,t}|^2\,\mathrm{d}x$$

with respect to s in the interval $[0,1]$. The integral provides an upper bound of the $\mathcal{W}_{\mathsf{m},\Omega}$-distance between μ_0 and $\mu_{1,t} = \rho_{1,t}\mathscr{L}^d$, which corresponds to the solution of the nonlinear diffusion equation (4.3) with initial datum ρ_1. As it was shown in [16], evaluating the time derivative of the action $A_{s,t}$ is a crucial step to prove that (4.3) satisfies the EVI formulation (4.16b). Next lemma, which does not require any convexity assumption on Ω, collects the main calculations.

Lemma 5.1. *Let ρ_s, $\rho_{s,t}$, and $\zeta_{s,t}$ be as in* (5.1), (5.2), *and* (5.3). *Then for every $(s,t) \in [0,1] \times (0,+\infty)$ we have*

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}A_{s,t} &= \frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |\nabla\zeta_{s,t}|^2\mathsf{m}(\rho_{s,t})\,\mathrm{d}x \\ &= -\int_\Omega \nabla P(\rho_{s,t})\cdot\nabla\zeta_{s,t}\,\mathrm{d}x \\ &\quad - s\int_\Omega \Big(\big(P'(\rho_{s,t})\mathsf{m}(\rho_{s,t}) - H(\rho_{s,t})\big)(\Delta\zeta_{s,t})^2 + H(\rho_{s,t})\big|D^2\zeta_{s,t}\big|^2\Big)\,\mathrm{d}x \\ &\quad + s\int_\Omega P'(\rho_{s,t})\mathsf{m}''(\rho_{s,t})|\nabla\rho_{s,t}|^2|\nabla\zeta_{s,t}|^2\,\mathrm{d}x \\ &\quad + \frac{1}{2}s\int_{\partial\Omega} H(\rho_{s,t})\nabla|\nabla\zeta_{s,t}|^2\cdot\boldsymbol{n}\,d\mathscr{H}^{d-1}, \end{aligned} \tag{5.5}$$

where H is defined in (4.9).

Proof. For keep the notation simple, we omit the explicit dependence of ρ, ζ on s, t. By the definition of ρ we easily get

$$\partial_t \rho = s\Delta P(\rho) \quad \text{and} \quad s\int_\Omega \nabla P(\rho)\cdot\nabla\eta \,\mathrm{d}x = -\int_\Omega \partial_t\rho\,\eta\,\mathrm{d}x \quad \forall\eta\in C^1(\bar{\Omega}). \tag{5.6}$$

Further differentiation with respect to s yields

$$-\int_\Omega \nabla P(\rho)\cdot\nabla\eta\,\mathrm{d}x + s\int_\Omega \partial_s P(\rho)\Delta\eta\,\mathrm{d}x = \int_\Omega \partial_{st}\rho\eta\,\mathrm{d}x \tag{5.7}$$

for all $\eta\in C^2(\bar{\Omega})$ with $\nabla\eta\cdot\boldsymbol{n}=0$ on $\partial\Omega$. On the other hand, differentiating (5.4) with respect to t we obtain

$$\int_\Omega \mathsf{m}(\rho)\partial_t\nabla\zeta\cdot\nabla\eta\,\mathrm{d}x = \int_\Omega \partial_{st}\rho\,\eta\,\mathrm{d}x - \int_\Omega \partial_t\mathsf{m}(\rho)\nabla\zeta\cdot\nabla\eta\,\mathrm{d}x \quad \forall\eta\in C^1(\bar{\Omega}). \tag{5.8}$$

The time derivative of the action functional is

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |\nabla\zeta|^2\mathsf{m}(\rho)\,\mathrm{d}x &= \frac{1}{2}\int_\Omega \partial_t\mathsf{m}(\rho)|\nabla\zeta|^2\,\mathrm{d}x + \int_\Omega \partial_t\nabla\zeta\cdot\nabla\zeta\,\mathsf{m}(\rho)\,\mathrm{d}x \\
&\overset{(5.8)}{=} \int_\Omega \partial_{st}\rho\zeta\,\mathrm{d}x - \frac{1}{2}\int_\Omega \partial_t\mathsf{m}(\rho)|\nabla\zeta|^2\,\mathrm{d}x \\
&\overset{(5.7)}{=} -\int_\Omega \nabla P(\rho)\cdot\nabla\zeta\,\mathrm{d}x + s\int_\Omega \partial_s P(\rho)\Delta\zeta\,\mathrm{d}x - \frac{1}{2}\int_\Omega \partial_t\mathsf{m}(\rho)|\nabla\zeta|^2\,\mathrm{d}x \\
&= I + II + III.
\end{aligned} \tag{5.9}$$

We evaluate separately the various contributions: concerning the second integral II we introduce the auxiliary function G

$$G(r) := P'(r)\mathsf{m}(r) - H(r), \quad \text{so that} \quad G'(r) = P''(r)\mathsf{m}(r), \tag{5.10}$$

and we get

$$\begin{aligned}
II &= s\int_\Omega \partial_s P(\rho)\Delta\zeta\,\mathrm{d}x \\
&= s\int_\Omega P'(\rho)\partial_s\rho\Delta\zeta\,\mathrm{d}x \\
&\overset{(5.4)}{=} s\int_\Omega P'(\rho)\mathsf{m}(\rho)\nabla\Delta\zeta\cdot\nabla\zeta\,\mathrm{d}x + s\int_\Omega \Delta\zeta\,P''(\rho)\mathsf{m}(\rho)\nabla\rho\cdot\nabla\zeta\,\mathrm{d}x
\end{aligned}$$

$$\overset{(5.10)}{=} s\int_\Omega P'(\rho)\mathsf{m}(\rho)\nabla\Delta\zeta\cdot\nabla\zeta\,\mathrm{d}x + s\int_\Omega \Delta\zeta\nabla G(\rho)\cdot\nabla\zeta\,\mathrm{d}x$$
$$= s\int_\Omega H(\rho)\nabla\Delta\zeta\cdot\nabla\zeta\,\mathrm{d}x - s\int_\Omega G(\rho)(\Delta\zeta)^2\,\mathrm{d}x.$$

The third integral of (5.9) is

$$\begin{aligned} III &= -\frac{1}{2}\int_\Omega \mathsf{m}'(\rho)\partial_t\rho|\nabla\zeta|^2\,\mathrm{d}x \\ &\overset{(5.6)}{=} \frac{s}{2}\int_\Omega P'(\rho)\mathsf{m}''(\rho)|\nabla\rho|^2|\nabla\zeta|^2\,\mathrm{d}x + \frac{s}{2}\int_\Omega P'(\rho)\mathsf{m}'(\rho)\nabla|\nabla\zeta|^2\cdot\nabla\rho\,\mathrm{d}x \\ &\overset{(4.9)}{=} \frac{s}{2}\int_\Omega P'(\rho)\mathsf{m}''(\rho)|\nabla\rho|^2|\nabla\zeta|^2\,\mathrm{d}x + \frac{s}{2}\int_\Omega \nabla|\nabla\zeta|^2\cdot\nabla H(\rho)\,\mathrm{d}x. \end{aligned} \tag{5.11}$$

A further integration by parts in the last integral and the Bochner formula

$$\nabla\zeta\cdot\nabla\Delta\zeta - \frac{1}{2}\Delta|\nabla\zeta|^2 = -|D^2\zeta|^2$$

yield (5.5). □

Corollary 5.2. *Under the same notation and assumptions of Lemma* 5.1*, if Ω is convex and U satisfies the generalized McCann condition $GMC(\mathsf{m}, d)$* (4.10a), (4.10b)*, then*

$$\frac{1}{2}\frac{\partial}{\partial t}A_{s,t} = \frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |\nabla\zeta_{s,t}|^2\mathsf{m}(\rho_{s,t})\,\mathrm{d}x \leqslant -\frac{\partial}{\partial s}\mathscr{U}(\mu_{s,t}) = -\frac{\partial}{\partial s}\int_\Omega U(\rho_{s,t})\,\mathrm{d}x, \tag{5.12}$$

and

$$\begin{aligned} \frac{1}{2}\mathcal{W}^2_{\mathsf{m},\Omega}(\mu_{1,t},\mu_0) + t\mathscr{U}(\mu_{1,t}) &\leqslant \frac{1}{2}\int_0^1 A_{s,t}\,\mathrm{d}s + \int_0^t \mathscr{U}(\mu_{1,\tau})\,\mathrm{d}\tau \\ &\leqslant \frac{1}{2}\int_0^1 A_{s,0}\,\mathrm{d}s + t\mathscr{U}(\mu_0). \end{aligned} \tag{5.13}$$

Proof. We determine the sign of the terms in the right-hand side of (5.5) thanks to (4.10a), (4.10b) and the convexity of Ω. Recalling that $|D^2\zeta| \geqslant \frac{1}{d}(\Delta\zeta)^2$ and $H \geqslant 0$ we obtain that the second integral in the right-hand side of (5.5) is nonpositive

$$-\int_\Omega \big(\big(P'(\rho)\mathsf{m}(\rho) - H(\rho)\big)(\Delta\zeta)^2 + H(\rho)\big|D^2\zeta\big|^2\big)\,\mathrm{d}x \overset{(4.10a)}{\leqslant} 0. \tag{5.14}$$

Since P is increasing and m is concave, we have $P'(\rho)\mathsf{m}''(\rho) \leqslant 0$ which yields

$$\int_\Omega P'(\rho)\mathsf{m}''(\rho)|\nabla\rho|^2|\nabla\zeta|^2\,\mathrm{d}x \leqslant 0. \tag{5.15}$$

Since H is nonnegative, the smoothness and convexity of Ω and the smoothness of ζ yields, see [24,25,34],

$$\nabla|\nabla\zeta|^2 \cdot \boldsymbol{n} \leqslant 0, \quad \text{on } \partial\Omega, \quad \int_{\partial\Omega} H(\rho)\nabla|\nabla\zeta|^2 \cdot \boldsymbol{n}\, d\mathscr{H}^{d-1} \leqslant 0. \tag{5.16}$$

Combining (5.14), (5.15) and (5.16), (5.5) yields the inequality

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |\nabla\zeta|^2\mathsf{m}(\rho)\,\mathrm{d}x \leqslant -\int_\Omega \nabla\zeta\cdot\nabla P(\rho)\,\mathrm{d}x. \tag{5.17}$$

On the other hand

$$\frac{\partial}{\partial s}\int_\Omega U(\rho)\,\mathrm{d}x = \int_\Omega U'(\rho)\partial_s\rho\,\mathrm{d}x \overset{(5.4)}{=} \int_\Omega \mathsf{m}(\rho)\nabla U'(\rho)\cdot\nabla\zeta\,\mathrm{d}x = \int_\Omega \nabla P(\rho)\cdot\nabla\zeta\,\mathrm{d}x, \tag{5.18}$$

so that (5.12) follows by (5.17) and (5.18). Integrating (5.12) with respect to s and t, we obtain the second inequality in (5.13). The first inequality in (5.13) follows from the definition of $\mathcal{W}_{\mathsf{m},\Omega}$ and the monotonicity of $\tau \mapsto \mathscr{U}(\mu_{1,\tau})$ (see the energy identity (4.6)). □

6. Proof of the main theorems

6.1. The generation result

Recall that $\mathcal{S}_t(\mu_0) = S_t(\rho_0)\mathscr{L}^d|_\Omega$ when $\mu_0 = \rho_0\mathscr{L}^d|_\Omega$; Theorem 4.10 is an immediate consequence of the following result.

Theorem 6.1. *Let Ω be a bounded, convex open set of $\mathbb{R}^d$ and let us assume that $\mu_i \in \mathcal{M}^+(\bar{\Omega})$, $i = 0, 1$, have finite distance $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) < +\infty$ and satisfy $\mathscr{U}(\mu_0) < +\infty$ and $\mu_1 = \rho_1\mathscr{L}^d|_\Omega \ll \mathscr{L}^d$. If U satisfies the generalized McCann condition GMC(m, d) (4.10a), (4.10b) then*

$$\frac{1}{2}\mathcal{W}^2_{\mathsf{m},\Omega}(\mathcal{S}_t\mu_1, \mu_0) + t\mathscr{U}(\mathcal{S}_t\mu_1) \leqslant \frac{1}{2}\mathcal{W}^2_{\mathsf{m},\Omega}(\mu_1, \mu_0) + t\mathscr{U}(\mu_0) \quad \forall t \geqslant 0.$$

Proof. Since $\mathcal{W}_{\mathsf{m},\Omega}(\mu_0, \mu_1) < +\infty$ there exists a geodesic curve $(\mu, \boldsymbol{\nu}) \in \mathcal{CE}_\Omega(\mu_0 \to \mu_1)$ such that

$$\Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s) \equiv \int_0^1 \Phi_{\mathsf{m},\Omega}(\mu_s, \boldsymbol{\nu}_s)\,\mathrm{d}s = \mathcal{W}^2_{\mathsf{m},\Omega}(\mu_0, \mu_1).$$

Applying Lemma 3.6, we find a family of approximating curves $(\mu^{\varepsilon,\delta}, \boldsymbol{\nu}^{\varepsilon,\delta})$ and smooth convex open sets Ω_ε satisfying (3.7), (3.8) and

$$\int_0^1 \Phi_{\mathsf{m},\Omega_\varepsilon}\big(\mu^{\varepsilon,\delta}, \boldsymbol{\nu}^{\varepsilon,\delta}\big)\,\mathrm{d}s \leqslant c_\varepsilon^2 \mathcal{W}^2_{\mathsf{m},\Omega}(\mu_0,\mu_1).$$

Let m_η, P_η as in Lemma 4.6, for constants $0 < M' < M'' < M$ such that $M' \leqslant \rho^{\varepsilon,\delta} \leqslant M''$ in Ω_ε, and let $\zeta_s^{\varepsilon,\delta,\eta} \in C^\infty(\mathbb{R}^d)$ obtained by solving (5.3) with respect to m_η and $\partial_s \rho^{\varepsilon,\delta}$ in Ω_ε. Since $\mathsf{m}_\eta \geqslant \mathsf{m}$, by Theorem 5.21 of [19] we easily have

$$\int_{\Omega_\varepsilon} \big|\nabla \zeta_s^{\varepsilon,\delta,\eta}\big|^2 \mathsf{m}_\eta\big(\rho^{\varepsilon,\delta}\big)\,\mathrm{d}x = \Phi_{\mathsf{m}_\eta,\Omega_\varepsilon}\big(\mu^{\varepsilon,\delta}, \boldsymbol{\nu}^{\varepsilon,\delta}\big) \leqslant \Phi_{\mathsf{m},\Omega_\varepsilon}\big(\mu^{\varepsilon,\delta}, \boldsymbol{\nu}^{\varepsilon,\delta}\big) \quad \forall \eta > 0,\ s \in [0,1].$$

If $\mathcal{S}^{\varepsilon,\eta} = \mathcal{S}(\mathscr{U}_\eta, \mathsf{m}_\eta, \Omega_\varepsilon)$ is the semigroup associated with $S(P_\eta, \Omega_\varepsilon)$ and the corresponding integral functional $\mathscr{U}_\eta$, (5.13) gives

$$\begin{aligned}
\frac{1}{2}\mathcal{W}^2_{\mathsf{m}_\eta,\Omega_\varepsilon}\big(\mathcal{S}_t^{\varepsilon,\eta}\mu_1^{\varepsilon,\delta}, \mu_0^{\varepsilon,\delta}\big) + t\mathscr{U}_\eta\big(\mathcal{S}_t^{\varepsilon,\eta}\mu_1^{\varepsilon,\delta}\big) &\leqslant \frac{1}{2}\int_0^1\int_{\Omega_\varepsilon} \big|\nabla \zeta_s^{\varepsilon,\delta,\eta}\big|^2 \mathsf{m}_\eta\big(\rho^{\varepsilon,\delta}\big)\,\mathrm{d}x\,\mathrm{d}s + t\mathscr{U}_\eta\big(\mu_0^{\varepsilon,\delta}\big) \\
&\leqslant \frac{c_\varepsilon^2}{2}\mathcal{W}^2_{\mathsf{m},\Omega}(\mu_1,\mu_0) + t\mathscr{U}_\eta\big(\mu_0^{\varepsilon,\delta}\big).
\end{aligned}$$

Passing to the limit as $\eta \downarrow 0$ (notice that the functions $S_t^{\varepsilon,\eta}\rho^{\varepsilon,\delta}$ take their values in $[M', M'']$), we get

$$\frac{1}{2}\mathcal{W}^2_{\mathsf{m},\Omega_\varepsilon}\big(\mathcal{S}_t^{\varepsilon}\mu_1^{\varepsilon,\delta}, \mu_0^{\varepsilon,\delta}\big) + t\mathscr{U}\big(\mathcal{S}_t^{\varepsilon}\mu_1^{\varepsilon,\delta}\big) \leqslant \frac{c_\varepsilon^2}{2}\mathcal{W}^2_{\mathsf{m},\Omega}(\mu_1,\mu_0) + t\mathscr{U}\big(\mu_0^{\varepsilon,\delta}\big)$$

where $\mathcal{S}^\varepsilon = \mathcal{S}(\mathscr{U}, \mathsf{m}, \Omega_\varepsilon)$ is associated with $S(P, \Omega_\varepsilon)$. We can then pass to the limit as $\delta \downarrow 0$: since $\rho^{\varepsilon,\delta} \to \rho^\varepsilon$ in $L^\infty(\Omega_\varepsilon)$ we immediately have

$$\frac{1}{2}\mathcal{W}^2_{\mathsf{m},\Omega_\varepsilon}\big(\mathcal{S}_t^{\varepsilon}\mu_1^{\varepsilon}, \mu_0^{\varepsilon}\big) + t\mathscr{U}\big(\mathcal{S}_t^{\varepsilon}\mu_1^{\varepsilon}\big) \leqslant \frac{c_\varepsilon^2}{2}\mathcal{W}^2_{\mathsf{m},\Omega}(\mu_1,\mu_0) + t\mathscr{U}\big(\mu_0^{\varepsilon}\big).$$

Finally as $\varepsilon \downarrow 0$ we conclude, recalling Proposition 4.3. □

6.2. *Geodesic convexity*

The proof of Theorem 4.12 follows immediately from the generation result Theorem 6.1 and Theorem 4.9 if every measure μ_s of the geodesic curve is absolutely continuous with respect to $\mathscr{L}^d|_\Omega$ (see also Remark 4.14). On the other hand, this property is not known *a priori*, so we need a more refined argument.

Proof. As in Proposition 3.5 we set $\mu_s^\varepsilon := \mu_s * k_\varepsilon$, $\boldsymbol{\nu}_s^\varepsilon := \boldsymbol{\nu}_s * k_\varepsilon$ and we denote by $\mathcal{S}^\varepsilon = \mathcal{S}(\mathscr{U}, \mathsf{m}, \Omega_\varepsilon)$. By (3.5) and the contraction property given by Theorem 4.10 in Ω_ε we have

$$\begin{aligned}\mathcal{W}_{\mathsf{m},\Omega_\varepsilon}\big(\mathcal{S}^\varepsilon_t\mu^\varepsilon_{s_1},\mathcal{S}^\varepsilon_t\mu^\varepsilon_{s_2}\big)&\leqslant \mathcal{W}_{\mathsf{m},\Omega_\varepsilon}\big(\mu^\varepsilon_{s_1},\mu^\varepsilon_{s_2}\big)\\&\leqslant \mathcal{W}_{\mathsf{m},\Omega}(\mu_{s_1},\mu_{s_2})\\&=|s_1-s_2|\mathcal{W}_{\mathsf{m},\Omega}(\mu_0,\mu_1)\end{aligned}\tag{6.1}$$

and by (3.6), Theorem 4.9 and (4.18) we have

$$\delta^2_\varepsilon:=\mathcal{W}^2_{\Omega,\mathsf{m}}(\mu_0,\mu_1)-\mathcal{W}^2_{\mathsf{m},\Omega_\varepsilon}\big(\mu^\varepsilon_0,\mu^\varepsilon_1\big)\to 0\quad\text{as }\varepsilon\downarrow 0,$$

$$\mathscr{U}\big(\mathcal{S}^\varepsilon_t\mu^\varepsilon_s\big)\leqslant(1-s)\mathscr{U}\big(\mu^\varepsilon_0\big)+s\mathscr{U}\big(\mu^\varepsilon_1\big)+\frac{\delta^2_\varepsilon}{2t}s(1-s),\tag{6.2}$$

$$\mathcal{W}^2_{\mathsf{m},\Omega_\varepsilon}\big(\mathcal{S}^\varepsilon_t\mu^\varepsilon_i,\mu_i\big)\leqslant t\big(\mathscr{U}\big(\mu^\varepsilon_i\big)-\inf\mathscr{U}\big)\leqslant t\big(\mathscr{U}(\mu_i)-\inf\mathscr{U}\big),\tag{6.3}$$

where the second inequality in (6.3) follows from Jensen's inequality.

We choose now a countable set $\mathscr{C}$ dense in $[0,1]$ and containing 0 and 1, a vanishing sequence $(t_k)_{k\in\mathbb{N}}$ and another vanishing sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ so that $\lim_{k\uparrow+\infty}t_k^{-1}\delta^2_{\varepsilon_k}=0$. By compactness and a standard diagonal argument, up to extracting a further subsequence, we can find limit points $\tilde\mu_s$ for $s\in\mathscr{C}$ such that

$$\mathcal{S}^{\varepsilon_k}_{t_k}\mu^{\varepsilon_k}_s\rightharpoonup\tilde\mu_s\quad\text{weakly as }k\uparrow+\infty.$$

By (6.1) and Proposition 3.3 we get

$$\mathcal{W}_{\mathsf{m},\Omega}(\tilde\mu_{s_1},\tilde\mu_{s_2})\leqslant|s_1-s_2|\mathcal{W}_{\mathsf{m},\Omega}(\mu_0,\mu_1)\quad\forall s_1,s_2\in\mathscr{C}.\tag{6.4}$$

(6.3) yields $\tilde\mu_0=\mu_0,\ \tilde\mu_1=\mu_1$ so that we can extend $\tilde\mu$ to a continuous curve (still denoted by $\tilde\mu$) connecting μ_0 and μ_1 still satisfying (6.4) for every $s_1,s_2\in[0,1]$. The triangular inequality shows that (6.4) is in fact an equality and the curve $\tilde\mu$ is a constant speed minimizing geodesic. On the other hand, the lower semicontinuity of $\mathscr{U}$ with respect to weak convergence and (6.2) yields

$$\mathscr{U}(\tilde\mu_s)\leqslant(1-s)\mathscr{U}(\mu_0)+s\mathscr{U}(\mu_1)\quad\forall s\in\mathscr{C}.\tag{6.5}$$

A further lower semicontinuity and density argument shows that (6.5) holds for every $s\in[0,1]$. □

7. Final remarks and open problems

This paper is a first step towards the investigation of the geometry of spaces of measures metrized by $\mathcal{W}_{\mathsf{m},\Omega}$, the induced convexity notions for integral functionals and the corresponding generation of gradient flows with applications to various nonlinear evolutionary PDE's.

Since a sufficiently general theory is far from being developed and understood, it is in some sense surprising that one can reproduce in this setting the celebrated McCann convexity result. On the other hand, many interesting and basic problems remain open: here is just a provisional list.

– At the level of the distance $\mathcal{W}_{\mathsf{m},\Omega}$ only partial results on some basic properties (such as density of regular measures or necessary and sufficient conditions ensuring the finiteness of the distance), are known and a complete and accurate picture is still missing.
– The situation is even less clear in the case of unbounded domains: in this paper we restricted our attention to the bounded case only.
– The study of other integral functionals is completely open, as well as applications to different types of evolution equations, like scalar conservation laws (2.4), interaction equation (2.5) or nonlinear fourth order equation (1.12).
– It would be interesting to study other metric quantities (e.g. the metric slope) and the pseudo-Riemannian structure (tangent space, Alexandrov curvature, etc.) connected with the distance and the energy functionals, see [2,12].
– The regularization properties and asymptotic behaviour of the gradient flow $\mathscr{U}$ and its perturbation can be studied as well: in the Wasserstein case the geodesic convexity of a functional yields many interesting estimates.
– The convergence of the so called "Minimizing movement" or JKO-scheme could be exploited in this and other situations: in the case of geodesically convex energies, further information on the Alexandrov curvature of the distance $\mathcal{W}_{\mathsf{m},\Omega}$ would be crucial, see [2,38].

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