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Functional equations on finite groups of substitutions

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Abstract

Motivated by some investigations of Babbage, we study a class of single variable functional equations. These are functional equations involving one unknown function and a finite set of known functions that form a group under the operation of composition. It turns out that the algebraic structure of a stabilizer determines the number of initial value conditions for the functional equation. In the proof of the main result, the Implicit Function Theorem and, when the stabilizer is nontrivial, the Global Existence and Uniqueness Theorem play a key role.

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1. Introduction

The source of the present investigations can be traced back to a class of single variable functional equations studied by Charles Babbage in a long series of papers [3–7]. In the focus of these publications, as Babbage himself points out, "*is this inverse method with respect to functions, which I at present propose to consider*". Roughly speaking, if a function is given, then, by applying a direct calculation to the function one can easily determine particular equations that the function satisfies. However, the inverse problem,

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when we start with a given equation and our aim is to determine the family of functions satisfying that equation, is highly nontrivial. This problem led Babbage to the functional equation

$$F(f \circ g_1(t), \ldots, f \circ g_r(t), t) = 0, \tag{1}$$

where functions $g_1, \ldots, g_r$ and F are given, and f is to be determined. However, under such general circumstances, there is no hope to give the set of solutions. As Babbage illustrates it via several examples, it may occur that, having a particular solution, infinitely many functions f can be created satisfying the equation above, although the family of such functions may not be the complete collection of all solutions. For further interesting mathematical and historical comments, one can refer to the book of Small [22].

The problem becomes more tractable if the functions $g_1, \ldots, g_r$ form a group G under composition on their domain. Throughout the paper, we always use this assumption. According to the best of our knowledge, the first systematic investigations in this direction are due to Presić. He characterized all solutions of (1) when F is linear and does not depend on its last variable [16,19]. Another nonlinear variant is also due to him [20].

Finally we mention that special cases of (1) which are solvable by elementary methods arose as mathematical competition problems for secondary school students [10,18]. In this setting, F is linear in its first r arguments, but the coefficients are allowed to be (known) functions of t. We illustrate this and the general idea of the method via the subsequent examples.

Example. Let $H \subseteq \mathbb{R}$ be a nonempty set symmetric about zero, and let $\alpha, \beta, h : H \to \mathbb{R}$ functions satisfying $\alpha(t)\alpha(-t) \neq \beta(t)\beta(-t)$ for all $t \in H$. Then there exists a function $f : H \to \mathbb{R}$ satisfying the functional equation

$$\alpha(t)f(t) + \beta(t)f(-t) = h(t).$$

Proof. Set $u(t) := f(t)$ and $v(t) := f(-t)$. Replacing t by $-t$ (which is allowed due to the symmetry of H), we arrive at the next inhomogeneous system of linear equations:

$$\begin{aligned} &\alpha(t)u(t) + \beta(t)v(t) = h(t);\\ &\beta(-t)u(t) + \alpha(-t)v(t) = h(-t). \end{aligned}$$

The base determinant of the system is $D(t) = \alpha(t)\alpha(-t) - \beta(t)\beta(-t)$, which is nonsingular by assumption. Hence, applying Cramer's Rule,

$$\begin{aligned} u(t) &= \frac{1}{D(t)} \begin{vmatrix} h(t) & \beta(t) \\ h(-t) & \alpha(-t) \end{vmatrix} = \frac{h(t)\alpha(-t) - h(-t)\beta(t)}{\alpha(t)\alpha(-t) - \beta(t)\beta(-t)};\\ v(t) &= \frac{1}{D(t)} \begin{vmatrix} \alpha(t) & h(t) \\ \beta(-t) & h(-t) \end{vmatrix} = \frac{h(-t)\alpha(t) - h(t)\beta(-t)}{\alpha(t)\alpha(-t) - \beta(t)\beta(-t)} \end{aligned}$$

follows. That is, if there exists a solution f, then $f = u$. To make sure that f indeed solves the equation, one can either check it via a direct substitution, or test the compatibility property $v(t) = u(-t)$. □

For slightly less trivial examples, one can consult the Putnam Competition Problem A3 (1959) [13] or Problem B2 (1971) [2]. Let us recall here the latter one.

Example. Find all solutions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ of the functional equation

$$f(t) + f\left(\frac{t-1}{t}\right) = 1 + t.$$

Proof. Set $g_1(t) = t$, $g_2(t) = (t-1)/t$, $g_3(t) = 1/(1-t)$ and compose the sides of the equation above with g_1, g_2, g_3, respectively. Since these functions generate a cyclic group, we arrive at

$$\begin{aligned}
f\big(g_1(t)\big) + f\big(g_2(t)\big) &= 1 + t \\
f\big(g_2(t)\big) + f\big(g_3(t)\big) &= (2t-1)/t \\
f\big(g_1(t)\big) + f\big(g_3(t)\big) &= (2-t)/t
\end{aligned}$$

with unknowns $f\big(g_k(t)\big)$. Simple calculations show that the base determinant of the system equals 2; hence, by Cramer's Rule,

$$f(t) = \frac{1}{2}\begin{vmatrix} 1+t & 1 & 0 \\ (2t-1)/t & 1 & 1 \\ (2-t)/t & 0 & 1 \end{vmatrix} = \frac{t^2 - 2t + 3}{2t}.$$

A direct substitution into the original equation shows that f is indeed a solution. $\square$

The method sketched above can also be applied in more general settings. Composing both sides of (1) by the elements of the group, a system of nonlinear equations is obtained where $\Phi_i = f \circ g_i$ play the role of unknowns. Then, under additional regularity assumptions, the Implicit Function Theorem offers a possible approach to provide a solution.

Such an approach is presented in [8] under the assumptions that the group is an Abelian one and its elements have a common fixed point in the interior of the domain. However, that paper concerns only a special case of (1). The general form of (1) is investigated in [9] where the Abelian structure is omitted but stronger regularity is required. The assumption on the existence of a common fixed point is also kept.

As we shall see, the required regularity properties of [8,9] turn out to be quite restrictive: If the group elements have a common fixed point, then the group contains at most two elements. On the other hand, there exist functions forming a group and having no common fixed point. The aim of this paper is to handle these problems in a unified manner that generalizes the previous results too. Our Main Theorem clarifies the connection of the algebraic and analytic feature of the problem, contains the previous results as special cases, and offers further applications.

In Section 2 we investigate the structure of the group G. The most important consequence of these results justifies the required initial value conditions for (1): It turns out that the number of conditions is determined by the structure of a stabilizer. The Main Theorem is presented in Section 3. The proof can be split into two parts according to the structure of the stabilizer. If the stabilizer is trivial, then the Implicit Function Theorem gives a necessary condition for the representation of the solution. In this case, simply the choice of the domain guarantees that this representation is indeed a solution.

If the stabilizer is nontrivial, the question of compatibility arises. It may occur that the natural candidate for the solution (constructed by the Implicit Function Theorem) is only a relation but not a function. Assuming Lipschitz property on a suitable mapping, this problem can be avoided. The key idea is as follows: If functions (which are the components of the relation obtained) satisfy the same Cauchy problem possessing (local) uniqueness, then the functions (locally) coincide. That is, the relation itself is in fact a function.

2. Preliminaries

The aim of this Section is to justify some assumptions of the Main Theorem. In the investigations, the behavior of functions that form a group under the operation of composition plays a crucial role. The subsequent propositions describe the most important properties of group of functions.

If $\emptyset \neq H \subseteq \mathbb{R}$, then we use the notation $G(H)$ for a set of functions $\{g_i : H \to H \mid i \in L\}$ forming a group under composition, and say that $G(H)$ is a group of functions on H. Of course, on a set H several different, even non-isomorphic groups can exist, thus the notation $G(H)$ does not specify the group but rather only the domain of its elements. It turns out that a finite group of monotonic functions has very simple structure.

Proposition 1. *Let $H \subseteq \mathbb{R}$ be a nonempty set, $G(H)$ be a group of strictly monotonic functions. If $G(H)$ is finite, then $|G(H)| \leq 2$.*

Proof. Let $g \in G(H)$, $g \neq \mathrm{id}$ be arbitrary (if such element exists). Let n be the order of g and $t \in I$ be an element such that $g(t) \neq t$. We indirectly prove that g is not increasing. If g is increasing and $t < g(t)$, then composing this inequality by g several times yields to a contradiction

$$t < g(t) < g^2(t) < \cdots < g^n(t) = t,$$

where g^k means the k-times iterative composition of g. If g is increasing and $t > g(t)$, we get a contradiction similarly. Thus an arbitrary nontrivial element of $G(H)$ is decreasing. Now, if $g, h \in G(H)$, $g \neq \mathrm{id}$, $h \neq \mathrm{id}$, then g, g^{-1}, h and h^{-1} are all decreasing. Then $g \circ h^{-1}$ is increasing, hence $g \circ h^{-1} = \mathrm{id}$, yielding $g = h$. □

Note, that if $H = I$ is an interval, then $g \in G(H)$ is continuous if and only if g is strictly monotonic. Hence the previous proposition immediately implies the following simple result.

Proposition 2. *Let $I \subseteq \mathbb{R}$ be an interval, $G(I)$ be a group of continuous functions. If $G(I)$ is finite, then $|G(I)| \leq 2$.*

In the Main Theorem, quite strong regularity properties are assumed on the elements of $G(H)$. A natural problem arises immediately, namely whether every finite group can be represented as a group of (continuous or differentiable) functions over some open set $H \subseteq \mathbb{R}$. The answer to this question is positive.

Proposition 3. *If G is a finite group, then there exists an open set $H \subseteq \mathbb{R}$ and a group of C^∞ functions $G(H)$ such that $G \simeq G(H)$.*

Proof. It is enough to represent the symmetric group S_n for every $n \in \mathbb{N}$, since every subgroup of a representable group is representable, as well. Let $I_1, \ldots, I_n \subseteq \mathbb{R}$ be disjoint bounded open intervals and let $H = \cup_{i=1}^{n} I_i$. Let $h_{i,j}: I_i \to I_j$ be the increasing linear isomorphism between I_i and I_j. For an arbitrary permutation $\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}$ let $g_\pi: H \to H$ be the function for which $g_\pi(t) = h_{i,\pi(i)}(t)$ whenever $t \in I_i$. Then $g_\pi \in C^\infty(H)$ and $G(H) = (\{g_\pi \mid \pi \in S_n\}, \circ)$ forms a group isomorphic to S_n. $\square$

This construction requires H to be the union of (at least) n intervals if n is the smallest number such that G has an isomorphic copy in S_n. The minimum number of intervals required can be decreased for certain finite groups. Let $C_2 = (\{0, 1\}, +)$ be the two-element group. For $\varepsilon \in \{0, 1\}$ let $h_{i,j}^{\varepsilon}: I_i \to I_j$ the linear isomorphism between intervals I_i and I_j which is increasing if $\varepsilon = 0$ and decreasing if $\varepsilon = 1$. For every $\pi \in S_n$ and every $w \in \{0, 1\}^n$ let $g_{w,\pi}: H \to H$ be the function for which $g_{w,\pi}(t) = h_{i,\pi(i)}^{w(i)}(t)$ if $t \in I_i$. Then $g_{w,\pi} \in C^\infty(H)$ and $K(H) = \left(\left\{g_{w,\pi} \mid \pi \in S_n\right\}, \circ\right)$ forms a group isomorphic to the wreath product $C_2 \wr S_n$. This group is called the hyperoctahedral group, and is the symmetry group of the n-dimensional cube (see e.g. [14] for more details). For example, any subgroup of $C_2 \wr S_2 \simeq D_4$ can be represented on two bounded open intervals, or any subgroup of $C_2 \wr S_3 \simeq C_2 \times S_4$ can be represented on three bounded open intervals. In fact, on the union of n bounded open intervals $C_2 \wr S_n$ is the maximal finite representable group.

Proposition 4. *Let $I_1, \ldots, I_n \subseteq \mathbb{R}$ be disjoint open intervals and $H = \cup_{i=1}^{n} I_i$. If $G(H)$ is a finite group of continuous functions, then $G(H)$ is isomorphic to a subgroup of $C_2 \wr S_n$.*

Proof. Let $K(H)$ be the above mentioned representation of $C_2 \wr S_n$ on H. Let $g \in G(H)$ be arbitrary, we prove that $g \in K(H)$. As g is continuous, $g^{-1}\left(I_j\right)$ is an open interval for every $1 \le j \le n$. Thus $\cup_{i=1}^{n} I_i = H = \cup_{j=1}^{n} g^{-1}\left(I_j\right)$ are two partitioning of H into n disjoint intervals. Hence there exists $\pi_g: \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $g|_{I_i}$ is an isomorphism between I_i and $I_{\pi_g(i)}$ for every $1 \le i \le n$. Let $h_{i,j}: I_i \to I_j$ be the increasing linear isomorphism between I_i and I_j. Let $h: H \to H$ be the continuous function for which $h(t) = h_{i,\pi_g(i)}(t)$ whenever $t \in I_i$. Note that $h \in K(H)$. Then $g \circ h^{-1}$ is a continuous function on I_i (for every $1 \le i \le n$) of finite order. Thus by Proposition 1 the element $g \circ h^{-1}$ is either the increasing or the decreasing linear isomorphism on I_i for every $1 \le i \le n$, and hence $g \circ h^{-1} \in K(H)$. Composing by h we obtain $g \in K(H)$. $\square$

Note that by extending these functions as id on the interior of $\mathbb{R} \setminus H$, one can have the domain of the functions differ from $\mathbb{R}$ by only finitely many points. In particular, if n is the smallest number for which G has an isomorphic copy in $C_2 \wr S_n$, then there exists a set of C^∞ functions $G(H)$ on $H = \mathbb{R} \setminus \{0, 1, \ldots, n\}$ isomorphic to G.

The last proposition of the Section plays a crucial role in the proof of the Main Theorem in Section 3. It is known as the orbit-stabilizer theorem, and can be found in any textbook on permutation groups (see e.g. [12, Theorem 1.4A]).

Proposition 5. *Let $H \subseteq \mathbb{R}$ be a nonempty set, $G(H)$ be a group of functions. Let $\xi \in H$ be arbitrary. Then $G_\xi(H) = \{g \in G(H) \mid g(\xi) = \xi\}$ is a subgroup of $G(H)$. Consider the left cosets of $G_\xi(H)$ in $G(H)$. Then, the elements g and h of $G(H)$ belong to the same left coset if and only if $g(\xi) = h(\xi)$.*

3. The main result

For our convenience, introduce the following concept. Assume $G = \{g_i \mid i \in L\}$ is an arbitrary finite group. Define the operation $*$ on the L via the conventions $i * j = k$ if and only if $g_i g_j = g_k$ holds. Then $(L, *)$ is a group and $\varphi : G \to L$ given by $\varphi(g_i) = i$ is an isomorphism between the groups G and L. The inverse of $i \in L$ is denoted by i^{-1}.

For all $i \in L$, the mapping $\psi_i : L \to L$ defined by $\psi_i(j) = j * i$ is an automorphism of the group L. Let $x \in \mathbb{R}^{|L|}$ be fixed. Then, x_{*i} stands for the vector whose jth component is x_{j*i}. Since ψ_i is, in particular, a bijection, $*i$ permutes the coordinates of vectors.

In what follows, $G(H)$ stands for a finite set of functions defined on H that form a group under the operation of composition. For a fixed $\xi \in H$, we assume that the stabilizer $G_\xi(H)$ of ξ in $G(H)$ has m elements and is given by

$$G_\xi(H) = \{g_{kn+1} \mid k = 0, \ldots, m-1\}.$$

As Proposition 5 shows, the stabilizer is a subgroup; let $\{g_1, \ldots, g_n\}$ be a representation set of $G(H)$ with respect to $G_\xi(H)$. Then, by Lagrange's theorem, $|G(H)| = mn$. Moreover, without loss of generality, we may assume that the elements of $G(H)$ are indexed according to the following table.

$$\begin{array}{c|cccccc} \circ & g_1 & g_{n+1} & \cdots & g_{kn+1} & \cdots & g_{(m-1)n+1} \\ \hline g_1 & g_1 & g_{n+1} & \cdots & g_{kn+1} & \cdots & g_{(m-1)n+1} \\ g_2 & g_2 & g_{n+2} & \cdots & g_{kn+2} & \cdots & g_{(m-1)n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ g_l & g_l & g_{n+l} & \cdots & g_{kn+l} & \cdots & g_{(m-1)n+l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ g_n & g_n & g_{2n} & \cdots & g_{(k+1)n} & \cdots & g_{mn} \end{array} \tag{2}$$

Evaluating all elements at ξ, the obtained values in a particular row coincide (that is, they form a left coset of $G_\xi(H)$) while the values in a particular column are pairwise distinct. Moreover, we have

$$l * (kn+1) = kn + l \quad (l = 1, \ldots, n;\ k = 0, \ldots, m-1).$$

In these settings, due to Proposition 5 again, $g_i(\xi) = g_j(\xi)$ if and only if $i \equiv j \pmod{n}$. Hence if we require initial value conditions for the unknown f, then we have to take into consideration the number of left cosets induced by the stabilizer: The number of conditions must divide the order of the group $G(H)$. Therefore, in the rest of the paper, (1) shall be studied in the form

$$\begin{aligned} &F(f \circ g_1(t), \ldots, f \circ g_{mn}(t), t) = 0, \\ &f \circ g_{kn+l}(\xi) = \eta_l. \end{aligned} \tag{3}$$

As usual, if a matrix A is given, then $a_{i,j}$ denotes the common element of the ith row and jth column. We shall use the notation $\mathscr{M}_r$ for the set of quadratic matrices of type $r \times r$. If A is a quadratic matrix and $*i$ is a permutation on its type set, then A_{*i} stands for the matrix which is obtained by permuting the rows and then the columns (or equivalently: the columns and then the rows) of A by $*i$. Similarly, if a vector B is given, then B_{*i} is given as the vector arranged by the components of B by $*i$.

The main result of the paper is presented in the following theorem. In the proof, two cases are distinguished according to the structure of the stabilizer. If $G_\xi(H)$ is trivial (i.e., contains only the identity), then the Implicit Function Theorem guarantees the existence of the solution. The simple fact that the group elements take pairwise distinct values at ξ guarantees compatibility on a sufficiently small domain. If $G_\xi(H)$ is nontrivial, then, different from the previous case, the compatibility cannot be guaranteed by the construction. There will be part of the domain where the constructed candidate for the solution is only a relation but not necessarily a function. However, under some Lipschitz property, this problem can be avoided. The key idea is as follows: If functions (which are the components of the relation obtained) satisfy the same Cauchy problem possessing (local) uniqueness, then the functions (locally) coincide. That is, the relation itself is a function, indeed. For technical details about the theorems applied, consult the books of Rudin [21] and Chicone [11].

Theorem. *Let $H \subseteq \mathbb{R}$ be a nonempty open subset, $\xi \in H$, and $G(H) = \{g_1, \dots, g_{mn}\}$ be a group of continuously differentiable functions such that $g_i(\xi) = g_j(\xi)$ if and only if $i \equiv j \pmod n$. Let $\eta \in \mathbb{R}^n$, $p = (\eta, \dots, \eta) \in \mathbb{R}^{mn}$ and let $F\colon \mathbb{R}^{mn} \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $F(p_{*i}, g_i(\xi)) = 0$ hold for all $i = 1, \dots, mn$. Define the mappings $A\colon \mathbb{R}^{mn} \to \mathscr{M}_{mn}$ and $B\colon \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ by*

$$A(x,t) := \left[\partial_{j*i^{-1}} F(x_{*i}, g_i(t))\right],$$
$$B(x,t) := \left[\partial_{mn+1} F(x_{*i}, g_i(t)) g_i'(t)\right]$$

and assume that A is regular at (p, ξ). If either $m = 1$ or $m \geq 2$ and the mapping $x \to A^{-1}B(x,t)$ is Lipschitz in a neighborhood of (p, ξ) then there exist a $G(H)$-invariant open set $H_0 \subseteq H$ containing ξ and a unique differentiable function $f : H_0 \to \mathbb{R}$ satisfying (3).

Proof. By Proposition 5, the assumptions $g_i(\xi) = g_j(\xi)$ if and only if $i \equiv j \pmod n$ imply that the stabilizer $G_\xi(H)$ is of order m. Therefore, without loss of generality we may assume that the elements of $G(H)$ are listed as in (2), where $G_\xi(H) = \left\{g_1, \dots, g_{(m-1)n+1}\right\}$ and the representation set of the left cosets of $G_\xi(H)$ in $G(H)$ is $\{g_1, \dots, g_n\}$. Let

$$\Psi(x,t) := \left[F(x_{*i}, g_i(t))\right].$$

Then, $\Psi\colon \mathbb{R}^{mn} \times \mathbb{R} \to \mathbb{R}^{mn}$ is continuously differentiable, and the properties $F(p_{*i}, \xi) = 0$ imply $\Psi(p, \xi) = 0$. Hence, the definition of the operation $*$ and the Chain Rule yield

$$D_1 \Psi(p, \xi) = \left[\partial_{j*i^{-1}} F(p_{*i}, g_i(\xi))\right] = A(p, \xi).$$

The last term is nonsingular by assumption, hence the Implicit Function Theorem guarantees the existence of a neighborhood U of (p, ξ), an open interval V containing ξ and a continuously differentiable function $\Phi : V \to \mathbb{R}^{mn}$ such that $(\Phi(t), t) \in U$ for all

$t \in V$, further $\Psi(\Phi(t), t) = 0$ and $\Phi(\xi) = p$ are fulfilled. Let

$$H_0 := \bigcup_{l=1}^{n} \bigcap_{k=0}^{m-1} g_{kn+l}(V).$$

Since $g_i(V) = g_j^{-1}(V)$ for $j = i^{-1}$ and the elements of $G(H)$ are continuous, the sets $g_i(V)$ are open. If $l \in \{1, \dots, n\}$ is fixed, then the properties $g_l(\xi) \in g_{kn+l}(V)$ ensure that H_0 is nonempty. The construction also guarantees that H_0 is invariant under the action of the elements of $G(H)$. For simplicity we may assume that H_0 has exactly n components, that is, the sets $\cap_{k=0}^{m-1} g_{kn+l}(V)$ are pairwise disjoint for all $j = 1, \dots, n$.

Assume first that $m = 1$, that is, the stabilizer $G_\xi(H)$ is a trivial subgroup of $G(H)$. For simplicity we may assume also, that the neighborhood where A^{-1} exists and U coincide (or else we can replace them by their nonempty intersection).

Define the function $f : H_0 \to \mathbb{R}$ by the formula $f(t) := \Phi_i \circ g_i^{-1}(t)$ for $t \in g_i(V)$. The structure of H_0 guarantees that f is a function indeed; on the other hand, by the construction, f is a solution of (3) on H_0.

Assume now that $m \geq 2$ and choose the neighborhood U such that $A^{-1}B$ exists and is continuous on U. The next goal is to verify the compatibility identities $\Phi_{kn+l} = \Phi_l \circ g_{kn+1}$. To do this, first we derive a differential equation for the function Φ satisfied on V. The sides of the implicit equation $\Psi\big(\Phi(t), t\big) = 0$ are continuously differentiable on V; hence, after differentiating and applying the Chain Rule on the components of the latter term, we obtain

$$\begin{aligned} 0 &= \sum_{j=1}^{mn} \partial_j F(\Phi_{*i}, g_i)\Phi'_{j*i} + \partial_{mn+1} F(\Phi_{*i}, g_i) g'_i \\ &= \sum_{j=1}^{mn} \partial_{j*i^{-1}} F(\Phi_{*i}, g_i)\Phi'_j + \partial_{mn+1} F(\Phi_{*i}, g_i) g'_i. \end{aligned}$$

Considering the definitions of the matrix A and the vector B, these equations reduce to the first order implicit differential equation

$$A\big(\Phi(t), t\big)\Phi'(t) + B\big(\Phi(t), t\big) = 0. \tag{4}$$

We shall prove, that the function $\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}$ satisfies the same differential equation. Composing both sides of (4) by g_{kn+1} (which is allowed due to the construction of H_0), and then multiplying by g'_{kn+1} we arrive at

$$A\big(\Phi \circ g_{kn+1}, g_{kn+1}\big)(\Phi' \circ g_{kn+1})g'_{kn+1} + B\big(\Phi \circ g_{kn+1}, g_{kn+1}\big)g'_{kn+1} = 0. \tag{5}$$

The Chain Rule yields

$$(\Phi' \circ g_{kn+1})g'_{kn+1} = (\Phi \circ g_{kn+1})' = \big((\Phi_{*(kn+1)^{-1}}) \circ g_{kn+1}\big)'_{*(kn+1)}.$$

The ith component of the vector $B_{*(kn+1)}\big(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}, \mathrm{id}\big)$ can be obtained in a similar way, applying the Chain Rule again:

$$\begin{aligned} &\partial_{mn+1} F(\Phi_{*i*(kn+1)^{-1}*(kn+1)} \circ g_{kn+1}, g_{i*(kn+1)})g'_{i*(kn+1)} \\ &\quad = \partial_{mn+1} F(\Phi_{*i} \circ g_{kn+1}, g_i \circ g)(g_i \circ g_{kn+1})' \\ &\quad = \partial_{mn+1} F(\Phi_{*i} \circ g_{kn+1}, g_i \circ g)(g'_i \circ g_{kn+1})g'_{kn+1}. \end{aligned}$$

That is,

$$B\big(\Phi \circ g_{kn+1}, g_{kn+1}\big) g'_{kn+1} = B_{*(kn+1)}\big(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}, \mathrm{id}\big). \tag{6}$$

Finally, we determine the (i, j)-member of the matrix $A_{*(kn+1)}\big(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}, \mathrm{id}\big)$. According to the associativity of the operation $*$,

$$\begin{aligned} &\partial_{(j*(kn+1))*(i*(kn+1))^{-1}} F(\Phi_{*i*(kn+1)^{-1}*(kn+1)} \circ g_{kn+1}, g_{i*(kn+1)}) \\ &\quad = \partial_{j*i^{-1}} F(\Phi_{*i} \circ g_{kn+1}, g_i \circ g_{kn+1}); \end{aligned}$$

or equivalently,

$$A\big(\Phi \circ g_{kn+1}, g_{kn+1}\big) = A_{*(kn+1)}\big(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}, \mathrm{id}\big). \tag{7}$$

Substituting (6) and (7) into (5), we obtain that $\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}$ satisfies the differential equation

$$\begin{aligned} 0 = {} & A_{*(kn+1)}\big(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}, \mathrm{id}\big)(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1})'_{*(kn+1)} \\ & + B_{*(kn+1)}\big(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}, \mathrm{id}\big). \end{aligned}$$

Note that this equation can be derived from (4) by applying the permutation $*(kn+1)$ thus they are equivalent. That is, $\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}$ is a solution of differential equation (4). On the other hand, $G_\xi(H)$ is a subgroup hence for every $k \in \{0, \dots, m-1\}$ there exists $r \in \{0, \dots, m-1\}$ such that $(kn+1)^{-1} = rn+1$. Therefore, using the definition of p and the table in (2), we have

$$\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}(\xi) = \Phi_{*(kn+1)^{-1}}(\xi) = p_{*(rn+1)} = p.$$

However, (4) can be written into the equivalent form $\Phi'(t) = -A^{-1}B(\Phi(t), t)$ with the initial value condition $\Phi(\xi) = p$. The Global Existence and Uniqueness Theorem and the calculations above guarantee that $\Phi = \Phi_{*(kn+1)^{-1}} \circ g_{kn+1}$ holds; in particular, comparing the $(kn+l)$th components we get the desired compatibility properties

$$\Phi_{kn+l} = \big(\Phi_{*(kn+1)^{-1}} \circ g_{kn+1}\big)_{kn+l} = \Phi_{(kn+l)*(kn+1)^{-1}} \circ g_{kn+1} = \Phi_l \circ g_{kn+1}.$$

To complete the proof, define the function $f : H_0 \to \mathbb{R}$ by $f(t) := \Phi_l \circ g^{-1}_{kn+l}(t)$ provided that $t \in \bigcap_{k=0}^{m-1} g_{kn+l}(V)$. The compatibility properties above and the fact that a particular component of H_0 remains unchanged under the action of g_{kn+1} ensure that f is well-defined. Moreover, the rows of the implicit equation $\Psi\big(\Phi(t), t\big) = 0$ and the first row of $\Psi\big(\Phi(\xi), \xi\big) = 0$ show that (3) holds on H_0, indeed. □

Observe, that the "existence" part of the Global Existence and Uniqueness Theorem does not play any role in the proof. The main point is its "uniqueness" part for verifying the compatibility equations. Hence the Lipschitz property can be changed to any suitable one that guarantees a unique solution for a Cauchy problem. For such settings, one can consult [17,23].

4. Special cases

In the rest of the paper, two applications are given. The first corollary concerns the case when F is linear in its arguments except for the last one. In this setting, keeping the notations of the Main Theorem, the matrix $A(x, t)$ depends only on its second variable. In particular, the Lipschitz property of the mapping $x \to A(x, t)$ is straightforward.

In the second corollary, the form of (1) is nonlinear, but does not depend on the free variable. This phenomenon is reflected in the fact that the matrix $A(x, t)$ depends only on the first variable.

Corollary 1. *Let $H \subseteq \mathbb{R}$ be a nonempty open set, $\xi \in H$, and $G(H) = \{g_1, \ldots, g_{mn}\}$ be a group of continuously differentiable functions such that $g_i(\xi) = g_j(\xi)$ if and only if $i \equiv j \pmod n$. Let $\alpha_1, \ldots, \alpha_{mn}\colon H \to \mathbb{R}$ be continuous, $h\colon H \to \mathbb{R}$ be continuously differentiable functions and*

$$D(t) := \left[\alpha_{j*i^{-1}} \circ g_i(t)\right].$$

If D is regular at ξ, then there exist a $G(H)$-invariant open set $H_0 \subseteq H$ containing ξ and a unique differentiable function $f : H_0 \to \mathbb{R}$ satisfying the functional equation

$$\alpha_1 f \circ g_1 + \cdots + \alpha_{mn} f \circ g_{mn} = h.$$

Proof. Keeping the notations of the Main Theorem, let $F(x_1, \ldots, x_{mn}, t) := \sum_{k=1}^{mn} \alpha_k(t)x_k - h(t)$. Simple calculations show that $A(x, t) = D(t)$ and hence the regularity of $A(x, t)$ at (p, ξ) means the regularity of $D(t)$ at ξ. Since $B(x, t)$ does not depend on x, the mapping $x \to A^{-1}B(x, t)$ is constant and, in particular, is Lipschitz. Let $p := D^{-1}(\xi)h_g(\xi)$ where h_g denotes the vector whose components are $h \circ g_1, \ldots, h \circ g_{mn}$. Then, the choice of p implies $F(p_{*i}, g_i(\xi)) = 0$ since this is the ith equation of the linear system $D(\xi)p = h_g(\xi)$. Finally we show that the coordinates of p are n-periodic. Let $k \in \{0, \ldots, m-1\}$ be fixed. Then,

$$\begin{aligned}\left(D_{*(kn+1)}(\xi)\right)_{i,j} &= \alpha_{(j*(kn+1))*(i*(kn+1))^{-1}} \circ g_{i*(kn+1)}(\xi) = \alpha_{j*i^{-1}} \circ g_i(\xi)\\ &= (D(\xi))_{i,j}\,.\end{aligned}$$

The middle equation holds since i and $i * (kn+1)$ belong to the same coset of $G_\xi(H)$. That is, $D_{*(kn+1)}(\xi) = D(\xi)$. Similarly, $(h_g)_{*(kn+1)}(\xi) = h_g(\xi)$. On the other hand, $D(\xi)p = h_g(\xi)$ is equivalent to its arranged form $D_{*(kn+1)}(\xi)p_{*(kn+1)} = (h_g)_{*(kn+1)}(\xi)$ showing that p and $p_{*(kn+1)}$ solve simultaneously the same system. Therefore $p = p_{*(kn+1)}$ as it was desired. The assumptions of the Main Theorem are fulfilled hence the statement follows. $\square$

Corollary 2. *Let $H \subseteq \mathbb{R}$ be a nonempty open set, $\xi \in H$, and $G(H) = \{g_1, \ldots, g_{mn}\}$ be a group of continuously differentiable functions such that $g_i(\xi) = g_j(\xi)$ if and only if $i \equiv j \pmod n$. Let $\eta \in \mathbb{R}^n$, $p = (\eta, \ldots, \eta) \in \mathbb{R}^{mn}$ and let $F\colon \mathbb{R}^{mn} \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $F(p_{*i}) = h \circ g_i(\xi)$ hold for all $i = 1, \ldots, mn$. Assume that the mapping $D\colon \mathbb{R}^{mn} \to \mathscr{M}_{mn}$ defined by*

$$D(x) := \left[\partial_{j*i^{-1}} F(x_{*i})\right]$$

is regular at p. If either $m = 1$ or $m \geq 2$ and the mapping $x \to D(x)$ is Lipschitz in a neighborhood of p, then there exist a $G(H)$-invariant open set $H_0 \subseteq H$ containing ξ and a unique differentiable function $f\colon H_0 \to \mathbb{R}$ satisfying the conditions $f \circ g_{kn+l}(\xi) = \eta_l$

and functional equation

$$F\big(f \circ g_1(t), \ldots, f \circ g_{mn}(t)\big) = h(t).$$

Proof. Define $G(x,t) := F(x) - h(t)$. Then, $A(x,t) = D(x)$ and $G(p_{*k}, \xi) = 0$ if and only if $F(p_{*k}) = h(g_k(\xi))$ hold for all $k = 1, \ldots, mn$. The Lipschitz property of $x \to A^{-1}B(x,t)$ is equivalent to that of $x \to D^{-1}(x)$. Hence the assumptions of the Main Theorem are satisfied, and the proof is completed. □

In fact, Corollary 1 remains true under much more general circumstances. Namely, if $D(t)$ is nonsingular on an arbitrary subset H of the reals, then the corresponding form of (1) can be solved uniquely under considerably weaker regularity properties on the known functions. An explicit representation for the solution can also be given via Cramer's Rule. This phenomenon can be traced through the motivating example of the present paper; for precise details, consult [8]. However, the method using Cramer's Rule is not applicable if (1) is nonlinear. Note also that in Corollary 1 the assumptions $F(p_{*i}, g_i(\xi)) = 0$ of the Main Theorem turn out to be redundant.

Considering the second theorem of [8], its regularity properties are weaker but the algebraic assumptions are stronger than in Corollary 2. The method presented here does not rely on an Abelian structure; moreover, the existence of a common fixed point is also dropped. Of course, the special setting $n = 1$ of the Main Theorem reduces to main result of [9]. Similarly, Corollaries 1 and 2 generalize the corollaries of [9]. For further examples, both in the linear and in the nonlinear cases, see [8,9]. Those who are interested in the basic results of functional equations should look at [1,15].

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