ROTTING MAGNETOHYDRODYNAMIC FLOW ABOUT A
STATIONARY DISK IN A CIRCULAR MAGNETIC FIELD

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Abstract—This paper examines the effects of a circular magnetic field on a rotating conducting fluid. The fluid is rotating with a uniform angular velocity \( \omega \) at infinity and is in contact with a stationary infinite disk. The equations of motions are solved numerically using a least change secant update quasi-Newton technique. The effects of the parameters \( \alpha \) (ratio of kinematic viscosity to magnetic diffusivity) and \( \beta \) (ratio of magnetic field strength to angular velocity of the fluid) on the flow are presented graphically.

1. INTRODUCTION

Bodewadt [1] first studied the interaction between a rotating fluid and a stationary surface. There has been renewed interest in this problem recently, due to such applications as the flow inside vortex reactors, magnetohydrodynamic vortex power generators, cylindrical chambers, hurricanes and tornadoes. Surveys of this type of flow were given by Rott and Lewellen [2] and Lewellen [3].

Bodewadt [1] matched a Taylor series expansion from the origin with the asymptotic expansion at infinity to solve the above problem. His solutions demonstrated oscillations in the velocity profiles before reaching their asymptotic values at the edge of the boundary layers. This behavior of the fluid's velocity was doubted at first, however subsequent theoretical and experimental studies by Rogers and Lance [4], Bein and Penner [5], Kuo [6], and Millsaps and Nadahl [7] support Bodewadt's solution.

Recently, King and Lewellen [8] studied Bodewadt's problem with and without an axial magnetic field. They found that the oscillations in the velocity field were damped by the magnetic field. Nath and Venkatachala [9] studied the effect of suction on the problem in King and Lewellen [8]. Suction tends to reduce the velocity overshoot and dampen the oscillations.

This paper studies the effects of an axial electric current of uniform current density \( J_0 \) at a large distance from an electrically insulated stationary disk. The fluid is rotating with a uniform angular velocity \( \omega \) far from the disk. The governing equations for this problem were given by Pao [10], who studied the flow of a conducting fluid over a rotating disk. The governing partial differential equations are reduced to nonlinear ordinary differential equations using similarity transformations. The nonlinear system is solved numerically using a least change secant update quasi-Newton method. The effects of \( \alpha \) (kinematic viscosity \( v \) over magnetic diffusivity \( \eta \)) and \( \beta \) (magnetic field strength \( \Omega \) over fluid angular velocity at infinity \( \omega_0 \)) on the flow field are presented graphically.

2. GOVERNING EQUATIONS

From Pao [10] the governing equations in cylindrical polar coordinates \((r, \theta, z)\) for an incompressible, viscous, electrically conducting fluid, assuming axisymmetric flow, are

\[
\frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} + \frac{\partial P}{\partial r} - \frac{\partial f}{\partial r} - \frac{h \partial f}{\partial z} + \frac{g^2}{r} - v \left( \nabla^2 u - \frac{u}{r^2} \right) = 0, \tag{1}
\]

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\[ u \left( \frac{\partial v}{\partial r} + \frac{v}{r} \right) + w \frac{\partial v}{\partial z} - f \left( \frac{\partial g}{\partial r} + \frac{g}{r} \right) - h \frac{\partial g}{\partial z} - v \left( \nabla^2 v - \frac{v}{r^2} \right) = 0, \]

\[ u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} - f \frac{\partial h}{\partial r} - h \frac{\partial h}{\partial z} - v \nabla^2 w = 0, \]

\[ \frac{\partial}{\partial r} (ru) + \frac{\partial}{\partial z} (rw) = 0, \]

\[ \frac{\partial}{\partial r} (rf) + \frac{\partial}{\partial z} (rh) = 0, \]

\[ \frac{\partial}{\partial r} (vf - ug) + \frac{\partial}{\partial z} (vh - wg) + \eta \left( \nabla^2 g - \frac{g}{r^2} \right) = 0 \]

\[ uh - wf - \eta \left( \frac{\partial h}{\partial r} - \frac{\partial f}{\partial z} \right) = 0, \]

where

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \]

and $u, v, w, f, g$ and $h$ are the components of velocity and normalized magnetic field strength respectively in the $r$, $\theta$ and $z$ directions. Also, $v$ is the kinematic viscosity, $p$ is the fluid pressure, $\rho$ is the uniform density, $\mu$ is the magnetic permeability, $\sigma$ is the electrical conductivity, $\eta = 1/\mu \sigma$ is the magnetic diffusivity, $(f, g, h) = (f_1, g_1, h_1)(\mu/\rho)^{1/2}$, and $P = (f_2^2 + g_2^2 + h_2^2)/2 + p/\rho$, where $f_1$, $g_1$ and $h_1$ are the components of the magnetic field in the $r$, $\theta$, and $z$ directions respectively. It is assumed that $v$, $\sigma$ and $\mu$ are constants and the net charge density is zero.

3. SIMILARITY SOLUTION AND BOUNDARY CONDITIONS

The stationary insulated disk surface is in the plane $z = 0$. The fluid is rotating with a constant angular velocity $\omega$ at infinity. An axial electric current of uniform current density $J_0$ is imposed at $z = \infty$. This yields a tangential magnetic field component at $z = \infty$ of $g = \Omega r$, where $\Omega = (J_0/2)(\mu/\rho)^{1/2}$ is constant and $f$ and $h$ are absent.

Using the following similarity transformations:

\[ u = \omega rm'(\zeta), \quad v = \omega rG(\zeta), \quad w = -2(\nu \omega)^{1/2}m(\zeta), \]

\[ \zeta = z(\omega/\nu)^{1/2}, \quad g = \Omega rM(\zeta), \quad P = \nu \omega S(\zeta) + \frac{\omega \Omega^2 r^4}{2}, \]

equations (1)-(3) and (6) reduce to

\[ m'' + 2mm'' - (m')^2 + G^2 - \beta^2 M^2 - \lambda = 0, \]

\[ 2(mG' - m'G) + G'' = 0, \]

\[ M'' + 2Mm' = 0, \]

\[ S' + 4mm' + 2m'' = 0, \]

where $\alpha = \nu/\eta$ and $\beta = \Omega/\omega$.

The boundary conditions of the problem are

\[ m = 0, \quad m' = 0, \quad G = 0, \quad M = 0, \quad S = 0 \quad \text{at} \quad \zeta = 0 \]

and

\[ m' \rightarrow 0, \quad G \rightarrow 1, \quad M \rightarrow 1, \quad \text{as} \quad \zeta \rightarrow \infty. \]

Using the boundary conditions (14) and assuming $m'' = m''' = 0$ at infinity, the constant $\lambda$ in equation (9) is $\lambda = 1 - \beta^2$. Integrating equation (12) once yields $S + 2(m^2 + m') = C$. Using the boundary conditions (13), $C$ is determined to be 0, yielding $S + 2(m^2 + m') = 0$. 

4. ASYMPTOTIC SOLUTIONS

For large values of $\zeta$, the functions $m, G,$ and $M$ can be written as

$$m = c/2 + m_1(\zeta), \quad G = 1 + G_1(\zeta), \quad M = 1 + M_1(\zeta).$$  \hspace{1cm} (15)

Substituting (15) into (9)-(11) yields

$$m_1 = e^{\psi}[A \cos(\psi \zeta) - B \sin(\psi \zeta)] + \frac{2c_1 \beta^2(\zeta^2 - c^2) e^{-c\zeta}}{(c^2 - c^2 \zeta)^2 + 4},$$  \hspace{1cm} (16)

$$G_1 = e^{\psi}[B \cos(\psi \zeta) + A \sin(\psi \zeta)] + \frac{4\beta^2 c_1 e^{-c\zeta}}{(c^2 - c^2 \zeta)^2 + 4},$$  \hspace{1cm} (17)

$$M_1 = c_1 e^{-c\zeta},$$  \hspace{1cm} (18)

where

$$\varphi = \frac{1}{2} \left[ -c - \left( \frac{c^4 + 64}{2} \right)^{1/2} \right], \quad \psi = -\frac{1}{2} \left[ \frac{(c^4 + 64)^{1/2} - c^2}{2} \right],$$  \hspace{1cm} (19)

and $A, B,$ and $c_1$ are constants of integration. The oscillatory nature of the asymptotic solutions is consistent with the earlier results of other authors, e.g. Bodewadt [1].

5. NUMERICAL METHOD

Since $S$ is easily determined after solving (9)-(11), it will not be discussed. Following the format in Heruska [11] define

$$X = (m'(0), G'(0), M'(0)).$$  \hspace{1cm} (20)

Let $m(\zeta; X), G(\zeta; X), M(\zeta; X)$ be the solution of the initial value problem given by (9)-(11) with initial conditions (13) and (20). The original two-point boundary value problem described by equations (9)-(11), (13) and (14) is numerically equivalent to solving the nonlinear system of equations:

$$F(X) = (m'(\tau; X), G(\tau; X) - 1, M(\tau; X) - 1)' = 0,$$  \hspace{1cm} (21)

where $\tau$ is chosen large enough so that

$$|m(\tau) - m(\zeta)| < \epsilon, \quad |G(\tau) - G(\zeta)| < \epsilon, \quad |M(\tau) - M(\zeta)| < \epsilon,$$  \hspace{1cm} (22)

for $\tau < \zeta < \infty$ and a given $\epsilon > 0$. Equation (21) is derived from the boundary conditions (14).

Solving (21) amounts to shooting from 0 to $\tau$ until an appropriate $X$ is determined such that the boundary conditions (14) are met at $\tau$. This process is repeated, increasing $\tau$ until (22) is satisfied.

Algorithms for solving systems such as (21) typically require partial derivatives, such as $\partial G/\partial X_k$. The functions needed can be written as

$$Y = \left( m, m', m'', G, G', M, M', \frac{\partial m}{\partial X_k}, \frac{\partial m'}{\partial X_k}, \frac{\partial m''}{\partial X_k}, \frac{\partial G}{\partial X_k}, \frac{\partial G'}{\partial X_k}, \frac{\partial M}{\partial X_k}, \frac{\partial M'}{\partial X_k} \right).$$  \hspace{1cm} (23)

The vector $Y(\zeta)$ can be calculated from the system of first order ordinary differential equations:

$$Y_1 = Y_2, \quad Y_2 = Y_3, \quad Y_3 = \beta^2 Y_2 - Y_4 - 2Y_1 Y_5 + Y_2^2 + 1 - \beta^2,$$

$$Y_4 = Y_5, \quad Y_5 = 2(Y_2 Y_4 - Y_1 Y_5), \quad Y_6 = Y_7, \quad Y_7 = -2\alpha Y_1 Y_5,$$

$$Y_8 = Y_9, \quad Y_9 = Y_{10}, \quad Y_{10} = 2(\beta^2 Y_6 Y_{13} - Y_4 Y_{11} - Y_5 Y_8 - Y_7 Y_{10} + Y_2 Y_9),$$

$$Y_{11} = Y_{12}, \quad Y_{12} = 2(Y_5 Y_7 + Y_2 Y_{11} - Y_1 Y_{12} - Y_3 Y_8),$$

$$Y_{13} = Y_{14}, \quad Y_{14} = -2\alpha(Y_4 Y_7 + Y_1 Y_{14}),$$

$$Y(0) = (0, 0, X_1, 0, X_2, 0, X_3, 0, 0, \delta_{1x}, 0, \delta_{2x}, 0, \delta_{3x}).$$  \hspace{1cm} (24)
where $\delta_{ik}$ is the Kronecker delta. By solving this system three times (for $k = 1, 2, 3$), the Jacobian matrix $DF(X)$ of $F(X)$ can be calculated. To solve (21), a least change secant update quasi-Newton method was utilized. The code used for this was HYBRJ from the MINPACK subroutine package from Argonne National Laboratory [13]. These quasi-Newton algorithms are quite efficient and robust, but they sometimes fail by converging to a local minimum of $F(X)'F(X)$ which is not a zero of $F(X)$.

As $\tau$ is increased, the partial derivatives also increase. This causes severe numerical instability when $\beta$ is small and when $\alpha$ is large. Another reason for difficulty in these regions is that $m(\infty)$ gets very small and becomes negative. For $\beta = 0$, $m(\infty) = -0.674$ [7]. From equation (18), as $c(=2m(\infty))$ becomes smaller, $\tau$ needs to be larger to satisfy (22). When $m(\infty) \leq 0$, no solution exists for (11) with boundary conditions (13) and (14). If $\beta = 0$ (Bodewadt's case [1]), then equation (11) has no contribution to equations (9) and (10). Since $\beta = 0$ implies no magnetic field [i.e. no equation (11) in the system], a solution to Bodewadt's [1] case with $m(\infty) < 0$ is possible.

6. DISCUSSION

Figure 1 shows the effect of $\beta$ on axial velocity $m$, radial velocity $m'$, tangential velocity $G$, and magnetic field strength $M$. Increasing $\beta$ causes an increase in all of these. The effect is less when $\beta$ is large. Figure 2 shows the effect of $\alpha$ on the axial, radial and tangential velocities and the magnetic field strength. As $\alpha$ increases, an opposite effect from increasing $\beta$ on axial and radial velocities is observed. Increasing $\alpha$ decreases the axial and radial velocities but increases the tangential velocity and magnetic field strength. The effect of changing small $\alpha$ is more pronounced than changes in larger values of $\alpha$.

Figures 1 and 2 do not show any oscillations that would be expected from the asymptotic equations (16) and (17). Figure 3 in [13] shows this oscillation in the graph of $H = e^{-\rho m'}$. Since $m'(\xi)$ has decayed to 0 when $\xi = 4$, the important area of Fig. 3 of [13] is from $\xi = 4$ to infinity.

The oscillatory nature of this area is apparent even though the amplitude is small.
Table 1 of [13] gives some values for $m(\infty) = c/2$, $m''(0)$, $G'(0)$, and $M'(0)$. When $\beta = 0$, we have the classical case from Bodewadt [1]. When $\beta = 0$, equation (11) decouples from equations (9) and (10). In this case, $m(\infty) < 0$. Using the asymptotic formula (18), there obviously is then no solution for (11), (13) and (14). Furthermore, no solution exists for (11), (13) and (14) with values of $\alpha$ and $\beta$ such that $m(\infty) \leq 0$.

To obtain an estimate for $\alpha$ and $\beta$ values where solutions do not exist, an extrapolation technique was employed. Viewing $m(\infty) = c/2$ as a function of $\beta$ and keeping $\alpha$ fixed, a least squares quadratic polynomial fit to the data points $(\beta, c(\beta))$ was computed, giving more weight to the points where $c(\beta)$ is closer to zero. Doing this for $\alpha = 1$ and $\beta = 2.0, 1.8, 1.6, 1.4, 1.2$ with corresponding weights 1, 2, 3, 4, 5 respectively yielded the estimate $\hat{\beta} = 0.88$ for the critical value of $\beta$. This technique can also be applied to estimate critical values of $\alpha$, beyond which no solution exists.

The addition of a strong circular magnetic field at infinity while requiring the magnetic field strength to zero at the disk surface yields no solution for cases that approximate Bodewadt’s problem ($\beta \approx 0$). By reorienting the magnetic field such that it originates at the disk surface and fades to zero at infinity may yield solutions that approximate Bodewadt’s case.

REFERENCES