JOURNAL OF COMBINATORIAL THEORY, Series B 25, 326–338 (1978)

# A Constructive Characterization of Trees with at Least $k$ Disjoint Maximum Matchings

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*Communicated by the Editors*

Received July 15, 1975

Let $H = F(v) \oplus G(w)$ denote the graph obtained from $F$ and $G$ by identifying vertices $v$ of $F$ and $w$ of $G$; $H$ will be said to be obtained by surgery on $F$ and $G$. A matching of a graph is a collection of edges, no two of which are incident with the same vertex. This paper presents a constructive characterization of the set $S_k$ ($k \geqslant 2$) of trees which have at least $k$ disjoint maximum matchings. There are three types of surgery such that, for each $k \geqslant 2$, $S_k$ is the set of all trees obtainable from a star $K_{1,n}$ ($n \geqslant k$) by a finite sequence of the specified surgical operations. A constructive characterization is also given for trees with two disjoint maximum independent vertex sets.

## 1. INTRODUCTION

Given a (finite, undirected) graph $G$, a *matching* is a collection of edges which are independent, that is, no two of them are incident with the same vertex. A *maximal* matching is one which is not a proper subset of any other matching. $\beta_1(G)$ denotes the number of edges in a *maximum* matching, that is, a maximal matching with the largest number of edges possible. In Fig. 1, {(2, 3), (4, 5)} and {(1, 2), (3, 4)} are disjoint maximal matchings, and {(1, 2), (4, 5), (3, 6)} is the only maximum matching.

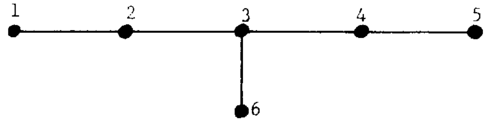


FIG. 1. A graph with two disjoint maximal matchings.

In [1] Cockayne and Hedetniemi have given a characterization of those

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0095-8956/78/0253–0326$02.00/0

trees which do not have two or more disjoint maximal matchings, and in [3] Hartnell has initiated a study of unicyclic graphs with disjoint maximal matchings. In this paper, a constructive characterization of the set of trees with $k$ ($k \geqslant 2$) or more disjoint maximum matchings is presented.

If $T$ is a *tree* (a connected, acyclic graph) containing vertex $v$, then a *branch* of $T$ at $v$ is a maximal subtree containing $v$ as an endpoint. A *path* on $n$ vertices, denoted $P_n$, is a tree with exactly two endpoints; a *star* on $n + 1$ vertices ($n \geqslant 2$), denoted $K_{1,n}$, is a tree with $n$ endpoints. A branch at $v$ which is a path will be called a *branch path* at $v$. It is easy to see that in a tree $T$ with at least one vertex $u$ of degree at least three (i.e., $\deg(u) \geqslant 3$), there is at least one vertex $v$, with $\deg(v) \geqslant 3$, such that $v$ has at least two branch paths, and, in fact, there is a vertex $v$ with $\deg(v) \geqslant 3$ such that at most one branch at $v$ is not a path. If $v_1, v_2, ..., v_n, v_{n+1}$ is a path in $T$ in $T$ with $v_i$ adjacent to $v_{i+1}$ for all $i \leqslant n$ and with $\deg_T(v_1) = 1$, $\deg_T(v_2) = \cdots = \deg_T(v_n) = 2$, then $v_1, ..., v_n, v_{n+1}$ will be called a *tail* of length $n$. Note that $\deg_T(v_{n+1})$ is unrestricted.

Let $V(G)$ and $E(G)$ denote the vertex and edge sets of graph $G$, respectively. Let $v$ and $w$ be specified vertices in graphs $F$ and $G$, respectively. Then $H = F(v) \oplus G(w)$ will denote the graph obtained from $F$ and $G$ by identifying vertices $v$ and $w$. That is, letting $x$ be a vertex not in $V(F)$ or $V(G)$, one has $V(H) = (V(F) - \{v\}) \cup (V(G) - \{w\}) \cup \{x\}$ and $E(H) = E(F - v) \cup E(G - w) \cup \{xu \mid vu \in E(F)$ or $wu \in E(G)\}$. Graph $H$ will be said to be obtained by *surgery* on $F$ and $G$.

Let $S_k$ ($k \geqslant 2$) denote the set of trees which have at least $k$ disjoint maximum matchings. It will be shown that there are three types of surgery such that, for each $k \geqslant 2$, $T$ is in $S_k$ if and only if $T$ can be obtained from a star $K_{1,n}$ ($n \geqslant k$) by a finite sequence of the specified surgical operations. The case for $k = 2$ will be handled in Section 2, and $k \geqslant 3$ will be done in Section 3.

A collection of vertices is called *independent* if no two are incident with the same edge (that is, no two are adjacent). It will be shown in Section 4 that there is one type of surgery such that tree $T$ has two disjoint maximum independent vertex sets if and only if $T$ can be obtained from $P_2$ by a finite sequence of surgical operations of that type.

## 2. Trees with Two Disjoint Maximum Matchings

Lemma 1. *If a tree $T$ has a tail $v_1, v_2, v_3, v_4$ of length three, then $T$ has at most two disjoint maximum matchings. Furthermore, $T$ has two disjoint maximum matchings if and only if $T - \{v_1, v_2\}$ does.*

*Proof.* If $M$ is a matching without edges $v_2v_1$ or $v_2v_3$, then $M \cup \{v_2v_1\}$

is also a matching. Thus every maximum matching contains an edge incident with $v_2$. Since $\deg(v_2) = 2$, there are at most two disjoint maximum matchings.

It is clear that $\beta_1(T) = \beta_1(T - \{v_1, v_2\}) + 1$. Given disjoint maximum matchings $M_1$ and $M_2$ for $T$ (that is, disjoint sets each with $\beta_1(T)$ independent edges), one can assume $v_1v_2 \in M_1$. Now $M_1 - v_1v_2$ and $M_2 - v_2v_3$ are disjoint sets in $T - \{v_1, v_2\}$, each with $\beta_1(T) - 1$ independent edges. Thus $T - \{v_1, v_2\}$ has two disjoint maximum matchings. Given disjoint maximum matchings $M_1$ and $M_2$ for $T - \{v_1, v_2\}$, one can assume that $v_3v_4 \notin M_2$. Now $M_1 \cup \{v_1v_2\}$ and $M_2 \cup \{v_2v_3\}$ are disjoint maximum matchings for $T$. ∎

Since the subgraph induced by $\{v_1, v_2, v_3\}$ is $K_{1,2}$, one can write $T$ as $(T - \{v_1, v_2\})(v_3) \oplus K_{1,2}(v_3)$. In general, let $H$ be called obtainable from $G$ by *$K_{1,2}$-surgery* if $H$ can be written as $G(v) \oplus K_{1,2}(w)$ where $v$ and $w$ are endpoints of $G$ and $K_{1,2}$, respectively. It is clear that path $P_k$ has two disjoint maximum matchings if and only if there are an even number of edges, that is, $k$ is odd ($k \geqslant 3$). Thus a path has two disjoint maximum matchings if and only if it can be obtained from $K_{1,2}$ (that is, $P_3$) by a finite sequence of $K_{1,2}$-surgeries.

Now suppose $T$ is a tree (with a vertex of degree at least three) in which there is no tail of length three. Let $w$ be a vertex of degree at least three which has at least two branch paths.

LEMMA 2. *If $w$ has three or more branch paths with exactly two edges, then $T$ does not have two disjoint maximum matchings.*

*Proof.* Suppose $u_1$, $u_2$, $w$ and $v_1$, $v_2$, $w$ and $x_1$, $x_2$, $w$ are tails as in Fig. 2. Let $M$ be a maximum matching. Since $M$ contains edges incident

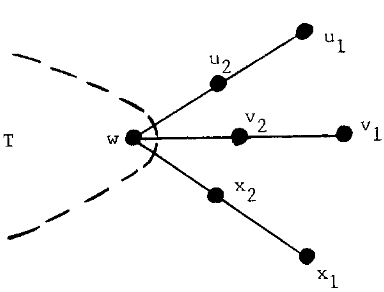


FIG. 2. Substructure at a vertex with three branch paths of length two.

with $u_2$, $v_2$, and $x_2$ and has at most one edge incident with $w$, then $M$ contains at least two of the edges $u_1u_2$, $v_1v_2$, and $x_1x_2$. As this is true for any maximum matching, $T$ does not have two disjoint maximum matchings. ∎

LEMMA 3. *If $w$ has a branch path with exactly two edges and another with exactly one edge, then $T$ does not have two disjoint maximum matchings.*

*Proof.* Suppose $u_1$, $u_2$, $w$ and $v$, $w$ are tails as in Fig. 3. Let $M$ be a maximum matching. If $u_1u_2 \notin M$, then $u_2w \in M$, $vw \notin M$, and no other edge incident with $w$ is in $M$. If $M' = M + u_1u_2 + wv - u_2w$, then $M'$ is a larger matching than $M$. This contradiction implies that $u_1u_2$ is in every maximum matching, and $T$ cannot have two disjoint maximum matchings. ▮

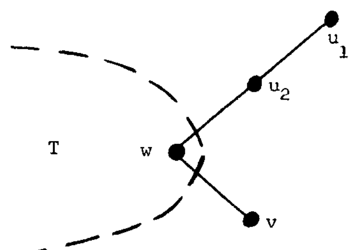


FIG. 3. Substructure at a vertex with branch paths of length one and two.

Vertex $w$ ($\deg(w) \geqslant 3$) can be selected so that at most one branch is not a path. If one branch is not a path, let $x$ be the vertex adjacent to $w$ on that branch. If $T$ has two disjoint maximum matchings (and, by assumption, has no tail of length three), then, by Lemmas 2 and 3, either every branch path from $w$ is of length one or there are exactly two branch paths and each is of length two.

Note that if $T$ is a star $K_{1,n}$ ($n \geqslant 2$), then $T$ has $n$ disjoint maximum matchings.

LEMMA 4. *If tree $T$ is not a star,* $\deg(w) = d \geqslant 3$, *and $w$ has $d - 1$ branch paths, $wu_1$, $wu_2$,..., $wu_{d-1}$, each of length one, then $T$ has two disjoint maximum matchings if and only if $T - \{w, u_1, u_2, ..., u_{d-1}\}$ does.*

Since every maximum matching of $T$ has exactly one edge incident with $w$, the proof of Lemma 4 is like the proof of Lemma 1. Note that one can write $T$ as $(T - \{w, u_1, ..., u_{d-1}\})(x) \oplus K_{1,d}(x)$. In general, let $H$ be called obtainable from $G$ by *$K_{1,n}^*$-surgery* if $H = G(x) \oplus K_{1,n}(v)$ where $v$ is an endpoint of $K_{1,n}$ and $x$ may be any vertex of $G$.

LEMMA 5. *If $T$ is a tree,* $\deg(w) = 3$, *$w$ has exactly two branch paths of length two (say $u_1$, $u_2$, $w$ and $v_1$, $v_2$, $w$) and* $\deg(x) = 2$ *where $x$ is the third vertex adjacent to $w$, then $T$ does not have two disjoint maximum matchings.*

*Proof.* Let $y$ be the other vertex adjacent to $x$ (Fig. 4, $j = 1$). Let $M$

be a maximum matching. If $xy$ is not in $M$, then $xw$, $u_1u_2$, and $v_1v_2$ are in $M$; if $xy$ is in $M$, then at least one of $u_1u_2$ and $v_1v_2$ is in $M$. So at least two of the three edges $xy$, $u_1u_2$, and $v_1v_2$ are in $M$. The same is true of any other maximum matching, which therefore cannot be disjoint from $M$. ▌

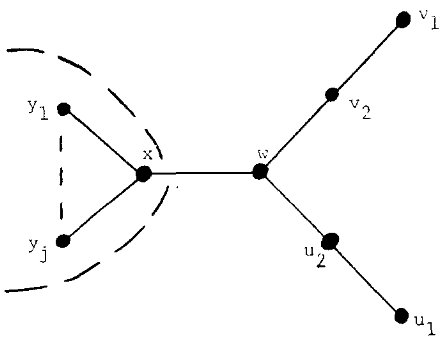


FIG. 4. Structure at a vertex with two branch paths of length two.

Suppose $w$, $u_1$, $u_2$, $v_1$, and $v_2$ are as in Lemma 5 with $\deg(x) = j + 1 \geqslant 3$ (as in Fig. 4). If $T' = T - \{u_1, u_2, v_1, v_2, w\}$ has a maximum matching with no edge incident with $x$, then $\beta_1(T) = \beta_1(T') + 3$, and every maximum matching of $T$ contains $xw$, $u_1u_2$, and $v_1v_2$. It follows that $T$ having two disjoint maximum matchings implies that $\beta_1(T) = \beta_1(T - \{u_1, u_2, v_1, v_2 w\}) + 2$, and one easily obtains the next lemma.

LEMMA 6. *If $T$ is as in Fig. 4 with $j \geqslant 2$, then $T$ has two disjoint maximum matchings if and only if tree $T' = T - \{u_1, u_2, v_1, v_2, w\}$ has two disjoint maximum matchings and every maximum matching of $T'$ has an edge incident with $x$.*

COROLLARY 6.1. *If $T$ is as in Fig. 4 with $j \geqslant 2$ and if at least one $y_i$ adjacent to $x$ is an endpoint, then $T$ has two disjoint maximum matchings if and only if $T - \{u_1, u_2, v_1, v_2, w\}$ does.* ▌

COROLLARY 6.2. *If $T$ is as in Fig. 4 and there is a $y_i$ adjacent to $x$ such that* $\deg(y_i) = 2$, *$y_i$ is adjacent to $x$ and $y$, and* $\deg(y) = 1$, *then $T$ does not have two disjoint maximum matchings.*

*Proof.* Suppose $T$ has two disjoint maximum matchings. By Lemma 6, $T - \{u_1, u_2, v_1, v_2, w\}$ has a maximum matching $M$ which does not include $y_iy$. It therefore includes $y_ix$, and hence no other edge incident with $x$. Then $M + y_iy - y_ix$ is an independent set of edges in $T - \{u_1, u_2,$

$v_1, v_2, w\}$ with as many edges as $M$ and no edge incident with $x$. This contradicts Lemma 6. ▮

Let $Z_2$ be the graph with $V(Z_2) = \{u_1, u_2, v_1, v_2, w, x\}$ and $E(Z_2) = \{u_1u_2, u_2w, v_1v_2, v_2w, wx\}$. If $T$ is as in Corollary 6.1, then $T = (T - \{v_1, v_2, w, u_1, u_2\})(x) \oplus Z_2(x)$. In general, let $H$ be said to be obtainable from $G$ by *$Z_2$-surgery* if $H$ can be written as $G(v) \oplus Z_2(x)$ where $x$ is the endpoint of $Z_2$ adjacent to the vertex of degree three and $v$ is a vertex which is adjacent to an endpoint of $G$.

Assume $T'$ is a tree (not a path or star) with two disjoint maximum matchings, and assume there is not a tree $T$ such that $T'$ can be obtained from $T$ by $K_{1,2}$-surgery or by $K_{1,n}^*$-surgery for any $n \geqslant 3$. (See Lemmas 1 and 4.) Now every vertex $w$ of $T'$, with $\deg(w) = d \geqslant 3$ and at most one branch which is not a path, must have $d = 3$ and exactly two branch paths, each of which has length two. Let $w_1, w_2, \ldots, w_t$ be a listing of all such $w$'s in $T'$. and let $\gamma_i$ be the graph consisting of $w_i$ and its two branch paths. (Each $\gamma_i$ is a $P_5$.) Also, let $x_i$ be the vertex adjacent to $w_i$ which is not in $\gamma_i$. The $x_i$'s may not be distinct, as in Fig. 5 where $x_1 = x_2 = x_3 = x$.

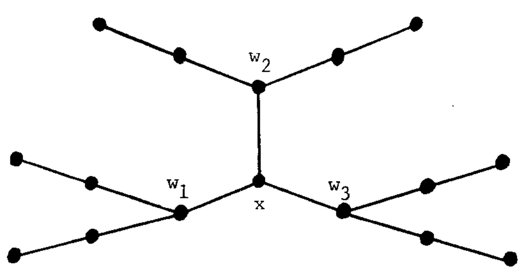


FIG. 5. A tree for which $T_t'$ is an isolated vertex.

From Lemma 5 one has $\deg(x_i) \geqslant 3$ for $1 \leqslant i \leqslant t$. By assumption, $T'$ has no branch paths of length three, and by Corollary 6.2 there are no branch paths at an $x_i$ of length two. To show that $T'$ is obtainable from some $T' - \gamma_i$ by $Z_2$-surgery for some $i$ $(1 \leqslant i \leqslant t)$, it suffices to show that some $x_i$ is adjacent to an endpoint of $T'$.

Since $\deg(x_i) \geqslant 3$ and there are no branch paths at $x_i$ of length two, $x_i \notin \bigcup_{j=1}^{t} V(\gamma_j)$. Letting $T_0' = T'$ and $T_i' = T_{i-1}' - \gamma_i$, it is easy to see that each $T_i'$ is a tree $(1 \leqslant i \leqslant t)$. In particular, $T_t'$ is a tree. If an $x_i$ of degree zero remains (that is, $x_1 = x_2 = \cdots = x_t$, as in Fig. 5), then $\beta_1(T') = 2t + 1$ and any maximum matching will use both edges incident with endpoints from some $\gamma_i$. Since any maximum matching uses at least one edge of $\gamma_i$ incident with an endpoint, there cannot be two disjoint maximum matchings. Thus $\deg(x_i) \geqslant 1$ in $T_t'$.

LEMMA 7. *If $T'$ is as described above, then at least one $x_i$ is adjacent to an endpoint.*

*Proof.* Assume no $x_i$ is adjacent to an endpoint. One now has that every branch of $T'$ from each $x_i$ contains a vertex of degree at least three. It will first be shown that some $x_i$ $(1 \leqslant i \leqslant t)$ has degree one in $T_t'$. Select a value $j1$ $(1 \leqslant j1 \leqslant t)$. If $\deg(x_{j1}) \geqslant 2$ in $T_t'$, then let $p_1'$ and $p_1''$ be vertices of $T_t'$ which are adjacent to $x_{j1}$. On the branch $B_1$ of $T'$ at $x_{j1}$ which contains $p_1''$ there is a vertex of degree at least three in $T'$. This implies that $B_1$ contains a $w_i$ $(i \neq j1)$, and $p_1'' \in T_t'$ implies that $w_i$ is not adjacent to $x_{j1}$. Now one can select $x_{j2}$ on $B_1$ with $j2 \neq j1$. If $\deg(x_{j2}) \geqslant 2$ in $T_t'$, then let $p_2'$ and $p_1''$ be vertices of $T_t'$ which are adjacent to $x_{j2}$. One can assume that the branch $B_2$ of $T'$ at $x_{j2}$ which contains $p_2''$ does not contain $x_{j1}$. Repeating the above argument, one obtains $x_{j3}$ on $B_2$ with $x_{j3} \neq x_{j1}$ and $x_{j3} \neq x_{j2}$. Iterating, one obtains a sequence of distinct vertices $x_{j1}, x_{j2}, x_{j3}, \ldots$. Since $T'$ is finite, some $x_{jk}$ must have degree one in $T_t'$.

Select $i$ such that $x_i$ has degree one in $T_t'$, and let $y$ be the vertex of $T_t'$ adjacent to $x_i$ (as in Fig. 6). Let $M$ be a maximum matching of $T'$ which does not contain edge $yx_i$. (One exists since $T'$ is assumed to have two

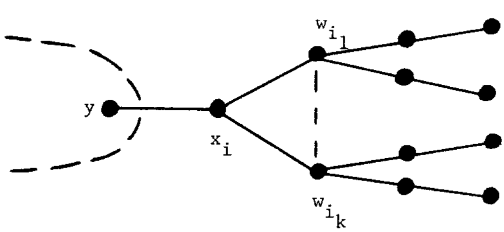


FIG. 6. Structure at an $x_i'$ of degree one in $T_i'$.

disjoint maximum matchings.) Let $\deg(x_i) = k + 1$ in $T'$ $(k \geqslant 1)$. In the component of $T' - yx_i$ containing $x_i$, $M$ has $2k + 1$ edges, two of which are adjacent to endpoints in one $y_{i_j}$ where $x_i w_{i_j} \in M$. Since any maximum matching uses at least one of these two edges, there cannot be two disjoint maximum matchings.

This contradiction shows that at least one $x_i$ is adjacent to an endpoint. ∎

THEOREM 8. *A tree $T$ has two disjoint maximum matchings if and only if $T$ can be obtained from a star $K_{1,m}$ (for some $m \geqslant 2$) by a finite sequence of the following operations:*

(1) *$K_{1,2}$-surgery,*

(2) *$K_{1,n}^*$-surgery $(n \geqslant 3)$, and*

(3) *$Z_2$-surgery.*

*Proof.* Star $K_{1,m}$ has $m \geqslant 2$ disjoint maximum matchings. By Lemmas 1 and 4 and Corollary 6.1, each operation produces a tree with at least two disjoint maximum matchings.

Conversely, assume $T$ has two disjoint maximum matchings. If $T$ is a path, say $P_{2n+1}$, then $T$ can be obtained from $K_{1,2}$ by $n - 1$ $K_{1,2}$-surgeries. If $T$ has exactly one vertex of degree at least three, say $\deg(v) = d \geqslant 3$, then using Lemmas 2 and 3 one can see that $T$ is obtainable from $K_{1,d}$ by a sequence of $K_{1,2}$-surgeries.

Employing induction on the number of vertices of $T$, assume $T$ is a tree with $p$ vertices and any tree with at most $p - 1$ vertices which has two disjoint maximum matchings can be obtained by a suitable sequence of operations. Suppose $T$ has at least two vertices of degree at least three. One may assume that $T$ cannot be obtained from another tree $T^*$ with two disjoint maximum matchings by $K_{1,2}$-surgery or $K_{1,n}^*$-surgery or else, applying the induction hypothesis to $T^*$, it is clear that $T$ can be obtained from a star by a suitable sequence of operations. Applying Lemma 7, one obtains tree $T^*$ such that $T = T^*(x_i) \oplus Z_2(x_i)$. By Corollary 6.1, $T^*$, and hence $T$, can be obtained from a star by a suitable sequence of operations. ∎

## 3. Trees with $k$ Disjoint Maximum Matchings

For $k \geqslant 2$, the set of trees which have at least $k$ disjoint maximum matchings will be denoted by $S_k$. For $k \geqslant 3$, if tree $T$ has a tail of length at least two then, by the first paragraph in the proof of Lemma 1, $T \notin S_k$. Such trees will be excluded for the balance of this section, and it will be assumed that $k \geqslant 3$. Note that a star $K_{1,n}$ is in $S_k$ if and only if $n \geqslant k$.

Suppose $T$ is a tree with at least two vertices that have degree at least three. Let $w$ be a vertex with $\deg(w) \geqslant 3$ for which all but one of its branches are tails. (Each tail is necessarily of length one.) Since $w$ is adjacent to an endpoint, each maximum matching must have an edge incident with $w$. Thus $\deg(w) \leqslant k - 1$ implies $T \notin S_k$. Letting $u_1, u_2, ..., u_t$ be the endpoints adjacent to $w$, it is easy to show that $\beta_1(T) = \beta_1(T - \{w, u_1, ..., u_t\}) + 1$. Let $v$ be the vertex which is adjacent to $w$ and which is not an endpoint. One easily obtains the following two lemmas.

LEMMA 9. *If* $\deg(w) = k$ (*that is,* $t = k - 1$) *and* $\deg(v) \leqslant k$, *then* $T$ *has* $k$ *disjoint maximum matchings if and only if* $T - \{w, u_1, u_2, ..., u_{k-1}\}$ *does.*

LEMMA 10. *If* $\deg(w) \geqslant k + 1$, *then* $T$ *has* $k$ *disjoint maximum matchings if and only if* $T - \{w, u_1, ..., u_t\}$ *does.*

Let $H$ be called obtainable from $G$ by $K_{1,k}$-*surgery* if $H$ can be written as

$G(v) \oplus K_{1,k}(x)$ where $x$ is an endpoint of $K_{1,k}$ and $\deg(v) \leqslant k-1$ in $G$. Recall that $H$ is said to be obtainable from $G$ by $K_{1,n}^*$-surgery if $H = G(v) \oplus K_{1,n}(x)$ where $x$ is an endpoint of $K_{l,n}$ and $v$ is any vertex of $G$.

Define a *k-constellation*, denoted $X_k$, to be the graph obtained from $k$ copies of the star $K_{1,k}$ by identifying one endpoint of each star. For example, $X_2 = P_5$. $X_k$ is as in Fig. 7. Call the vertex of distance two from all the

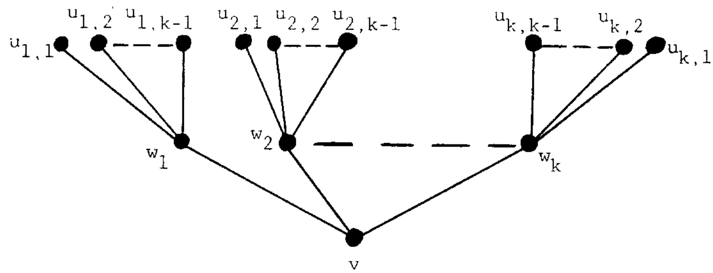


FIG. 7. The $k$-constellation $X_k$.

endpoints the *base vertex* of $X_k$. Note that $\beta_1(X_k) = k$, and $X_k \in S_k$. Furthermore, in any collection $M_1, M_2, \ldots, M_k$ of $k$ disjoint maximum matchings of $X_k$, each $M_i$ must contain an edge incident with the base vertex.

Let $Z_k$ be the graph obtained from $X_k$ by adding another vertex of degree one adjacent to the base vertex. $\beta_1(Z_k) = k + 1$, and $Z_k \notin S_2$.

Let $H$ be said to be obtainable from $G$ by *$Z_k$-surgery* if $H$ can be written as $G(y) \oplus Z_k(x)$ where $x$ is the endpoint of $Z_k$ adjacent to the base vertex and $y$ is a vertex of $G$ which is adjacent to an endpoint.

LEMMA 11. *Suppose tree $T = T'(y) \oplus Z_k(x)$ where $X_k$ is the k-constellation with base vertex $v$ contained in $Z_k$. (Note that* $\deg(v) = k + 1$ *in $Z_k$, and $y$ is the vertex not in $X_k$ which is adjacent to $v$.) If $y$ is adjacent to an endpoint $s$, then $T \in S_k$ if and only if $T' = T - X_k \in S_k$.*

*Proof.* As $y$ is adjacent to endpoint $s$, each maximum matching of $T - X_k$ contains an edge incident with $y$. This implies $\beta_1(T) \leqslant \beta_1(T - X_k) + k$. Assume $M_1, M_2, \ldots, M_k$ are disjoint maximum matchings for $T - X_k$. Since one has $M_1, \ldots, M_k$ and $X_k \in S_k$, one easily obtains $k$ disjoint matchings of $T$, each with $\beta_1(T - X_k) + k$ elements. Thus $T \in S_k$.

Conversely, assume disjoint maximum matchings $M_1, M_2, \ldots, M_k$ for $T$. Assume one $M_i$ contains edge $vy$, say $vy \in M_1$. Then $vw_i \notin M_1$ $(1 \leqslant i \leqslant k)$. Since $\deg(w_i) = k$, each $vw_i$ must appear in one $M_j$ where $2 \leqslant j \leqslant k$. This would imply two edges incident with $v$ are in one $M_j$. Thus $vy \notin M_i$ $(1 \leqslant i \leqslant k)$. This implies $\beta_1(T - X_k) \geqslant \beta_1(T) - k$. Removing all edges of $X_k$ from each $M_i$, one obtains $k$ disjoint maximum matchings for $T - X_k$. ∎

THEOREM 12. *Tree $T \in S_k$ if and only if $T$ can be obtained from a star $K_{1,m}$ (for some $m \geqslant k$) by a finite sequence of the following operations:*

(1) *$K_{1,k}$-surgery,*

(2) *$K_{1,n}^*$-surgery ($n \geqslant k + 1$), and*

(3) *$Z_k$-surgery.*

*Proof.* By Lemmas 9–11 each operation will produce a tree with at least $k$ disjoint maximum matchings.

Conversely, assume $T$ is a tree with $k$ disjoint maximum matchings, namely, $M_1$, $M_2$,..., $M_k$. Since $T$ has no tails of length two, $T$ is not a path, and if $T$ has exactly one vertex of degree at least three, then $T$ is a star $K_{1,m}$ with $m \geqslant k$.

We proceed by induction on the number of vertices of $T$. Suppose $T$ has at least two vertices of degree at least three, and assume that $T$ cannot be obtained from another tree by $K_{1,k}$-surgery or $K_{1,m}^*$-surgery ($m > k$). Let $w_1$, $w_2$,..., $w_t$ be the vertices with degree at least three for which all but one of the branches are tails (of length one), and let $v_i$ be the vertex which is not an endpoint and which is adjacent to $w_i$. (The $v_i$'s may not be distinct.) By assumption, each $w_i$ has degree exactly $k$, and each $v_i$ has degree at least $k + 1$. Letting $L_i$ be the set of vertices containing $w_i$ and each endpoint adjacent to $w_i$, one has $v_i \notin L_j$ ($1 \leqslant j \leqslant t$).

Consider the tree $F = T - \bigcup_{i=1}^{t} L_i$. If one $v_i$ were adjacent to $k + 1$ or more $w_j$'s, say $w_1$, $w_2$,..., $w_s$ ($s \geqslant k + 1$), then since $\deg(w_j) = k$ each edge $v_i w_j$ ($1 \leqslant j \leqslant s$) must appear in one $M_h$. This would imply that some $M_h$ contains two edges incident with $v_i$. Thus each $v_i$ is adjacent to at most $k$ of the $w_j$'s. This implies $F$ has more than one vertex. If $F$ were a $P_2$, then the edge of $F$ would appear in every maximum matching.

Also, $F$ has no tail of length two, for suppose $u_1$, $u_2$, $u_3$ is a tail. Since $k \geqslant 3$, at least one $M_i$ does not contain edges $u_1u_2$ or $u_2u_3$, say $M_1$. If $M_1$ has an edge incident with $u_i$, label it $e_i$ ($i = 1$ or 2). As $M_1$ is maximum, $M_1$ has at least one of $e_1$ and $e_2$. Suppose $e_1 = u_1 w_{i1}$ and $e_2 = u_2 w_{i2}$ are in $M_1$. Let $e_1'$ and $e_2'$ be other edges incident with $w_{i1}$ and $w_{i2}$, respectively. Now $M_1 + e_1' + e_2' - e_1 - e_2 + u_1u_2$ would be a larger matching than $M_1$. Suppose $e_1 = u_1 w_{i1} \in M_1$ and no edge incident with $u_2$ is in $M_1$. Let $e_1'$ be another edge incident with $w_{i1}$, and $M_1 + e_1' - e_1 + u_1u_2$ would be a larger matching than $M_1$. Thus there is not a tail of length two, and $F$ is not a path.

Select a vertex $y$ in $F$ of degree $h + 1 \geqslant 3$ with $y$ adjacent to $x_1$, $x_2$,..., $x_h$ and $y'$ where $x_i$ is an endpoint of $F$($1 \leqslant i \leqslant h$, $h \geqslant 2$). Now $F$ is as in Fig. 8, where the dashed lines indicate edges in $T$ but not $F$.

If endpoint $x_i$ of $F$ is not an endpoint of $T$, then $x_i$ is adjacent to some $w_j$, that is, $x_i = v_j$ for some $j$, $1 \leqslant j \leqslant t$. As has been shown, $\deg(x_i) = \deg(v_j) \geqslant$

$k+1$ and $x_i$ is adjacent to at most $k$ $w_i$'s. This implies that $\deg(x_i) = k+1$ and $x_i$ is the base vertex of a $k$-constellation, $X_k^i$.

At least one $x_i$ is an endpoint of $T$, for suppose there exists an $X_k^i$ for each $i$, $1 \leqslant i \leqslant h$. As in Fig. 8, let $b$ be the number of $w_i$'s adjacent to $y$. Select an $M_d$ ($1 \leqslant d \leqslant k$) which does not contain edge $y'y$. In the component of $T - y'y$ containing $y$ there are $hk + 1 + b$ edges of $M_d$. One can obtain this many independent edges only if $M_d$ has an edge of the form $yx_i$

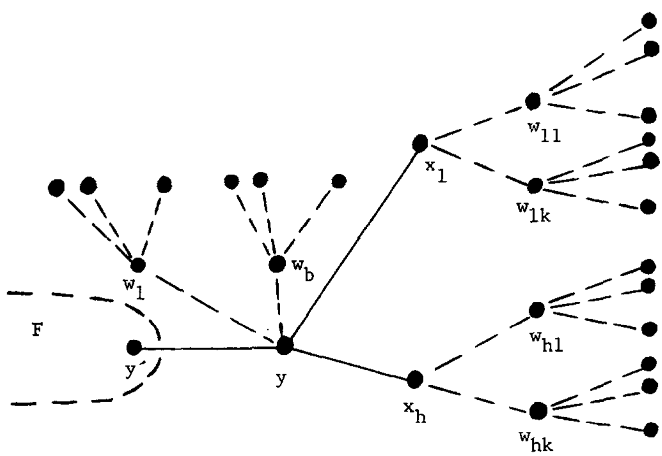


FIG. 8. Substructure at vertex $y$ of graph F.

for some $i$, $1 \leqslant i \leqslant h$. As each edge of $X_k^i$ incident with $x_i$ appears in one $M_j$, each $M_j$ must contain one of these edges, so that $M_d$ would contain two edges incident with $x_i$. Thus one can assume $x_1$ is an endpoint of $T$.

Now $b = 0$, for suppose $b \geqslant 1$. Since $\deg(w_1) = k$, in some $M_i$ the edge incident with $w_1$ would have to be $yw_1$; say this $M_i$ is $M_1$. Let $e_1$ be an edge incident with $w_1$ other than $yw_1$. Now $M_1 + e_1 + yx_1 - yw_1$ would be a matching with more edges than $M_1$.

Since $b = 0$, if each $x_i$ ($1 \leqslant i \leqslant h$) is an endpoint of $T$, then $y$ would be a $w_j$ for some $j$, $1 \leqslant j \leqslant t$, but $y \in F$ implies $y \neq w_j$ for any $j$. Suppose $x_h$ is a base vertex of $k$-constellation $X_k^h$.

The maximal subgraph with $V(X_k^h) \cup \{y\}$ as vertex set is a $Z_k$, and $T = (T - X_k^h)(y) \oplus Z_k(y)$ where vertex $y$ in $Z_k$ is the endpoint adjacent to the base vertex $x_h$, and vertex $y$ in $T - X_k^h$ is adjacent to endpoint $x_1$. Using Lemma 11 and the induction hypothesis, the theorem is proved. ∎

## 4. PARAMETERS OTHER THAN $\beta_1$

The *edge independence number*, $\beta_1(G)$, is the maximum number of edges in an independent set; the *vertex independence number*, $\beta_0(G)$, is the maximum number of vertices in an independent set (no two of the vertices are adjacent); *the vertex covering number*, $\alpha_0(G)$ is the minimum number of vertices in a set $S$ such that every edge is incident with at least one vertex in $S$; the *edge covering number*, $\alpha_1(G)$, is the minimum number of edges in a set $S$ such that every vertex is incident with at least one edge in $S$. Gallai [2] has shown that $\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$ for any nontrivial connected graph where $p$ is the number of vertices.

Since each edge of tree $T$ incident with an endpoint must be in every $\alpha_1$-set, no tree has two disjoint $\alpha_1$-sets. For maximum independent sets of vertices, that is, $\beta_0$-sets, one easily derives the next two lemmas which can be used to prove the following theorem.

LEMMA 13. *If $x_1$, $x_2$, $x_3$ is a tail of tree $T$, then $T$ has two disjoint maximum independent vertex sets if and only if $T - x_1 - x_2$ does.*

LEMMA 14. *If $x$ is a vertex of tree $T$ and $x$ is adjacent to two endpoints, then $T$ does not have two disjoint maximum independent vertex sets.*

THEOREM 15. *A tree $T$ has two disjoint maximum independent vertex sets if and only if $T$ can be obtained from $P_2$ by a finite sequence of $K_{1,2}^*$-surgeries.*

COROLLARY 15.1. *A tree $T$ has two disjoint maximum independent vertex sets if and only if it has a 1-factor.*

COROLLARY 15.2. *A tree $T$ has two disjoint minimum vertex covering sets if and only if $T$ can be obtained from $P_2$ by a finite sequence of $K_{1,2}^*$-surgeries.*

*Proof.* If $T$ has two disjoint $\beta_0$-sets and is obtainable from tree $T'$ by a $K_{1,2}^*$-surgery, then $T$ has exactly two more vertices than $T'$. Label them $u$ and $v$ where $u$ is adjacent to vertex $w \in V(T')$. By induction on the number of vertices in trees with two disjoint $\beta_0$ sets, it can be assumed that $V(T') = S_1' \cup S_2'$ where $S_1'$ and $S_2'$ are disjoint $\beta_0$-sets. Assume $w \in S_i'$. Now $S_1 = S' \cup \{v\}$ and $S_2 = S_2' \cup \{u\}$ are disjoint $\beta_0$-sets of $T$ with $V(T) = S_1 \cup S_2$. Each $S_i$ ($i = 1$ or 2) is an $\alpha_0$-set since it is the complement in $V(T)$ of a $\beta_0$-set. If $A_1$ and $A_2$ are $\alpha_0$-sets in graph $G$ and $u \notin A_1$ and $u \notin A_2$, then every vertex adjacent to $u$ is in $A_1 \cap A_2$. Thus, if a graph $G$ has two disjoint $\alpha_0$-sets, $A_1$ and $A_2$, then $V(G) = A_1 \cup A_2$, and $G$ has two disjoint $\beta_0$-sets. (The complement of an $\alpha_0$-set is a $\beta_0$-set.) This implies that a tree has two disjoint $\beta_0$-sets if and only if it has two disjoint $\alpha_0$-sets. ▮

Let $v$ be an endpoint of tree $T$ with $v$ adjacent to $w$. Any $\beta_0$-set (respectively, $\alpha_0$-set) which does not contain $w$ must contain $v$. Thus no tree contains three or more disjoint $\beta_0$-sets (respectively, $\alpha_0$-sets).

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