The Construction and Classification of Self-Dual Spherical Polyhedra

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In this paper we consider spherical polyhedra, or equivalently 3-connected embedded planar graphs. A self-duality map sends vertices to faces and faces to vertices while preserving incidence. We give six constructions of polyhedra with self-duality maps and show that these constructions yield all such polyhedra. Included is the construction of polyhedra which admit only self-duality maps of large order. © 1992 Academic Press, Inc.

1. INTRODUCTION

One of the oldest concepts in the study of spherical polyhedra is that of duality. It was perhaps first noticed among the Platonic solids, where the hexahedron (or cube) is dual to the octahedron, the dodecahedron is dual to the icosahedron, while the tetrahedron is dual to itself. In Fig. 1.1 we show the isomorphism between the tetrahedron and its dual. Here we depict the polyhedron as a 3-connected planar graph (the equivalence is due to a Theorem of Steinitz [S]). The isomorphism is the permutation (0, A)(1, B)(2, C)(3, D), where the numbers are vertices and the letters are faces. The graph is shown with solid lines, while the dual is shown with dashed lines.

This example of self-duality is particularly intriguing. Although known for thousands of years, there has been no systematic study of such self-dual polyhedra. The purpose of this paper is:
THE MAIN PROBLEM. *Construct and classify all self-dual spherical polyhedra.*

More formally, let $P_1$ and $P_2$ be polyhedra with underlying graphs $G_1$ and $G_2$, respectively. A polyhedral isomorphism $\varphi$ from $P_1$ to $P_2$ is a graph isomorphism from $G_1$ to $G_2$ which preserves facial walks. A polyhedron $P$ is *self-dual* if there is a polyhedral isomorphism between $P$ and its dual $P^*$. A polyhedral isomorphism $\varphi$ which establishes self-duality of a polyhedron $P$ can be interpreted as mapping the vertices of $P$ to faces of $P$. But this map extends to a map sending faces to vertices as follows. Let $(v_0, ..., v_n)$ be the vertices in the boundary walk of some face $f$ of $G$. Then $(\varphi(v_0), ..., \varphi(v_n))$ is the boundary walk of some face of $G^*$. This face of $G^*$ corresponds to a vertex $v$ of $G$. Define $\varphi(f) = v$. Note that under this extension $\varphi$ preserves incidence, that is, if vertex $v$ is incident with face $f$ then face $\varphi(v)$ is incident with vertex $\varphi(f)$. Conversely, for spherical polyhedra any such incidence-preserving map defines a polyhedral isomorphism between $P$ and $P^*$.

Define a *self-duality map* on a polyhedron $P$ as a function $\varphi$ which sends vertices to faces and faces to vertices while preserving incidence. Note that $\varphi^2$ maps vertices to vertices and faces to faces, and hence is an automorphism of the polyhedron. However, despite the fact that "the dual of the dual is the original," $\varphi^2$ need not be the identity. For example, Fig. 1.2 shows a labeling of the vertices and faces of a self-dual polyhedron. The self-duality map $(1)(25D4C3B)(6J9H8G7F)$ is of order 8.

Grünbaum and Shephard [GS] noted that it had been incorrectly stated that the square of a self-duality map was always the identity, and provided
They defined the rank of a self-dual polyhedron as the minimum order of its duality maps. They asked if every polyhedron was of rank 2. (Since the polyhedron of Fig. 1.2 also admits an order 2 duality map $(1A)(2D)(3C)(4B)(5E)(6H)(7G)(8F)(9J)$, it is of rank 2.) This question was negatively answered by Jendrol [J] who gave a polyhedron of rank 4. The problem was completed by McCanna [McC], who found a polyhedron of rank $2^n$ for every $n$, and showed that these were the only possible ranks. Servatius et al. [SC] constructed self-dual polyhedra and wondered if they had all of them. We discuss this in Section 6, showing that they did not.

In this paper we will study both involutory (order 2) and noninvolutory self-duality maps. While we construct all self-dual polyhedra, we do not construct all possible pairs $(G, \varphi)$ where $\varphi$ is a self-duality map. We do, however, discuss ways to restrict the possible self-duality maps and include a systematic method for constructing examples of polyhedra having arbitrarily large rank.

The paper is organized as follows. In Section 2 we give the requisite background material. Included is a reformulation of the main problem to radial graphs, some material on voltage graphs, relevant results on quotients of the sphere by group actions, and a summary of our technique. In Section 3 we give three constructions of polyhedra with involutory self-duality maps. We then give a proof that any such polyhedron comes from one of these constructions. In Section 4 we give three constructions of polyhedra with noninvolutory self-duality maps, and show that these constructions give all such polyhedra. In Section 5 we prove our Main Theorem, describe some easy special cases, and fit the previously known constructions into our classification scheme. In Section 6 we show how to restrict the self-duality maps for the polyhedra of Section 4, and give examples of polyhedra of large rank. Finally, in Section 7 we give some concluding remarks and discuss directions for future research.
2. BACKGROUND MATERIAL

In this section we discuss the requisite background material. The reader is also referred to [GT] where some of the material is developed in more detail.

**Radial and Medial Graphs.** Let $G$ be an embedded graph. The vertices of the radial graph, $R(G)$, are the vertices of $G$ together with the faces of $G$. An edge of the radial graph joins two vertices of $R$ which represent incident elements of $G$. Note that the radial graph is naturally bipartite, with the vertex bipartition of $R$ induced by vertices versus faces of $G$. Note also that $R$ embeds in a natural manner in the same surface as does $G$. In particular, the faces of this embedding are quadrilaterals and are in one-to-one correspondence with the edges of $G$.

The dual of the radial graph in the surface is the medial graph, denoted $M(G)$. Note that the medial graph is 4-regular, and again has a natural embedding in the same surface. The faces of the medial graph may be properly 2-colored, with color classes corresponding to the vertices of $G$ and the faces of $G$. If $G^*$ denotes the dual of the graph $G$, then $M(G) = M(G^*)$ and $R(G) = R(G^*)$. The following lemma is easily established.

**Lemma 2.1.** Let $G$ be a 3-connected graph embedded in the plane. Then both the radial graph and the medial graph are 3-connected.

We ask under what conditions is an embedded graph $R$ a radial graph? As noted above, it is necessary that the graph be bipartite and have every face a quadrilateral. In fact, these conditions are sufficient. For in such a graph we can 2-color the vertices, black and white. Let $G$ be the graph having only the black vertices, with edges joining the two black vertices in each quadrilateral face. Then $G$ is embedded naturally in the same surface, and $R(G) = R$. If we had chosen the white vertices in place of the black vertices, we would have obtained the dual graph $G^*$. Similarly, any 4-regular embedded graph $M$ whose faces are 2-colored is a medial graph of some dual pair. Note that any plane quadrangulation is necessarily bipartite, and hence a radial graph. Likewise any planar 4-regular graph necessarily has a proper 2-coloring of the faces, and hence is a medial graph.

We reformulate the main problem in terms of the radial graph. Recall that a self-duality map $\varphi$ of an embedded 3-connected graph $G$ is a mapping from vertices to faces and from faces to vertices of $G$ which preserves incidence. As the vertices of $R(G)$ form the domain of $\varphi$, and the edges of $R(G)$ record incidences, it follows that $\varphi$ induces an automorphism on $R(G)$ which reverses parts in the bipartition. Conversely, any such part-
reversing automorphism induces a self-duality map on $G$. Define a radial self-duality map as such a part-reversing automorphism.

The main problem may be reformulated as follows:

**The Main Problem Reformulated.** Construct and classify all quadrangulations of the plane which admit a radial self-duality map.

*Voltage and Quotient Graphs.* We begin with a graph $G$ and a cyclic group $\mathbb{Z}_n$. (The general theory of voltage graphs allows an arbitrary group, but cyclic groups suffice for our purposes.) A voltage assignment is a map $v$ from the directed edges of $G$ to $\mathbb{Z}_n$ such that oppositely directed arcs receive inverse group elements. In practice, we usually direct the edges of $G$ and assign a group element; it is understood that the oppositely directed edge receives the inverse element.

Given a graph $G$ with a voltage assignment $v$ we form the derived graph, $\tilde{G}$, with vertex set $V(G) \times \mathbb{Z}_n$ and an edge joining $(u, a)$ to $(v, b)$ if and only if $uv \in E(G)$ and $v(uv) = b - a$. There is a natural projection map $\rho$ from $\tilde{G} \to G$ defined by suppressing group elements. Observe that $\tilde{G}$ has $n$ times the vertices and edges of $G$, moreover $\rho$ maps these vertices and edges to $G$ in an $n$-to-1 fashion. For $v \in V(G)$ we call $\rho^{-1}(v)$ the fiber above $v$.

Let $C$ be a directed cycle of $G$. We can suppose that the edges of $C$ are directed so as to agree with the direction on $C$. Define the net voltage on $C$ as the sum of the voltages on its edges.

**Lemma 2.2** Two voltage assignments which give the same net voltage on each directed cycle in the graph generate isomorphic derived graphs.

*Proof.* Let $v$ be a vertex of the voltage graph. Suppose that the edges incident with $v$ are all directed outward. Modify the voltage assignment by adding $i$ to each edge incident with $v$. Note that the derived graph of this new assignment is isomorphic to the old graph, we merely map the vertex $(v, j)$ to $(v, j - i)$ for each $j$ and fix the vertices in other fibers. Also note that we have not changed the voltage assigned to any cycle.

Following a sequence of these modifications we may assume that the voltage assignment is identically zero on the edges of any particular spanning tree $T$. Let $e$ be an edge not in this tree. Let $C$ be the unique cycle in $T \cup e$, and assign a direction to $C$. There is some voltage $x$ assigned to the directed $C$. As edges in $T$ received zero voltage, $e$ must receive voltage $x$ in the direction induced by that on $C$.

If we begin with two different voltage assignments on a graph $G$ the above process reduces them to voltage assignments which are identically zero on $T$, without changing the isomorphism class of the derived graph. If these assignments give the same net voltage assignment on the directed cycles, then the preceding paragraph shows that they must give the same
voltage assignment on $E(G) - E(T)$. It follows that the two reduced voltage assignments are identical, and hence that the two original voltage assignments generate isomorphic derived graphs.

The group $\mathbb{Z}_n$ acts on $\tilde{G}$ in a natural manner which respects the quotient map $\rho$. Namely, the automorphism $j$ sends vertex $(v, i)$ to $(v, i + j)$. Call this group of automorphisms the \textit{deck transformations}. Note that no vertex or edge is fixed under any non-identity deck transformation.

More generally, we say that an automorphism acts \textit{freely} on $\tilde{G}$ if no vertex or edge is fixed by any power of the automorphism. An automorphism (or a power thereof) which fixes no vertex may still fix an edge by switching the endpoints. In this case, we call the edge \textit{reflexive}.

Suppose that we are given a graph $G$ and a cyclic group of automorphisms $\mathbb{Z}_n$ acting freely on $G$. When are these the deck transformations of some voltage construction? Define the \textit{quotient graph} as the graph whose vertices and edges are the orbits of the vertices and edges of $G$ under the group action. Note that since the action is free, the projection map $\rho$ maps the vertices and edges of $G$ to the vertices and edges of the quotient graph in an $n$-to-$1$ manner.

\textbf{Lemma 2.3}. If a group acts freely on $\tilde{G}$ then there is a voltage assignment on the quotient graph $G$ which constructs $\tilde{G}$. Moreover, the group acts on $\tilde{G}$ as the deck transformations of the covering.

\textit{Proof}. See Theorem 2.2.2 in [GT].

Loosely speaking, constructing a covering graph from a voltage assignment, and constructing a quotient graph under a free group action are inverse operations.

\textit{Embeddings and Quotient Surfaces}. An embedding of a graph in an orientable surface can be described by a rotation scheme, a cyclic permutation of the incident edges at each vertex. Given an embedded $G$ with a voltage assignment $v$ from $\mathbb{Z}_n$, we can embed the derived graph $\tilde{G}$ by lifting the rotation scheme. Let $f$ be a face of $G$ receiving a net voltage of order $i$. Then $f$ lifts to $n/i$ faces in $\tilde{G}$, each $i$ times as long (see Theorem 4.1.1 in [GT]). This fact makes it easy to count the number of faces in the derived embedding, and hence to calculate the Euler characteristic of its surface. Similarly, we may lift embeddings of base graphs in nonorientable surfaces. The reader is referred to [GT] for details.

As with graphs, we can define a group action on a surface. The action is \textit{free} if it fixes no point of the surface, and is \textit{pseudofree} if there are a finite number of fixed points. We can define the quotient space under the equivalence relation defined by the group action. It can be shown that if the
action is pseudofree, then this quotient space is again a surface. Moreover the quotient map is a covering projection on the surface minus the fixed points. These fixed points are called prebranch points, their image in the quotient space are called branch points. The quotient map \( \rho \) is called a branched covering. The order of a prebranch point \( v \) is \( k \) if \( \rho \) is \( k \)-to-1 in a small neighborhood of \( v \).

Let \( G \) be a graph embedded in an orientable surface. Let \( \varphi \) be a free group action of \( G \) and let \( \pi_v \) denote the cyclic rotation of edges about a vertex \( v \). Then \( \varphi \) is orientation-preserving if \( \pi_{\varphi(v)^{-1}} = \varphi \pi_v \) for each \( v \) and orientation-reversing if \( \pi_{\varphi(v)^{-1}} \varphi = \varphi \pi_v \) for each \( v \).

**Lemma 2.4.** Let \( G \) be an embedded graph with a free action such that every automorphism either preserves or reverses orientation. Then the group action extends to an action on the surface. Moreover, this extended action can be chosen to have at most one fixed point inside any face.

**Proof.** See Theorem 4.3.2 in [GT].

Observe that if an automorphism \( \varphi \) maps faces to faces, then it generates a group in which every element either preserves or reverses orientation. Moreover, if \( G \) is a 3-connected planar graph then the faces are precisely the cycles with exactly one bridge [T]. It follows that any graph automorphism maps faces to faces, and hence extends to an action on the sphere. In particular, the radial self-duality map of a 3-connected polyhedron extends to an action on the sphere. Lemma 2.5 gives the possible quotients of the sphere by this action.

**Lemma 2.5.** Suppose that \( \varphi \) generates a cyclic group of order \( n \) acting pseudofreely on the sphere. Let \( S \) be the quotient surface under this action. Then one of the following holds:

1. \( S \) is the sphere and there are exactly two branch points,
2. \( S \) is the real projective plane, there is exactly one branch point, and \( n > 2 \), or
3. \( S \) is the real projective plane, there are no branch points, and \( n = 2 \).

**Proof.** See Theorems 6.3.1 and 6.3.2 in [GT].

**Summary.** We can now describe the main technique of the proof. Suppose that \( G \) is a 3-connected planar graph with self-duality map \( \varphi \) of order \( n \). Then \( \varphi \) induces a radial self-duality map on the 3-connected radial graph \( R(G) \). Moreover, the medial of the radial graph, denoted \( M^2(G) \), has an automorphism induced by \( \varphi \). Since \( M^2(G) \) is 3-connected (Lemma 2.1), it has a unique embedding. We will show that (usually) this action is free on \( M^2(G) \), and hence extends to an action on the sphere (Lemma 2.4).
follows that the quotient graph of $M^2(G)$ lies on one of the surfaces in Lemma 2.5. We systematically examine the construction of self-dual polyhedra using embedded voltage graphs on these surfaces. Lemmas 2.2 and 2.3 are then used to show that these constructions give rise to all self-dual polyhedra.

We conclude this section with the following lemma.

**Lemma 2.6.** Let $G$ be an embedded 3-connected planar graph with radial graph $R(G)$. Then any orientation-preserving automorphism which fixes a directed edge of $R(G)$ is the identity map.

**Proof.** Let $\theta$ be any orientation-preserving automorphism that fixes the directed edge $xy$, so $\theta(x) = x$ and $\theta(y) = y$. Because $\theta$ preserves the orientation at $x$ and fixes the edge $xy$, it must fix each directed edge from $x$. But now $\theta$ also fixes the directed face containing $xy$, and hence fixes a directed edge at each vertex in that face boundary. As above, each directed edge incident with these vertices is fixed. Continuing in this manner, it follows by connectedness that $\theta$ is the identity.

**3. Involutions**

In this section we give three constructions of self-dual radial graphs with involutory duality maps. We then prove that these construction give rise to all such graphs.

**Construction 3.1. The folding construction.**

We begin with a graph $H$ embedded in the plane with all faces quadrilaterals, except possibly the unbounded face. Let $\bar{H}$ be another copy of $H$, embedded in the plane with a common unbounded face, such that all local rotations are reversed. Thus, for example, if the unbounded face of $H$ is bounded by $0, 1, ..., 7$ (read clockwise), then the unbounded face of $\bar{H}$ is bounded by $7, 6, ..., 0$ (again, read clockwise). It follows that we may join corresponding vertices of $H$ and $\bar{H}$ by edges in their common unbounded face. Note that every face of the resulting $R(G)$ is a quadrilateral, so that $R(G)$ is indeed a radial graph of some $G$.

We claim that the automorphism $\varphi$ which swaps corresponding vertices in $H$ and in $\bar{H}$ is a radial self-duality map. All that we need show is that it reverses parts of the vertex bipartition. But two corresponding vertices on the common incident face are switched by $\varphi$. Since these are adjacent it follows that $\varphi$ switches vertex parts.

We note that any radial self-duality map constructed as above is an orientation-reversing involution. We also note that there is a cut-set of
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Construction 3.3. The projective planar construction.

We begin with a nonbipartite quadrangulation of the real projective plane, say $H$. Since each face is a quadrilateral, any contractible cycle is of even length. However, the graph is nonbipartite, so that it follows that there is a noncontractible cycle of odd length. Even more strongly, since any noncontractible cycle can be written as the $\mathbb{Z}_2$ sum of a fixed noncontractible cycle and some face boundaries, it follows that any noncontractible cycle is of odd length.

Let $p$ be the quotient map of the 2-fold covering of the real projective plane by the sphere. Consider the planar graph $R(G) = p^{-1}(H)$ (the notation will be justified momentarily). This graph is connected since $H$ is. Moreover, since each face of $H$ was a quadrilateral, the same is true of $R(G)$. But in the plane the face boundaries generate the cycle space, so that $R(G)$ is bipartite. By the results of Section 2, $R(G)$ is the radial graph of some $G$, and the notation is justified.

For each $v \in V(H)$, there are two vertices $v_1, v_2$ in $p^{-1}(v)$. Let $\varphi$ be the deck transformation which reverses each such pair. This $\varphi$ is clearly a fixed-point free automorphism of $R(G)$ having order 2. We claim that $\varphi$ is in fact a radial self-duality map. We only need show that $\varphi$ reverses parts of the vertex bipartition. Recall that $H$ was nonbipartite. Hence for each $v$ there exist a noncontractible cycle $C$ of odd length containing $v$. Let $\tilde{C}$ be the lift of this cycle in $M(G)$. Then $v_1, v_2$ separate $\tilde{C}$ into two paths of equal length. Since the length of $\tilde{C}$ is twice the length of $C$, it follows that $v_1$ and $v_2$ are of odd distance in $R(G)$. Hence $\varphi$ reverses vertex parts as desired.

We note that any self-duality map constructed as above has an orientation-reversing involutory self-duality map. Moreover, there are no reflexive edges. In Fig. 3.4 we give an example of this construction. Here the graph

![Figure 3.2](image-url)
$H$ is $K_4$ (the vertices on the dotted circle are identified to give the real projective plane). The covering $R(G)$ shown in the right side of the figure is the hexahedron. This $R(G)$ is the radial graph of the tetrahedron $G$ with the self-duality map given in Fig. 1.1.

**Construction 3.5. The near quadrangulation construction.**

We begin with a near-quadrangulation, i.e., a planar graph in which every face is a quadrilateral except for two triangles. Moreover, we suppose that triangle $T_i$ ($i=1,2$) has a distinguished incident vertex $v_i$, and that these two vertices are distinct. We will construct the desired graph with a radial self-duality map from this near-quadrangulation $H$ using a voltage graph construction.

Place a voltage 1 from the voltage group $\mathbb{Z}_2$ on each edge of $H$. Under this voltage assignment every face of $H$ gets net voltage 0, except for the two distinguished triangles which receive net voltage 1.

Let $\tilde{H}$ be the graph derived using this voltage assignment on $H$. Then each face of the embedded $\tilde{H}$ is a quadrilateral, except for two hexagons which are lifts of the triangles. If $v$, $e$, and $f$ ($\tilde{v}$, $\tilde{e}$, $\tilde{f}$, respectively) are the numbers of vertices, edges, and faces of $H$ ($\tilde{H}$, respectively), it follows that $\tilde{v} = 2v$, $\tilde{e} = 2e$, and $\tilde{f} = 2 + 2(f - 2)$. Since $v - e + f = 2$, it follows that $\tilde{v} - \tilde{e} + \tilde{f} = 2$ and hence the derived surface is the sphere.

Note that in each hexagon there are now two distinguished vertices, $v_i$ and $\tilde{v}_i$, in the fiber above $v_i$. These vertices are of distance 3 in the hexagon. Let $R(G)$ be the graph formed by adding in a chord through the hexagon joining $v_i$ and $\tilde{v}_i$. In $R(G)$, the hexagon has become two quadrilaterals. So $R(G)$ is the radial graph of some $G$. We claim that the deck transformation of the derived graph extends to a radial self-duality map of $R(G)$. It is clear that it extends to an automorphism of $R(G)$. Moreover, this automorphism swaps the ends of the two added chords, so that it must be part-reversing.

We have constructed a radial self-duality map as desired. Any map constructed as above is an orientation-preserving involution. Moreover, this involution acts freely on the vertices, but there are exactly two reflexive edges. In Fig. 3.6 we give an example of Construction 3.5, starting with $H$ on the left and the derived $R(G)$ on the right.
THEOREM 3.7. Any 3-connected plane quadrangulation with involutory radial self-duality map arises from either Construction 3.1, 3.3, or 3.5.

Proof. Let $R = R(G)$ be such a graph with radial self-duality $\varphi$. Let $A_R$ be the subgraph of $R$ induced by the reflexive edges ($A_R$ may be the empty graph). Let $A_M$ be the subgraph induced in the dual, $M(G)$, by the edges corresponding to those in $A_R$.

Case 1

$A_M$ has a vertex of degree exceeding one. Then in $R$ there is a quadrilateral face incident with more than one reflexive edge. By 3-connection any vertex is incident with at most one reflexive edge. It follows that there are exactly two reflexive edges and that the face is fixed by $\varphi$. The other face incident with one of these reflexive edges must also be fixed by $\varphi$, and so it is again incident with exactly two reflexive edges. Continuing in this way, there is a cycle $C_M$ contained in $A_M$.

Let $C_R$ denote the edges in $R$ corresponding to $C_M$. Then $C_R$ is a minimal edge-cut of $R$. Let $H$ denote the component of this cut corresponding to the subgraph of $M(G)$ inside the cycle $C_M$. For each edge of $C_R$, $\varphi$ takes the incident vertex in $H$ to the incident vertex not in $H$. By minimality of the edge-cut, $H$ is connected, and so every vertex of $H$ is mapped to a vertex not in $H$. Similarly, every vertex not in $H$ is mapped to a vertex in $H$. Hence, the only reflexive edges are those in $C_R$, i.e., $A_R = C_R$. It follows that $R$ and $\varphi$ arise from the embedded $H$ by Construction 3.1.

Case 2

$A_M$ is of maximum degree at most one. In this case, $A_M$ is a matching. Let $e$ be a reflexive edge with ends $w$ and $\varphi(w)$ and let $F$ and $F'$ be the faces of $R$ incident with $e$. No other reflexive edge is incident with either of these faces. Let $x$ and $y$ be the other vertices of $R$ incident with $F$. Since $\varphi$ preserves faces, $\varphi(x)$ and $\varphi(y)$ are the other vertices of $F'$. The graph $R - e$
has a part-reversing automorphism, namely the restriction of \( \varphi \). The two quadrilaterals \( F \) and \( F' \) have become a single hexagon, which is preserved under \( \varphi \). It follows that \( R - E(A_R) \) is an embedded bipartite graph with each face a quadrilateral or a hexagon. Moreover, the automorphism \( \varphi \) fixes each hexagon, does not fix any quadrilateral, and there are no fixed vertices or edges.

We form the embedded quotient \( H \) of \( R - E(A_R) \) using the group action of order 2 generated by \( \varphi \). The quadrilateral faces of \( R - E(A_R) \) map to quadrilateral faces of \( H \) in a two-to-one manner. The hexagonal faces of \( R - E(A_R) \) map to triangular faces of \( H \) in a one-to-one manner. Moreover there is a branch point of order 2 in each of these triangular faces.

Subcase 2.1. The quotient surface is the real projective plane. We will show that \( H \) is as in Construction 3.3. Since \( \varphi \) is an involution, the covering is 2-to-1. Lemma 2.5 implies that there are no branch points, so there are no reflexive edges. Hence every face of \( H \) is a quadrilateral. Since \( v \) and \( \varphi(v) \) are in different vertex parts of \( R - E(A_R) \) there is a walk of odd-length joining them. The image of this walk under the quotient map is a closed odd-length walk in \( H \). It follows that \( H \) is nonbipartite. Thus \( H \) is as in Construction 3.3; moreover, \( R \) arises from \( H \) in the same manner.

Subcase 2.2. The quotient surface is the sphere. By Lemma 2.5 there are exactly two branch points. In particular, \( R \) has exactly two reflexive edges. Hence \( R \) arises from \( H \) using Construction 3.5. The distinguished vertex in a triangular face of \( R \) is the vertex orbit incident with the reflexive edge.

By Lemma 2.5 these two subcases exhaust the possibilities, and the theorem is demonstrated.

4. Noninvolutions

We turn our attention to polyhedra with noninvolutory self-duality maps. These arise from three basic constructions. Two of these constructions give self-duality maps of order 4, while the third constructs self-dualities of any even order. We begin with the most general unwrapping construction, 4.1. Following are the two-order-four constructions, 4.3 and 4.5. In Theorem 4.7 we show that any self-dual polyhedron with no involutory self-duality map comes from one of these three constructions.

Construction 4.1. The unwrapping construction.

We begin with a nonbipartite quadrangulation of the projective plane \( H \) having a distinguished vertex \( v \). Let \( M(H) \) be the medial graph of \( H \), and let \( f_v \) be the face corresponding to \( v \). The dual of \( M(H) \) is the radial graph \( R(H) \).
Pick a noncontractible cycle $C_R$ in $R(H)$ which contains the vertex $f^*_v$ corresponding to $f_v$. Let $C_M$ be the set of edges in $M(H)$ corresponding to those in $C_R$. Since $C_R$ is noncontractible, it is unilateral. However, we can distinguish a left and a right side of $C_R - f^*_v$. Direct the edges of $C_M$ so that they cross $C_R - f^*_v$ from left to right. Assign these edges voltage 1 in the group $\mathbb{Z}_{2n}$. All other edges receive voltage zero. Observe that each face receives voltage 0, except for the distinguished face $f_v$ which receives voltage 2. Moreover, each directed noncontractible cycle receives voltage ±1. (If the distinguished vertex is of degree 1 then we first assign voltages as above so that its neighbour $u$ corresponds to the unique face $f_u$ with net voltage 2. By then adding a voltage of 2 to the loop in $M(H)$, we get a net voltage of 0 on $f_u$ and 2 on $f_v$ as desired.)

Using the above voltage assignement $v$ we construct the derived graph $M^2(G)$ (we will justify this notation for the derived graph momentarily). By Lemma 2.2 this derived graph is determined by the assignment of net voltages to cycles. This assignment depends only on the distinguished face, and is independent of the choice of $C_R$.

**Claim.** The derived embedding is planar. Let $p$, $q$, $r$ be the number of vertices, edges, and faces of the embedded $M(H)$, so that $p - q + r = 1$. Recall that $2n$ is the order of the voltage group. The derived graph has $2np$ vertices and $2nq$ edges. Each face except $f_v$ receives zero voltage, hence lifts to $2n$ faces of the same length. The distinguished $f_v$ gets voltage 2, so it lifts to 2 faces each $n$ times as long as $f_v$. The claim follows by calculating the Euler characteristic of the derived embedding:

$$
\chi - 2np - 2nq + 2n(r - 1) + 2 = 2n(p - q + r) - 2n + 2 = 2.
$$

**Claim.** The derived embedding is the second medial of a planar graph $G$. Since $M^2(G)$ is planar and 4-regular, the faces can be 2-colored. Thus it is a medial graph. The faces of $M(H)$ fall naturally into two parts. The ones corresponding to faces of $H$ are all quadrilaterals. Moreover, these receive zero voltage, and hence lift to quadrilaterals. So $M^2(G)$ is the medial graph of some quadrangulation $R(G)$. Since any planar quadrangulation is bipartite, $R(G)$ is the radial graph of some planar graph $G$. This establishes the claim and justifies the notation $M^2(G)$.

It is this $G$ which is the desired self-dual polyhedron. We will show momentarily that the deck transformation induces a radial self-duality map on $R(G)$. We pause to observe that $R(G)$ does not cover $H$, as the vertices of $R(G)$ corresponding to the faces covering $f_v$ have degrees $n$ times too large. It is for this reason that we passed to the medial graph $M(H)$ and its cover $M^2(G)$ in the construction. This multiplication of degrees will later prove useful in restricting automorphisms of $R(G)$, and hence in restricting the possible self-duality maps on $G$.
Claim. The deck transformation of $M^2(G)$ induces an order $2n$ radial self-duality map on $R(G)$. Let $\varphi$ be the map which sends vertex $(u, i)$ to $(u, i + 1)$. Then $\varphi$ generates the deck transformations of $M^2(G)$. By the nature of the derived embedding, $\varphi$ carries faces to faces. It follows that $\varphi$ induces an order $2n$ automorphism on $R(G)$.

Let $f'_v$ and $f''_v$ denote the two faces above $f_v$, and let $u$ be a vertex incident with $f_v$. Since $f_v$ receives a net voltage of 2, one of the faces, say $f'_v$, is incident with vertices $(u, i)$ where $i$ is even and the other face, $f''_v$, is incident with those $(u, i)$ having $i$ odd. It follows that $\varphi$ interchanges $f'_v$ and $f''_v$. But $H$ was nonbipartite, so that $f'_v$ and $f''_v$ correspond to vertices in different vertex parts of $R(G)$. It follows that $\varphi$ switches the vertex parts in $R(G)$, and hence is a radial self-duality map as claimed.

The left portion of Fig. 4.2 shows a projective planar $M(H)$ (identifying antipodal points on the boundary circle). The underlying graph $H$ has 5 vertices, labeled $a, b, c, d,$ and $e$. Each unlabeled face of $M(H)$ corresponds to a face of $H$; these are all quadrilaterals. Finally, note that $(a, b, c)$ is a 3-cycle in $H$, so that it is nonbipartite. Thus this $H$ is a seed graph for the above construction. In this figure we also indicated a voltage assignment from $\mathbb{Z}_6$ in which $e$ is the only face with nonzero net voltage. The right portion of Fig. 4.2 shows the derived graph $M^2(G)$ from this voltage assignment. The quotient map sends face $x_i$ to face $x_i$, for $x \in \{a, b, c, d, e\}$. The deck transformation $\varphi$ sends face $x_i$ to $x_{i+1}$, where the indices are read modulo 6 (except for the modulo 2 indices on $e$). So $\varphi$ corresponds to a clockwise rotation of $2\pi/6$ radians together with an "inside-outside" flip. The square $\varphi^2$ induces an automorphism of the underlying $G$ represented by a clockwise rotation of $2\pi/3$ radians.
We have finished the unwrapping construction. Beginning with a non-bipartite quadrangulation of the projective plane and one distinguished vertex, we have constructed a planar graph \( G \) with a self-duality map of order \( 2n \). We note that any self-duality map thus constructed is orientation-reversing. (If it were orientation-preserving we could induce a consistent orientation on the quotient, contradicting that the projective plane is non-orientable.) Also note that these self-duality maps have no reflexive edges.

In Construction 4.1 the branch point was on a vertex of the quotient graph embedded in the real projective plane. The following two special constructions have the branch point(s) in the faces of the embedded quotient graph. These two constructions build only order four duality maps.

**Construction 4.3. The special projective planar construction.**

We begin with a nonbipartite graph \( H \) embedded in the real projective plane so that each face is a quadrilateral except for one digon. As in Construction 4.1, we can make a voltage assignment to \( H \) from \( \mathbb{Z}_4 \) such that every quadrilateral face gets 0 voltage, the digon gets voltage 2, and every directed noncontractible cycle gets voltage \( \pm 1 \). In the derived graph each quadrilateral face lifts to four quadrilaterals, and the digon lifts to a pair of quadrilaterals. It follows from Euler characteristic calculations that the derived surface is the sphere. Moreover, the derived graph is a bipartite quadrangulation, so it is a radial graph \( R(G) \). Again, \( H \) was nonbipartite, so that a generator of the deck transformations must switch parts. Hence this generator is a radial self-duality map.

We note that any such \( R(G) \) which arises from Construction 4.3 has an orientation-reversing radial self-duality map which is a free action of order 4.

An example of this construction is given in Fig. 4.4. As in Fig. 4.2, we show \( M(H) \) and the derived \( M^2(G) \). The vertices of \( H \) correspond to the faces labeled \( a, b, c, \) and \( d \); the vertices of \( R(G) \) are labeled accordingly.

![Figure 4.4](image-url)
Note that the digon in the projective plane embedding lifts to two quadrilateral faces, the square in the middle and unbounded outside face.

Construction 4.5. The special near quadrangulation construction.

We begin with a planar graph $H$ in which every face is a quadrilateral except for two faces bounded by loops. Then there is a voltage assignment from $\mathbb{Z}_4$ such that every quadrilateral gets net voltage 0 and the two loops from $\mathbb{Z}_4$ get net voltages $\pm 1$. It follows that in the derived graph each quadrilateral face lifts to 4 quadrilateral faces, and that each face of size 1 lifts to a single face of size 4. Since the derived graph is a quadrangulation of the sphere, it is a radial graph $R(G)$. Moreover, a generator of the deck transformations acts on a quadrilateral with a prebranch point as a rotation by $\pi/2$ radians. So this map reverses parts of the vertex bipartition, and is a radial self-duality map.

We note that any radial self-duality map constructed in this manner in orientation-preserving with a free action of order 4. Figure 4.6 illustrates this construction.

Theorem 4.7. Let $G$ be a 3-connected planar graph with self-duality map $\varphi$ of order $2n > 2$. Then either $G$ has an involutory self-duality map, or $G$ arises from one of Constructions 4.1, 4.3, or 4.5.

Proof. Let $R(G)$ be the radial graph of $G$, and let $M^2(G)$ be the medial graph of $R(G)$. By Lemma 2.1, $R(G)$ and $M^2(G)$ are 3-connected. Hence $\varphi$ induces a face-preserving automorphism on $M^2(G)$, which we shall also call $\varphi$.

Claim. Either $G$ has an involutory self-duality map, or the group generated by $\varphi$ acts freely on the vertices of $M^2(G)$. For suppose that the the group generated by $\varphi$ does not act freely on the vertices of $M^2(G)$. Let $v$ be a vertex of $M^2(G)$, let $e = xy$ be the corresponding edge of $R(G)$, and
let \( \varphi' \) fix \( v \) and \( e \). Then either \( \varphi'(x) = x \) and \( \varphi'(y) = y \) or \( \varphi'(x) = y \) and \( \varphi'(y) = x \). In the first case, \( \varphi' \) fixes parts of the bipartition, so \( r \) is even and \( \varphi' \) is orientation-preserving. But \( \varphi' \) fixes a directed edge, and so is the identity map by Lemma 2.6. In the second case, \( \varphi' \) switches parts of the vertex bipartition of \( R(G) \), and hence is a radial self-duality map. Since \( \varphi^{2r} \) fixes the directed edge \( xy \), it is the identity map. Thus \( \varphi' \) is an involutory self-duality map, and the claim is established.

We henceforth assume that the group generated by \( \varphi \) acts freely on the vertices of \( M^2(G) \).

Claim. The group generated by \( \varphi \) acts freely on the edges of \( M^2(G) \). For suppose that \( uv \) is any edge of \( M^2(G) \) fixed by \( \varphi' \). Let \( xy \) and \( yz \) be the edges of \( R(G) \) corresponding to \( u \) and \( v \), respectively. Then \( \{ \varphi'(xy), \varphi'(yz) \} = \{ xy, yz \} \). Hence \( \varphi'(y) = y \) and \( \varphi' \) fixes the parts of the bipartition. So \( r \) is even, and \( \varphi' \) is orientation-preserving. Either \( \varphi'(x) = x \) or \( \varphi'(x) = z \). In the first case, \( \varphi' \) is the identity by Lemma 2.6. In the second case, the path \( (x, y, z) \) is mapped to \( (z, y, x) \) by \( \varphi' \), so that the rotation at \( y \) must be reversed. This contradicts that \( \varphi' \) is orientation-preserving and establishes the claim.

We have shown that the group generated by \( \varphi \) acts freely on \( M^2(G) \). By Lemma 2.4 the free action extends to an action on the sphere. By Lemma 2.5, the quotient surface of this action is either the sphere with two branch points, the real projective plane with one branch point, or the real projective plane with no branch points and a 2-to-1 covering. Moreover, the quotient map induces an embedding of the quotient graph in the surface. We consider the three cases in reverse order.

Case 1

The quotient surface is the real projective plane and there are no branch points. Then by Lemma 2.5 the covering is 2-to-1. But by assumption \( \varphi \) is noninvolutory, so that the covering is \( 2n \)-to-1, where the order \( 2n \) of \( \varphi \) exceeds 2. This contradiction eliminates Case 1.

Case 2

The quotient surface is the real projective plane with one branch point. Since the action is free on \( M^2(G) \), this branch point must lie in a face of the embedded quotient graph. By Lemma 2.3 we can reconstruct \( M^2(G) \), from the quotient by a voltage construction. The faces not containing a branch point lift to faces of the same length, so they must get voltage 0. By Euler's formula it follows that the face with the branch point gets a voltage of order \( n \).

Color the faces of \( M^2(G) \) white if they correspond to faces of \( R(G) \), and black if they correspond to vertices. Each white face is a quadrilateral. But
$M^2(G)$ is a planar 4-regular map, so that by Euler's formula it must contain a face of size at most three. Hence the black faces must contain a nonquadrilateral. It follows that $\varphi$ maps black faces to black faces and white faces to white faces. Thus in the embedded quotient graph, we can color the faces black or white so that the quotient map respects face colors. Since this quotient is 4-regular and 2-face-colorable, it is a medial graph $M(H)$ for some pair of dual graphs $H$ and $H^*$. Suppose that $H$ is the graph whose vertices are the black faces of the quotient.

Subcase 2.1. The branch point lies in a black face of $M(H)$. All but one of the faces of the embedded quotient graph together with one noncontractible cycle form a basis of the $\mathbb{Z}_{2n}$-cycle space. (See [R] for the theory of cycle spaces over commutative rings.) Since the derived graph is connected, the net voltage assignments on these cycles must generate $\mathbb{Z}_{2n}$. If we exclude the distinguished face (i.e., the one containing the branch point), all basis elements but the essential cycle have net voltage zero. It follows that a noncontractible cycle gets assigned a generator of $\mathbb{Z}_{2n}$, without loss of generality. Moreover, because the sum of the voltages assigned to the face boundaries is twice that assigned to the noncontractible cycle, the net voltage on the distinguished face is 2. Finally, we conclude that $H$ is nonbipartite, or else the derived graph would not be connected. It follows from Lemma 2.2 that this voltage assignment gives the same derived graph as the one in Construction 4.1. Hence $M^2(G)$, and so $G$, must arise from that construction.

Subcase 2.2. The branch point lies in a white face of $M(H)$. This face lifts to a quadrilateral face of $M^2(G)$, so that its boundary has one, two, or four edges. One edge is impossible, as $H^*$ would then have a single vertex of odd degree. Four edges is impossible, as then there would be no branching. Hence the distinguished face must be a digon, and the branch point of order 2. By considerations similar to those in Subcase 2.1 we can construct $G$ from $H$ using Construction 4.3.

Case 3

The quotient surface is the sphere. Then there are two branch points in faces of the embedded quotient graph. As in Case 2, we can 2-color the faces of this quotient graph, so that it is a medial graph $M(H)$ where the vertices of $H$ correspond to the vertices of $R(G)$. The two distinguished faces must receive net voltages which generate the group, as all other faces receive net voltage zero.

Subcase 3.1. The two branch points lie in black faces of $M(H)$. In the derived graph the deck transformations correspond to the duality map. But the two distinguished black faces are fixed under the deck transformations,
so that they correspond to fixed vertices of $R(H)$. It follows that the deck transformations do not reverse parts of the vertex bipartition, and hence do not induce a radial self-duality map.

Subcase 3.2. The two branch points lie in white faces of $M(H)$. Then the faces lift to quadrilaterals. By assumption $\phi$ is of order at least 4, so at least one of the branch points lies in a loop which generates $\mathbb{Z}_4$. Since the other faces are all quadrilaterals and the sum of the number of edges incident with each face is even (twice the number of edges), the other branch point must also lie in a loop which generates $\mathbb{Z}_4$. It follows that we can construct $G$ from $H$ using Construction 4.5.

Subcase 3.3. The two branch points lie in one black face and one white face of $M(H)$. Then in $H^*$ each vertex is of degree 4, except for the vertex corresponding to the white face with the branch point. As above, this face must be a loop with a net voltage which generates $\mathbb{Z}_4$. Hence it corresponds to a vertex of degree 1 in $H^*$. Again we have a single vertex of odd degree, eliminating this case.

The three subcases exhaust Case 3. By Lemma 2.5, Cases 1, 2, and 3 cover the possible quotient surfaces. The theorem is demonstrated. 

THE MAIN THEOREM

We first state our main result:

**Classification Theorem 5.1.** Any self-dual spherical polyhedron comes from either Construction 3.1, 3.3, 3.5, 4.1, 4.3, or 4.5.

**Proof.** If the polyhedron admits an involutory self-duality map then it arises from Construction 3.1, 3.3, or 3.5 by Theorem 3.7. If the polyhedron does not admit an involutory self-duality map then it arises from Construction 4.1, 4.3, or 4.5 by Theorem 4.7.

We note that each of the six constructions are necessary in the statement of the Main Theorem (although Construction 3.3 can be considered as a special case of Construction 4.1). The authors have examples for each of the constructions which yield polyhedra that cannot arise from any of the other constructions. We digress for a moment to describe such an example.

The main tool in restricting possible self-duality maps is to use vertex degrees to restrict radial self-duality maps. For example, suppose that we build the radial graph of a self-dual polyhedron using the folding Construction 3.1. Suppose that there are reflexive edges $e_i$, $i = 1, 2, 3$, incident with pairs of vertices of degree $d_i$, and that there are no other vertices of degree $d_i$ in the graph (such examples are easy to construct). Since an
automorphism preserves vertex degrees, these must be switched in pairs in any radial self-duality map. It follows that the map must be an involution (the square fixes any of the directed reflexive edges to that we can apply Lemma 2.6) with at least three reflexive edges, hence it can only be built using Construction 3.1.

Similar considerations were involved in the other five examples. In Section 6 we treat with some care the construction of a polyhedron with a unique self-duality map of high order.

We turn our attention to known examples of self-dual polyhedra and how these examples fit into our classification scheme.

Figure 1.2 gives a self-dual polyhedron with a duality map of order 8 (due to [GS]). This comes from Construction 4.1. However, this same polyhedron also arises from Construction 3.5 with the involutory duality map given in Section 1.

Jendol [J] gave a polyhedron of rank 4. This example arises from Fig. 5.2 using Construction 4.5.

McCanna [McC] found a polyhedron with duality map of order $2n$ for every $n$; these arise from Construction 4.1 using the seed graph given in Fig. 5.3.
Shank [Sh1] gave the following interesting construction. Let $C$ be a circle in the plane and let $P_1,\ldots,P_k$ be a set of chords inside the circle with pairwise distinct endpoints such that at most two chords intersect at any given point in the interior. Now, place similar chords on the outside of the circle. Then the graph thus formed is the medial graph of a self-dual polyhedron, where the duality map switches the inside and outside of the circle. It is easy to see that the edges on the circle are fixed under this duality map. In fact, this is the dual of the folding Construction 3.1.

We note that the preceding construction shows that in practice it may be easier to construct the medial of the self-dual polyhedron rather than its radial. This comment applies to our constructions as well. For example, Construction 3.5 requires a graph on the plane with all faces of size 4 except for two distinguished triangles. The reader may find it convenient to first construct the dual graph, a planar graph which is 4-regular except for two distinguished vertices of degree 3. Similarly the reader may find it easier to find a 4-regular projective planar graph with an odd dual cycle rather than the nonbipartite quadrangulation needed in Construction 3.3.

B. Servatius et al. [SC] gave a construction of self-dual planar graphs. In fact, they asked whether their construction gave all self-dual graphs. It can be shown that their construction necessarily gives involutory self-duality maps with a pair of reflexive edges. From our classification scheme such polyhedra must come from Construction 3.5. But we can say something stronger in this case. It turns out that in the embedded radial graph of a polyhedron arising from the construction in [SC], the two reflexive edges must lie on the same left-right path (see [Sh2]). There is no such restriction in our Construction 3.5, and the authors have an example arising from Construction 3.5 which cannot come from the techniques in [SC].

Finally we note that McKee [McK] gives a construction which builds self-dual graphs, although it is unclear when the surface invoked is the sphere. These graphs always have involutory self-duality maps, and must come from a construction of Section 3.

6. POLYHEDRA OF LARGE RANK

Recall that Grünbaum and Shephard [GS] define the rank of a self-dual polyhedron as the minimum order of all of its self-duality maps. The following lemma is implicit in [GS] and given explicitly in [McC].

**Lemma 6.1.** The rank of a polyhedron is a power of 2.

**Proof.** For suppose that $\varphi$ is a duality map of order $2^km$, where $m$ is odd. Then $\varphi^m$ switches faces and vertices since $m$ is odd, so that it is also a duality map. But the order of $\varphi^m$ is $2^k$, and the lemma follows. □
McCanna [McC] first constructed a polyhedron of rank $2^k$ for each $k$, answering in the negative a question of Grünbaum and Shephard [GS] whether every polyhedron was of rank 2. It follows from our Classification Theorem that such examples must arise from Construction 4.1 (no other construction gives a self-duality map of order exceeding 4).

The main goal of this section is the systematic utilization of Construction 4.1 to build polyhedra with large rank. In particular, we will construct polyhedra in which the deck transformations are the only automorphisms. This takes some care, as the polyhedron of Fig. 1.2 comes from Construction 4.1 yet admits an involutory self-duality map in addition to the order 8 deck transformation.

If $G$ is an arbitrary regular covering of $H$, then the automorphism group of $G$ may be much richer than just the deck transformations. In our case, the following theorem allows us to conclude that any other such automorphism of $G$ induces an automorphism of $H$.

**THEOREM 6.2.** Let $H$ be a nonbipartite quadrangulation of the projective plane with a distinguished vertex of degree 1. Let $R(G)$ be the radial graph built from $H$ using Construction 4.1 with a voltage assignment from $\mathbb{Z}_{2n}$. Suppose that:

1. $n$ exceeds the maximum degree of $H$, and
2. $H$ is automorphism free.

Then the automorphism group of $R(G)$ is exactly the group of deck transformations.

**Proof.** We will use $\text{Aut}(R(G))$ to denote the automorphism group of $R(G)$, and $\langle \varphi \rangle$ for the group of deck transformations generated by the radial self-duality map $\varphi$. By way of contradiction, we suppose that there is an automorphism $\theta$ in $\text{Aut}(R(G)) - \langle \varphi \rangle$.

If $\theta$ fixes the vertex parts of $R(G)$, then $\theta \varphi$ does not. Moreover, $\theta \varphi$ is not in $\langle \varphi \rangle$. Hence, without loss of generality we may assume that $\theta$ is a radial self-duality map.

**Claim.** Any radial self-duality not in $\langle \varphi \rangle$ is orientation preserving. For suppose that such a $\psi$ reverses orientations. Let $x$ be the distinguished vertex of $H$ and set $\{v, \varphi(v)\} = \rho^{-1}(x)$. Now $x$ is of degree 1 in $H$ so that $v$ and $\varphi(v)$ are each of degree $n$. Moreover, $\varphi$ acts transitively on the $2n$ edges incident with these two vertices. Let $e$ be an edge incident with $v$. Since $\psi$ fixes no vertex and $v, \varphi(v)$ are the unique vertices of maximum degree ($n$ exceeds the maximum degree of $H$), $\psi(e)$ is incident with $\varphi(v)$. Moreover, since $\varphi$ acts transitively on such edges, $\psi(e) = \varphi^k(e)$ for some $k$. Now $\varphi^k(e)$ is incident with $\varphi(v)$, so that $k$ is odd and $\varphi^k$ reverses orienta-
tions. Hence $\varphi^{-k}\psi$ is orientation-preserving. But $\varphi^{-k}\psi$ fixes the directed edge $e$, and so is the identity by Lemma 2.6. Therefore, $\psi \in \langle \varphi \rangle$, a contradiction that establishes the claim.

**Claim.** $\langle \varphi \rangle$ has index 2 in $\text{Aut}(R(G))$. Let $\psi$ be in $\text{Aut}(R(G)) - \langle \varphi \rangle$. We consider separately the cases that $\psi$ is vertex part-reversing or part-preserving.

First suppose that $\psi$ is part-reversing. Then by the previous claim $\psi$ is orientation-preserving. Therefore, $\theta^{-1}\psi$ is both part- and orientation-preserving. Let $e$ be any edge incident with $v$. Then since $\varphi$ acts transitively on the edges there is some $k$ such that $\theta^{-1}\psi(e) = \varphi^k(e)$. Since both $e$ and $\theta^{-1}\psi(e)$ are incident with $v$, $k$ is even. Hence, $\varphi^k$ is both orientation- and part-preserving. It follows that the map $\varphi^{-k}\theta^{-1}\psi$ is both orientation- and part-preserving. But this map fixes the directed edge $e$, so it must be the identity. Thus $\psi \in \theta \langle \varphi \rangle$.

Secondly, suppose that $\psi$ is part-preserving. If $\psi$ is orientation-preserving, then $\theta\psi$ is an orientation-preserving radial self-duality. By the above, $\theta\psi \in \theta \langle \varphi \rangle$. Therefore, $\psi \in \langle \varphi \rangle$. If $\psi$ is orientation-reversing, then $\psi\varphi$ is an orientation-preserving radial self-duality. Thus $\psi\varphi \in \theta \langle \varphi \rangle$. It follows that $\psi \in \theta \langle \varphi \rangle$.

Having exhausted the possibilities, the claim is demonstrated.

**Claim.** $\theta$ induces an automorphism of $H$. We first show that $\theta$ respects vertex fibers. We must show that two vertices $u, \varphi'(u)$ map under $\theta$ to vertices in the same fiber. But because $\langle \varphi \rangle$ has index at most 2 in $\text{Aut}(R(G))$, $\theta\varphi' = \varphi'\theta$ for some $s$. Hence $\theta\varphi'(u) = \varphi'\theta(u)$, so $\theta\varphi'(u)$ is in the same fiber as $\theta(u)$.

Because $\theta$ respects vertex fibers it induces a permutation of these fibers, that is, it induces a permutation of the vertices in the quotient graph $H$. But if $e$ is an edge of $H$, then $\varphi^{-1}(e)$ consists of $n$ edges of $R(G)$, each joining vertices in the fibers over the ends of $e$. Since $R(G)$ is simple $\theta$ respects these edge orbits as well. Hence $\theta$ induces an automorphism of $H$ as claimed.

The proof of the theorem is completed by observing that the automorphism of the preceding claim contradicts hypothesis (2).

The hypotheses of Theorem 6.2 are not the most general possible. Our proof requires that any automorphism of $R(G)$ respects (i) the fiber above the distinguished vertex (we used $n$ large) and (ii) the fiber above each neighbourhood of the distinguished vertex (we used degree 1). We have used a similar result to prove that McCanna’s example has no other automorphisms. We chose to state above a less general, but clearer, version. It would be of interest to discover the most general hypotheses possible in Theorem 6.2.
Example 6.3. Figure 6.4 shows a seed graph $H$ with distinguished vertex $a$. We will show that this seed graph is automorphism free and hence satisfies the hypotheses of Theorem 6.2.

Any automorphism of $H$ must fix $a$, the unique vertex of degree 1, and hence must fix its neighbour $b$. The only other vertex of degree 5 is $d$, so it too is fixed. But the subgraph induced by vertices of distance 1 from $d$ is the 5-cycle with one edge duplicated. Since $b$ is fixed, so is $c$, the other vertex on this digon. It follows that each vertex of this 5-cycle is fixed. But this includes the rest of the vertices of $H$. Hence $H$ is automorphism free.

Corollary 6.5. There exists a polyhedron of rank $2^k$ for each $k \geq 1$.

Proof. If $k = 1$ we may use any example from Section 3. If $k = 2$ we can use the example of Jendrol [J]. In Fig. 6.6 we show the self-dual...
polyhedron $G$ with rank 8 (the case $k = 3$) derived from the seed graph of Example 6.3. While $n$ is not large enough to apply Theorem 6.2 directly in this case, we note that $a$ is the unique vertex of degree 4 incident only with vertices of degree 5. Hence the subgroup generated by $\varphi$ acts transitively on the edges covering $ab$, and the rest of the proof proceeds unhindered. Finally if $k \geq 4$ we can use the seed graph of Fig. 6.4 together with Construction 4.1 and Theorem 6.2.

The graph of Fig. 6.6 has roughly half the number of vertices in McCanna's example. We suspect that this may be the smallest such example of a rank 8 self-dual polyhedron. More strongly, we suspect that this seed graph gives the smallest example of a self-dual polyhedron of rank $2^k$ for any $k > 1$.

Finally, in Fig. 6.7 we give a seed graph which builds the polyhedron of Fig. 1.2 with an order 8 self-duality map. However, this seed graph admits an automorphism which switches the two parallel edges. This automorphism lifts to an element in a nontrivial coset of $\langle \varphi \rangle$ in $\text{Aut}(\mathcal{R}(G))$. By multiplying the lift of this automorphism by $\varphi$ one can construct an involutory self-duality map.

7. Conclusion

The main focus of this paper has been a complete set of constructions for all self-dual spherical polyhedra. In the case of an involutory self-duality map we have in fact done much more: we have constructed all pairs $(G, \varphi)$ where $\varphi$ is an involutory self-duality map of a 3-connected planar graph $G$. We have not, however, constructed all such pairs for noninvolutory $\varphi$'s. We ask if a modification of the techniques in this paper will give complete set of constructions for all such pairs. The main obstacle will be that the subgroup generated by $\varphi$ no longer need act freely on $M^2(G)$ as shown in Theorem 4.7.

Another natural question is the generalization of our results to other
surfaces. For example, could one find all self-dual 3-connected graphs on the real projective plane? In Fig. 7.1 we give the radial graph of a 3-connected graph in the real projective plane (where antipodal points on the boundary of the circle are to be identified). This radial graph has two radial self-duality maps, one a rotation by \( \pi \) radians around the origin, and the other a reflection about the \( x \)-axis.

We suspect that a classification of self-dual polyhedra is considerably more difficult on surfaces other than the sphere. The classification would arise from the nonspherical analogue of Lemma 2.5. But in the sphere, every graph automorphism of a 3-connected graph maps faces to faces, and hence extends to an action on the sphere. This is not the case in the projective plane or more complicated surfaces, where a graph automorphism might not map faces to faces. Also, for nonorientable surfaces we have lost the concept of orientation-preserving and orientation-reversing. This was necessary, among other things, in showing that a noninvolutory self-duality map acted freely on \( M^2(G) \).

In Section 6 we took some care to restrict the possible radial self-duality maps in order to build polyhedra of large rank. What can be said about graphs which admit radial self-duality maps of several different types? Must each arise from lifting an automorphism from the quotient graph?

Finally, we note that we have taken no great care in showing that the derived graphs are 3-connected. What conditions are needed on the seed graphs to guarantee that the derived graphs are 3-connected? Can one construct and classify all planar self-dual graphs, not just the 3-connected ones?

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