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Analysis of Optimal Control Problems for the 2-D Stationary Boussinesq Equations[1]

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This paper deals with optimal control problems associated with the 2-D Boussinesq equations. The controls considered may be of either the distributed or the Neumann type. These problems are first put into an appropriate mathematical formulation. Then the existence of optimal solutions is proved. The use of Lagrange multiplier techniques is justified and an optimality system of equations is derived. © 2000 Academic Press

1. INTRODUCTION

In this article we consider the minimization of some desired objective in viscous incompressible thermally convected flows using either boundary temperature or heat source as a control mechanism. The control of viscous flows for the purpose of achieving some desired objective is crucial to many technological and scientific applications. The problem we consider is a Bénard problem whose system is governed by the Boussinesq equations.

We now write the 2-D nondimensional Boussinesq equations as

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \alpha T\mathbf{g} + \mathbf{f} \qquad \text{in } \Omega, \tag{1.1}$$

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$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$-\kappa \Delta T + (\mathbf{u} \cdot \nabla) T = Q \qquad \text{in } \Omega, \tag{1.3}$$

with boundary conditions

$$\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega, \qquad T = h \text{ on } \Gamma_D, \qquad \frac{\partial T}{\partial \mathbf{n}} = g \text{ on } \Gamma_N, \tag{1.4}$$

where Ω is the regular bounded open set in $\mathbb{R}^2$ and $\partial\Omega \in C^2$. In (1.4), $\Gamma_D = \partial\Omega \setminus \Gamma_N$ where Γ_N is a regular nonempty open subset of $\partial\Omega$. In (1.1)–(1.4), $\mathbf{u}$, p, and T denote the velocity, pressure, and temperature fields, respectively, $\mathbf{f}$ a given body force, h a given function, and controls Q and g. The vector $\mathbf{g}$ is a unit vector in the direction of gravitational acceleration and $\kappa > 0$ is the thermal conductivity parameter. In this paper we consider, for simplicity, the case of constant κ. The vector $\mathbf{n}$ denotes the outward unit normal to Ω and $\nu > 0$ denotes the kinematic viscosity.

Next, we introduce the functionals

$$\mathcal{J}_1(\mathbf{u}, T, p, Q, g) = \frac{1}{2} \int_\Omega |\nabla \times \mathbf{u}|^2 \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega |Q|^2 \, d\mathbf{x} + \frac{\delta}{2} \int_{\Gamma_N} |g|^2 \, ds \tag{1.5}$$

and

$$\mathcal{J}_2(\mathbf{u}, T, p, Q, g) = \frac{1}{2} \int_\Omega |T - T_d|^2 \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega |Q|^2 \, d\mathbf{x} + \frac{\delta}{2} \int_{\Gamma_N} |g|^2 \, ds. \tag{1.6}$$

The optimal control problems we consider are to seek state variables $(\mathbf{u}, p, T)$ and controls Q and g such that the functional (1.5) or (1.6) is minimized subject to (1.1)–(1.4) where T_d is some desired temperature distribution. The functional (1.5) measures the vorticity of the flow. The control of vorticity has significant applications in science and engineering such as control of turbulence and control of crystal growth process. The functional (1.6) effectively measures the difference between the temperature field T and a prescribed field T_d. The real goal of optimization is to minimize the first term appearing in the definition (1.5) or (1.6). The second and third terms in the cost functionals (1.5) and (1.6) are added to limit the cost of controls. The positive penalty parameters γ and δ can be used to change the relative importance of the three terms appearing in the definitions of the functionals.

In past years, considerable progress has been made in mathematical analyses and computations of optimal control problems for viscous flows; see [2–4, 6–10, 13–18, 20] and references therein. Optimal control problems for the thermally coupled incompressible Navier–Stokes equation by Neumann and Dirichlet boundary heat controls were considered in [13, 17]. Also, optimal control problems for the time-dependent problems and related problems were considered in [4, 6, 18] and references therein. Exact controllability of the Boussinesq problem and related problems were considered in [10] and references therein.

The plan of the paper is as follows. In the remainder of this section, we introduce the notation that will be used throughout the paper. Then, in Section 2, we give a precise statement of a weak formulation of the Boussinesq equations and prove that a sufficiently smooth solution to the Boussinesq equations exists. In Section 3, we give a precise statement of the optimization problem and prove that an optimal solution exists. In Section 4, we prove the existence of Lagrange multipliers and then use the method of Lagrange multipliers to derive an optimality system. Some remarks and further discussions are also given.

1.1. *Notation*

We introduce some function spaces and their norms, along with some related notation used in subsequent sections; for details see [1].

Let Ω be a bounded domain of $\mathbb{R}^2$ with a C^2 boundary Γ. Let $L^2(\Omega)$ be the space of real-valued square integrable functions defined on Ω, and let $\|\cdot\|_{L^2(\Omega)}$ be the norm in this space. We define the Sobolev space $H^m(\Omega)$ for the nonnegative integer m by

$$H^m(\Omega) \stackrel{\text{def}}{=} \{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega) \text{ for } 0 \le |\alpha| \le m\},$$

where D^α is the weak (or distributional) partial derivative and α is a multi-index. The norm $\|\cdot\|_{H^m(\Omega)}$ associated with $H^m(\Omega)$ is given by

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)}.$$

Note that $H^0(\Omega) = L^2(\Omega)$. For the vector-valued functions, we define the Sobolev space $\mathbf{H}^m(\Omega)$ (in all cases, boldface indicates vector-valued) by

$$\mathbf{H}^m(\Omega) \stackrel{\text{def}}{=} \{\mathbf{u} = (u_1, u_2) \mid u_i \in H^m(\Omega) \text{ for } i = 1, 2\}$$

and its associated norm $\|\cdot\|_{\mathbf{H}^m(\Omega)}$ is given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^m(\Omega)} = \sum_{i=1}^{2} \|u_i\|^2_{H^m(\Omega)}.$$

We also define particular subspaces

$$L^2_0(\Omega) = \left\{ f \in L^2(\Omega) : \int_\Omega f \, d\mathbf{x} = 0 \right\}, \quad \mathbf{H}^1_0(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = \mathbf{0} \text{ on } \Gamma\}$$

and

$$H^1_D(\Omega) = \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_D\}.$$

We make use of the well-known space $\mathbf{L}^4(\Omega)$ equipped with the norm $\|\cdot\|_{\mathbf{L}^4(\Omega)}$.

We also define the solenoidal spaces

$$\mathbf{V} \stackrel{\text{def}}{=} \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \mid \nabla \cdot \mathbf{u} = 0\}.$$

If Ω is bounded and has a C^2 boundary (these are the kinds of domains under consideration here), Sobolev's embedding theorem yields that $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$, where $\hookrightarrow\hookrightarrow$ denotes compact embedding; i.e., a constant C exists such that

$$\|u\|_{L^4(\Omega)} \leq C\|u\|_{H^1(\Omega)}. \tag{1.7}$$

Obviously, a similar result holds for the spaces $\mathbf{H}^1(\Omega)$ and $\mathbf{L}^4(\Omega)$.

2. A WEAK FORMULATION OF THE BOUSSINESQ EQUATIONS

2.1. *A Weak Formulation of the Equations*

We introduce the following bilinear and trilinear forms, for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in \mathbf{H}^1(\Omega)$, $T, S \in H^1(\Omega)$,

$$\begin{aligned}
a_0(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x} && \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega),\\
a_1(T, S) &= \int_\Omega \kappa\nabla T \cdot \nabla S\, d\mathbf{x} && \forall T, S \in H^1(\Omega),\\
b(\mathbf{v}, q) &= -\int_\Omega q\nabla \cdot \mathbf{v}\, d\mathbf{x} && \forall \mathbf{v} \in \mathbf{H}^1(\Omega),\ \forall q \in L^2(\Omega),\\
c_0(\mathbf{u}, \mathbf{w}, \mathbf{v}) &= \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{w} \cdot \mathbf{v}\, d\mathbf{x} && \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega),\\
c_1(\mathbf{u}, T, S) &= \int_\Omega (\mathbf{u} \cdot \nabla)TS\, d\mathbf{x} && \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\ \forall T, S \in H^1(\Omega),
\end{aligned}$$

and

$$d(T, \mathbf{v}) = \int_\Omega T\mathbf{g} \cdot \mathbf{v}\, d\mathbf{x} \qquad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),\ \forall T \in H^1(\Omega).$$

We first note that the bilinear forms $a_0(\cdot, \cdot)$ and $a_1(\cdot, \cdot)$ are clearly continuous, i.e.,

$$|a_0(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \tag{2.1}$$

and

$$|a_1(T, S)| \leq \kappa\|T\|_{H^1(\Omega)}\|S\|_{H^1(\Omega)}. \tag{2.2}$$

We have the coercivity relations associated with $a_0(\cdot, \cdot)$ and $a_1(\cdot, \cdot)$

$$a_0(\mathbf{u}, \mathbf{u}) = \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \geq C_1 \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \qquad \forall \mathbf{u} \in \mathbf{H}^1_0(\Omega), \tag{2.3}$$

and

$$a_1(T, T) \geq \kappa \|\nabla T\|^2_{L^2(\Omega)} \geq C_2 \|T\|^2_{H^1(\Omega)} \qquad \forall T, S \in H^1_D(\Omega), \tag{2.4}$$

which are direct consequences of Poincaré inequality.

LEMMA 2.1. *For every* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ *and every* $T, S \in H^1(\Omega)$ *there are constants* $C_{1,2,3,4}$ *such that*

$$|c_0(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq C_1 \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)}, \tag{2.5}$$

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \qquad \textit{if } \mathbf{u} \in \mathbf{V}, \tag{2.6}$$

$$|c_1(\mathbf{u}, T, S)| \leq C_2 \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|T\|_{H^1(\Omega)} \|S\|_{H^1(\Omega)} \qquad \forall \mathbf{u} \in \mathbf{V}, \tag{2.7}$$

$$c_1(\mathbf{u}, T, T) = 0 \qquad \textit{if } \mathbf{u} \in \mathbf{V}, \tag{2.8}$$

and

$$|d(T, \mathbf{u})| \leq C_3 \|T\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C_4 \|\nabla T\|_{L^2(\Omega)} \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)}. \tag{2.9}$$

Proof. These follow from the Cauchy–Schwarz inequality, Hölder's inequality, and various embedding results, in particular the continuous embeddings of $\mathbf{H}^1$ into $\mathbf{L}^4$ and $\mathbf{L}^2$ and H^1 into L^4 and L^2, respectively. ∎

The weak form of the constraint equations (1.1)–(1.4) is then given as follows: seek $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$, $p \in L^2_0(\Omega)$, and $T \in H^1(\Omega)$ such that

$$\nu a_0(\mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \alpha d(T, \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{2.10}$$

$$b(\mathbf{u}, q) = 0 \qquad \forall q \in L^2_0(\Omega), \tag{2.11}$$

$$a_1(T, S) + c_1(\mathbf{u}, T, S) = \langle Q, S \rangle + \int_{\Gamma_N} \kappa g S \, ds \qquad \forall S \in H^1_D(\Omega), \tag{2.12}$$

and

$$T = h \qquad \text{on } \Gamma_D. \tag{2.13}$$

LEMMA 2.2 (Leray and Schauder). *Let E be a Banach space, and let G: $[0,1] \times E \to E$ be a continuous, compact map, such that $G(0, v) = v_0$ is independent of $v \in E$. Suppose there exists $M < \infty$ such that, for all $(\tau, x) \in [0,1] \times E$,*

$$G(\tau, x) = x \Rightarrow \|x\| < M.$$

Then the map G_1: $E \to E$ given by $G_1(v) = G(1, v)$ has a fixed point.

PROPOSITION 2.3. *For every $g \in L^2(\Gamma_N)$, $h \in H^1(\Gamma_D)$, $Q \in L^2(\Omega)$, and $\mathbf{f} \in \mathbf{L}^2(\Omega)$ the Boussinesq equations* (2.10)–(2.13) *have a solution $(\mathbf{u}, T, p) \in \mathbf{V} \times H^1(\Omega) \times L^2_0(\Omega)$. Moreover, if $(\mathbf{u}, T, p)$ is a solution to* (2.10)–(2.13), *then $(\mathbf{u}, T, p) \in \mathbf{V} \cap \mathbf{H}^2(\Omega) \times H^s(\Omega) \times L^2_0(\Omega) \cap H^1(\Omega)$ $(1 \le s < \frac{3}{2})$ and there is a continuous function P_s for each s such that*

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)} + \|T\|_{H^s(\Omega)} \\ &\quad \le P_s(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|Q\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)} + \|h\|_{H^1(\Gamma_D)}). \end{aligned} \tag{2.14}$$

Proof. By virtue of the trace theorem, let $\hat{T}$ in $H^1(\Omega)$ satisfy $\hat{T} = h$ on Γ_D and examine the following problem: for any given $\mathbf{u} \in \mathbf{V}$ find T in $H^1(\Omega)$ such that $T - \hat{T} \in H^1_D(\Omega)$ and

$$\begin{aligned} &a_1(T - \hat{T}, S) + c_1(\mathbf{u}, T - \hat{T}, S) \\ &\quad = \langle Q, S\rangle - a_1(\hat{T}, S) - c_1(\mathbf{u}, \hat{T}, S) + \int_{\Gamma_N} \kappa g S\, ds \quad \forall S \in H^1_D(\Omega). \end{aligned} \tag{2.15}$$

Let $\tilde{T} = T - \hat{T} \in H^1_D(\Omega)$. From (2.2), (2.4), (2.7), and (2.8), it follows that, for $\mathbf{u} \in \mathbf{V}$, $a_1(\cdot,\cdot) + c_1(\mathbf{u}, \cdot, \cdot)$ is a continuous, elliptic, bilinear form on $H^1_0(\Omega) \times H^1_0(\Omega)$ and thus on $H^1_D(\Omega) \times H^1_D(\Omega)$. Thus, for given $g \in L^2(\Gamma_N)$, $h \in H^1(\Gamma_D)$, and $Q \in L^2(\Omega)$, by the Lax–Milgram lemma and trace theorems there is a unique solution $\tilde{T} \in H^1_D(\Omega)$ satisfying (2.15) and there is a unique $T = \tilde{T} + \hat{T} \in H^1(\Omega)$ and the estimate

$$\|T\|_{H^s(\Omega)} + \|T\|_{L^2(\Gamma_N)} \le C(\|g\|_{L^2(\Gamma_N)} + \|h\|_{H^1(\Gamma_D)} + \|Q\|_{L^2(\Omega)}) \tag{2.16}$$

holds true for all $s \in [1, \frac{3}{2})$ (see [22]). Thus, we may define a mapping F: $\mathbf{V} \to H^1(\Omega)$ by $F(\mathbf{u}) = T$. The theorem will be proved if one can show that there is at least one $\mathbf{u} \in \mathbf{V}$ such that

$$\nu a_0(\mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \alpha d(F(\mathbf{u}), \mathbf{v}) + \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{2.17}$$

From inequality (2.3) it follows that $a_0(\cdot,\cdot)$ is a continuous elliptic bilinear form on $\mathbf{V} \times \mathbf{V}$ and

$$\begin{aligned} &|-c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + d(F(\mathbf{u}), \mathbf{v}) + \langle \mathbf{f}, \mathbf{v}\rangle| \\ &\quad \le (C_2\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \alpha C_4\|F(\mathbf{u})\|_{H^1(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)})\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}$ follows from (2.7) and (2.9). Thus, we may define a mapping $G: \mathbf{V} \to \mathbf{V}$ by

$$\nu a_0(G(\mathbf{u}), \mathbf{v}) = -c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \alpha d(F(\mathbf{u}), \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{2.18}$$

Clearly, $\mathbf{u}$ is a solution of (2.17) if it is a solution of

$$G(\mathbf{u}) = \mathbf{u}. \tag{2.19}$$

Now, we may apply the Leray–Schauder principle to prove the existence of the solution to (2.19). First, we verify the compactness of G. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V}$. Set $\mathbf{w} = G(\mathbf{u}_2) - G(\mathbf{u}_1)$. Subtracting the equations obtained from (2.18) by substituting $\mathbf{u}_2$ and $\mathbf{u}_1$ for $\mathbf{u}$ and $\mathbf{w}$ for $\mathbf{v}$, we get

$$\begin{aligned} \nu a_0(\mathbf{w}, \mathbf{w}) = & -c_0(\mathbf{u}_2 - \mathbf{u}_1; \mathbf{u}_2, \mathbf{w}) \\ & + c_0(\mathbf{u}_1; \mathbf{u}_2 - \mathbf{u}_1, \mathbf{w}) + \alpha d(F(\mathbf{u}_2) - F(\mathbf{u}_1), \mathbf{w}). \end{aligned} \tag{2.20}$$

Now, we estimate $\|F(\mathbf{u}_2) - F(\mathbf{u}_1)\|_{H^1(\Omega)}$. Substitute $\mathbf{u}_2$ and $\mathbf{u}_1$ in (2.12) and subtract to get

$$\begin{aligned} a_1(F(\mathbf{u}_2) - F(\mathbf{u}_1), S) = & -c_1(\mathbf{u}_2 - \mathbf{u}_1; F(\mathbf{u}_2), S) \\ & -c_1(\mathbf{u}_1; F(\mathbf{u}_2) - F(\mathbf{u}_1), S) \qquad \forall S \in H^1(\Omega). \end{aligned} \tag{2.21}$$

Substituting $F(\mathbf{u}_2) - F(\mathbf{u}_1)$ for S and using (2.4), (2.7), and (2.8)

$$\begin{aligned} & \|\nabla F(\mathbf{u}_2) - \nabla F(\mathbf{u}_1)\|_{L^2(\Omega)} \\ & \quad \le C(\|g\|_{L^2(\Gamma_N)} + \|h\|_{H^1(\Gamma_D)} + \|Q\|_{L^2(\Omega)})\|\mathbf{u}_2 - \mathbf{u}_1\|_{\mathbf{L}^4(\Omega)}. \end{aligned} \tag{2.22}$$

Thus,

$$\begin{aligned} \|\nabla \mathbf{w}\|_{\mathbf{L}^2(\Omega)} \le \nu^{-1}(& \|\mathbf{u}_2\|_{\mathbf{L}^4(\Omega)} + \|\mathbf{u}_1\|_{\mathbf{L}^4(\Omega)} \\ & + \alpha C(\|g\|_{L^2(\Gamma_N)} + \|h\|_{H^1(\Gamma_D)} + \|Q\|_{L^2(\Omega)}))\|\mathbf{u}_2 - \mathbf{u}_1\|_{\mathbf{L}^4(\Omega)} \end{aligned}$$

follows from (2.20) and (2.22) using (2.3), (2.5), and (2.8). Since $\mathbf{H}_0^1(\Omega)$ is compactly embedded in $\mathbf{L}^4(\Omega)$ so is $\mathbf{V}$. It follows that G is a continuous compact map.

Now, we define $G(\tau, \mathbf{v}) = \tau G(\mathbf{v})$ for all $(\tau, \mathbf{v}) \in [0, 1] \times \mathbf{V}$. Clearly, $G(0, \mathbf{v}) = \mathbf{0}$ is independent of $\mathbf{v}$.

Suppose $\tau \in (0, 1]$ and $\mathbf{v} \in \mathbf{V}$ satisfies $\tau G(\mathbf{v}) = \mathbf{v}$. Then

$$\tau^{-1} \nu a_0(\mathbf{v}, \mathbf{v}) = -c_0(\mathbf{v}; \mathbf{v}, \mathbf{v}) + \alpha d(F(\mathbf{v}), \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle. \tag{2.23}$$

From the above fact, we have

$$\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \le \tau\left(\frac{\alpha}{\nu} C_4 \|\nabla F(\mathbf{v})\|_{L^2(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\right)$$
$$\le C(\|g\|_{L^2(\Gamma_N)} + \|h\|_{H^1(\Gamma_D)} + \|Q\|_{L^2(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}),$$

which completes the proof of the existence of the solution.

Since $T \in H^s(\Omega)$ for all $s \in [1, \frac{3}{2})$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the regularity of $\mathbf{u}$ and p follows from well-known theories concerning the Navier–Stokes equations. By (2.16) and *a priori* estimates for the Stokes system (see [24]) there exists a continuous function S such that

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le S(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|Q\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)} + \|h\|_{H^1(\Gamma_D)}). \tag{2.24}$$

∎

We now state a global uniqueness criterion for the case of small data.

THEOREM 2.4. *Let* $\mathbf{u}$ *and* $F(\mathbf{u}) = T$ *be a solution of* (2.10)–(2.13) *and suppose* $N\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \alpha M < \nu$ *where*

$$N = \sup\left\{c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) : \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} = \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} = \|\nabla \mathbf{w}\|_{\mathbf{L}^2(\Omega)} = 1, \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}\right\}$$

and

$$M = \sup\left\{\frac{d(F(\mathbf{u}) - F(\mathbf{v}), \mathbf{u} - \mathbf{v})}{\|\nabla \mathbf{u} - \nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}} : \mathbf{u} \ne \mathbf{v},\ \mathbf{u}, \mathbf{v} \in \mathbf{V}\right\}.$$

Then $\mathbf{u}$ *and* $F(\mathbf{u}) = T$ *is the unique solution of* (2.10)–(2.13).

Proof. Suppose $\mathbf{w} \ne \mathbf{u}$ and $F(\mathbf{w})$ is a solution of (2.10)–(2.13). Then

$$\nu a_0(\mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \alpha d(F(\mathbf{u}), \mathbf{v}) + \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall \mathbf{v} \in \mathbf{V}$$

and

$$\nu a_0(\mathbf{w}, \mathbf{v}) + c_0(\mathbf{w}, \mathbf{w}, \mathbf{v}) = \alpha d(F(\mathbf{w}), \mathbf{v}) + \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall \mathbf{v} \in \mathbf{V}.$$

Subtracting with $\mathbf{v} = \mathbf{u} - \mathbf{w}$ and using the fact $c_0(\mathbf{w}, \mathbf{u} - \mathbf{w}, \mathbf{u} - \mathbf{w}) = 0$, we have

$$\nu a_0(\mathbf{u} - \mathbf{w}, \mathbf{u} - \mathbf{w}) = -c_0(\mathbf{u} - \mathbf{w}, \mathbf{u}, \mathbf{u} - \mathbf{w}) + \alpha d(F(\mathbf{u}) - F(\mathbf{w}), \mathbf{u} - \mathbf{w}).$$

Hence,

$$\nu\|\nabla(\mathbf{u} - \mathbf{w})\|^2_{\mathbf{L}^2(\Omega)} \le (N\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \alpha M)\|\nabla(\mathbf{u} - \mathbf{w})\|^2_{\mathbf{L}^2(\Omega)}$$
$$< \nu\|\nabla(\mathbf{u} - \mathbf{w})\|^2_{\mathbf{L}^2(\Omega)},$$

which is a contradiction. Therefore, $\mathbf{w} = \mathbf{u}$. ∎

3. THE OPTIMIZATION PROBLEM AND THE EXISTENCE OF OPTIMAL SOLUTIONS

3.1. *The Optimization Problem*

We state the optimal control problem. We look for a $(\mathbf{u}, T, p, Q, g) \in \mathbf{H}_0^1(\Omega) \times H^1(\Omega) \times L_0^2(\Omega) \times L^2(\Omega) \times \mathscr{V}$ such that the cost functional

$$(\mathscr{A}1) \quad \mathscr{J}_1(\mathbf{u}, T, p, Q, g) = \frac{1}{2}\int_\Omega |\nabla \times \mathbf{u}|^2 \, d\mathbf{x} + \frac{\gamma}{2}\int_\Omega |Q|^2 \, d\mathbf{x} + \frac{\delta}{2}\int_{\Gamma_N} |g|^2 \, ds$$

or

$$(\mathscr{A}2) \quad \mathscr{J}_2(\mathbf{u}, T, p, Q, g) = \frac{1}{2}\int_\Omega |T - T_d|^2 \, d\mathbf{x} + \frac{\gamma}{2}\int_\Omega |Q|^2 \, d\mathbf{x} + \frac{\delta}{2}\int_{\Gamma_N} |g|^2 \, ds$$

is minimized subject to the constraints

$$\nu a_0(\mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \alpha d(T, \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.1}$$

$$b(\mathbf{u}, q) = 0 \qquad \forall q \in L_0^2(\Omega), \tag{3.2}$$

$$a_1(T, S) + c_1(\mathbf{u}, T, S) = \langle Q, S \rangle - (\kappa g, S)_{\Gamma_N} \qquad \forall S \in H_D^1(\Omega), \tag{3.3}$$

$$T = h \qquad \text{on } \Gamma_D, \tag{3.4}$$

where $\mathscr{V}$ is a nonempty, closed, and convex subset of $L^2(\Gamma_N)$.

The *admissibility set* $\mathscr{U}_{\text{ad}}$ is defined by

$$\mathscr{U}_{\text{ad}} = \{(\mathbf{u}, T, p, Q, g) \in \mathbf{H}_0^1(\Omega) \times H^1(\Omega) \times L_0^2(\Omega) \times L^2(\Omega) \times \mathscr{V} : \\ \mathscr{J}(\mathbf{u}, T, p, Q, g) < \infty \text{ and (3.1)–(3.4) are satisfied}\}, \tag{3.5}$$

where $\mathscr{J}(\mathbf{u}, T, p, Q, g)$ is $\mathscr{J}_1(\mathbf{u}, T, p, Q, g)$ or $\mathscr{J}_2(\mathbf{u}, T, p, Q, g)$, depending on minimization problems. Then $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \in \mathscr{U}_{\text{ad}}$ is called an optimal solution if there exists $\varepsilon > 0$ such that

$$\mathscr{J}(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \leq \mathscr{J}(\mathbf{u}, T, p, Q, g) \qquad \forall (\mathbf{u}, T, p, Q, g) \in \mathscr{U}_{\text{ad}} \tag{3.6}$$

satisfying

$$\|\hat{\mathbf{u}} - \mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\hat{T} - T\|_{H^1(\Omega)} + \|\hat{p} - p\|_{L^2(\Omega)} \\ + \|\hat{Q} - Q\|_{L^2(\Omega)} + \|\hat{g} - g\|_{L^2(\Gamma_N)} < \varepsilon. \tag{3.7}$$

If, for optimal solution $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \in \mathscr{U}_{\text{ad}}$, inequalities (3.6) and (3.7) hold true with $\varepsilon = +\infty$, then we say that $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})$ is the *global minimum*. The optimal control problem can now be formulated as a constrained minimization in a Hilbert space:

$$\min_{(\mathbf{u}, T, p, Q, g) \in \mathscr{U}_{\text{ad}}} \mathscr{J}(\mathbf{u}, T, p, Q, g). \tag{3.8}$$

Problems ($\mathscr{A}1$) and ($\mathscr{A}2$) can be analyzed in exactly the same manner. From this section, we treat in detail the first problem ($\mathscr{A}1$).

3.2. *The Existence of an Optimal Solution*

We now show the existence of an optimal solution. The existence of an optimal solution can be proved based on the *a priori* estimates (2.14) and standard techniques.

THEOREM 3.1. *Let* $\mathbf{f} \in \mathbf{L}^2(\Omega)$ *and* $h \in H^1(\Gamma_D)$. *Then there is an optimal solution* $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \in \mathcal{U}_{\rm ad}$ *to problem* (3.8). *Moreover, any optimal solution satisfies* $\hat{\mathbf{u}} \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$, $\hat{T} \in H^s(\Omega)$ *for all* $s \in [1, \frac{3}{2})$, *and* $\hat{p} \in L^2_0(\Omega) \cap H^1(\Omega)$.

Proof. The set $\mathcal{U}_{\rm ad}$ is apparently nonempty because of Theorem 2.3. Thus, we may choose a minimizing sequence $\{\mathbf{u}^{(n)}, T^{(n)}, p^{(n)}, Q^{(n)}, g^{(n)}\}$ in $\mathcal{U}_{\rm ad}$ such that

$$\lim_{n\to\infty} \mathcal{J}_1\big(\mathbf{u}^{(n)}, T^{(n)}, p^{(n)}, Q^{(n)}, g^{(n)}\big) = \inf_{(\mathbf{v}, S, q, R, z)\in\mathcal{U}_{\rm ad}} \mathcal{J}_1(\mathbf{v}, S, q, R, z). \quad (3.9)$$

By the definition of $\mathcal{U}_{\rm ad}$, we have

$$\begin{aligned} \nu a_0\big(\mathbf{u}^{(n)}, \mathbf{v}\big) + c_0\big(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v}\big) + b\big(\mathbf{v}, p^{(n)}\big) \\ = \alpha d\big(T^{(n)}, \mathbf{v}\big) + \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \end{aligned} \quad (3.10)$$

$$b(\mathbf{u}^{(n)}, q) = 0 \qquad \forall q \in L^2_0(\Omega), \quad (3.11)$$

$$a_1(T^{(n)}, S) + c_1(\mathbf{u}^{(n)}, T^{(n)}, S) = \langle Q, S\rangle + \kappa(g^{(n)}, S) \qquad \forall S \in H^1_D(\Omega), \quad (3.12)$$

and

$$T^{(n)} = h \qquad \text{on } \Gamma_D. \quad (3.13)$$

From (1.5) and (3.5), we easily see that $\{\|g^{(n)}\|_{L^2(\Gamma_N)}\}$ and $\{\|Q^{(n)}\|_{L^2(\Omega)}\}$ are uniformly bounded. Also, by (2.14) we have that the sequences $\{\|\mathbf{u}^{(n)}\|_{\mathbf{H}^1(\Omega)}\}$, $\{\|T^{(n)}\|_{H^1(\Omega)}\}$, and $\{\|p^{(n)}\|_{L^2(\Omega)}\}$ are uniformly bounded. We may then extract subsequences such that

$$\begin{aligned}
Q^{(n)} &\rightharpoonup \hat{Q} && \text{in } L^2(\Omega),\\
g^{(n)} &\rightharpoonup \hat{g} && \text{in } L^2(\Gamma_N),\\
\mathbf{u}^{(n)} &\rightharpoonup \hat{\mathbf{u}} && \text{in } \mathbf{H}^1_0(\Omega) \quad \text{and} \quad \nabla\mathbf{u}^{(n)} \rightharpoonup \nabla\hat{\mathbf{u}} \quad \text{in } \mathbf{L}^2(\Omega),\\
T^{(n)} &\rightharpoonup \hat{T} && \text{in } H^1(\Omega) \quad \text{and} \quad \nabla T^{(n)} \rightharpoonup \nabla\hat{T} \quad \text{in } \mathbf{L}^2(\Omega),\\
p^{(n)} &\rightharpoonup \hat{p} && \text{in } L^2_0(\Omega),\\
\mathbf{u}^{(n)} &\to \hat{\mathbf{u}} && \text{in } \mathbf{L}^4(\Omega) \text{ and } \mathbf{L}^2(\Omega),\\
T^{(n)}|_{\Gamma_N} &\to \hat{T}|_{\Gamma_N} && \text{in } L^2(\Gamma_N)
\end{aligned}$$

for some $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \in \mathbf{H}_0^1(\Omega) \times H^1(\Omega) \times L_0^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N)$. The last two convergence results above follow from the compact embeddings $\mathbf{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$ and $H^{1/2}(\Gamma_N) \hookrightarrow\hookrightarrow L^2(\Gamma_N)$. We may pass to the limit in (3.10)–(3.12) to determine that $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})$ satisfies (3.1)–(3.4). Indeed, the only troublesome term when one passes to the limit is the nonlinearity $c_0(\cdot, \cdot, \cdot)$. However, note that

$$c_0(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v}) = \int_{\partial\Omega} (\mathbf{u}^{(n)} \cdot \mathbf{n})\mathbf{u}^{(n)} \cdot \mathbf{v}\, ds - \int_{\Omega} (\mathbf{u}^{(n)} \cdot \nabla)\mathbf{v} \cdot \mathbf{u}^{(n)}\, d\mathbf{x} \qquad \forall \mathbf{v} \in \mathscr{D}(\bar{\Omega}),$$

where $\mathscr{D}(\bar{\Omega})$ is the space of test functions. Then, since $\mathbf{u}^{(n)} \to \hat{\mathbf{u}}$ in $\mathbf{L}^2(\Omega)$ and $\int_{\partial\Omega} (\mathbf{u}^{(n)} \cdot \mathbf{n})\mathbf{u}^{(n)} \cdot \mathbf{v}\, ds = 0$ for all n, we have that

$$\lim_{k\to\infty} c_0(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) = -\int_{\Omega} (\hat{\mathbf{u}} \cdot \nabla)\mathbf{v} \cdot \hat{\mathbf{u}}\, d\mathbf{x} \qquad \forall \mathbf{v} \in \mathscr{D}(\bar{\Omega}).$$

Since $\mathscr{D}(\bar{\Omega})$ is dense in $\mathbf{H}_0^1(\Omega)$, we have that, for each $\hat{\mathbf{u}} \in \mathbf{H}_0^1(\Omega)$,

$$\lim_{k\to\infty} c_0(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) = c_0(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}^1(\Omega).$$

Thus, we have shown that $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})$ indeed satisfies (3.1)–(3.4) so that $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \in \mathscr{U}_{\text{ad}}$.

Finally, it is easy to see that $\mathscr{J}_1(\cdot, \cdot, \cdot, \cdot, \cdot)$ is weakly lower semicontinuous so that

$$\mathscr{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) = \inf_{(\mathbf{v}, S, q, R, z)\in\mathscr{U}_{\text{ad}}} \mathscr{J}_1(\mathbf{v}, S, q, R, z). \tag{3.14}$$

Thus, an optimal solution belonging to $\mathscr{U}_{\text{ad}}$ exists.

The regularity result easily follows by an argument similar to that in Proposition 2.3. ∎

4. THE EXISTENCE OF LAGRANGE MULTIPLIERS AND AN OPTIMALITY SYSTEM

4.1. *The Existence of Lagrange Multipliers*

This section is devoted to obtaining an optimality system to problem (3.8). We wish to use the method of Lagrange multipliers to turn the constrained optimization problem (3.8) into an unconstrained one. We also establish that the Lagrange multiplier with respect to the functional (1.5) is equal to 1.

THEOREM 4.1. *Let $h \in H^1(\Gamma_D)$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Assume that* $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \in \mathscr{U}_{\text{ad}}$ *is an optimal solution to the minimization problem* (3.8). *Then there* *exist Lagrange multipliers* $(\boldsymbol{\xi}, \Phi, \sigma) \in \mathbf{V} \cap \mathbf{H}^2(\Omega) \times H^s(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega)$ *for all* $s \in [0, \frac{3}{2})$ *such that*

$$-\kappa\,\Delta\Phi - (\hat{\mathbf{u}} \cdot \nabla)\Phi - \alpha\,(\boldsymbol{\xi}, \mathbf{g}) = 0 \text{ in } \Omega, \qquad \Phi = \gamma\hat{Q}, \tag{4.1}$$

$$\left.\frac{\partial\Phi}{\partial\mathbf{n}}\right|_{\Gamma_N} = 0, \qquad (\delta\hat{g} - \kappa\Phi, g - \hat{g})_{L^2(\Gamma_N)} \geq 0 \qquad \forall g \in \mathscr{V}, \qquad \Phi|_{\Gamma_D} = 0, \tag{4.2}$$

$$-\nu\,\Delta\boldsymbol{\xi} - (\hat{\mathbf{u}} \cdot \nabla)\boldsymbol{\xi} + B(\hat{\mathbf{u}}, \boldsymbol{\xi}) + \Phi\nabla\hat{T} + \mathbf{curl}^2\hat{\mathbf{u}} = \nabla\sigma \qquad \text{in } \Omega, \tag{4.3}$$

and

$$\nabla \cdot \boldsymbol{\xi} = 0, \qquad \boldsymbol{\xi}|_{\partial\Omega} = 0, \tag{4.4}$$

where

$$B(\hat{\mathbf{u}}, \boldsymbol{\xi}) = \left(\left(\boldsymbol{\xi}, \frac{\partial\mathbf{u}}{\partial x_1}\right), \left(\boldsymbol{\xi}, \frac{\partial\mathbf{u}}{\partial x_2}\right)\right)^T.$$

Proof. To prove the existence of Lagrange multipliers for the constrained minimization problem (3.8), we use a penalty method. Let us consider the auxiliary extremal problem: find $(\mathbf{u}, T, p, Q, g) \in \mathbf{V} \times H^1(\Omega) \times L^2_0(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N)$ which minimizes the functional

$$\begin{aligned} J_\varepsilon(\mathbf{u}, T, p, Q, g) = \mathscr{J}_1(\mathbf{u}, T, p, Q, g) & \\ + \frac{1}{2\varepsilon}\| -\nu\,\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \alpha T\mathbf{g} + \nabla p - \mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} & \\ + \frac{N}{2}\|\mathbf{u} - \hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \frac{N}{2}\|T - \hat{T}\|^2_{L^2(\Omega)} & \\ + \frac{N}{2}\|g - \hat{g}\|^2_{L^2(\Gamma_N)}, & \end{aligned} \tag{4.5}$$

with

$$-\kappa\,\Delta T + (\mathbf{u} \cdot \nabla)T = Q \qquad \text{in } \Omega, \tag{4.6}$$

$$T|_{\Gamma_D} = h, \qquad \left.\frac{\partial T}{\partial\mathbf{n}}\right|_{\Gamma_N} = g, \qquad g \in \mathscr{V}, \qquad \mathbf{u}|_{\partial\Omega} = 0, \qquad \nabla \cdot \mathbf{u} = 0, \tag{4.7}$$

where $(\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{Q}, \hat{g}) \in \mathbf{V} \cap \mathbf{H}^2(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega) \times H^s(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N)$ for all $s \in [1, \frac{3}{2})$ is a solution to the extremal problem (3.8), such that inequality (3.7) holds true with $\varepsilon = \hat{\varepsilon}$ and $N > 0$ and $\varepsilon \in (0, 1)$ are parameters. The existence of this solution $(\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{Q}, \hat{g})$ was established in Theorem 3.1. As in the proof of Theorem 3.1, one can prove that there exists a solution to the problem (4.5)–(4.7) $(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) \in$

$\mathbf{V} \cap \mathbf{H}^2(\Omega) \times H^s(\Omega) \times H^1(\Omega) \cap L_0^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N)$ for all $s \in [1, \frac{3}{2})$. Moreover, from the fact that $J_\varepsilon(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) \leq J_\varepsilon(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) = \mathcal{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})$ and inequality (2.14), we have

$$\begin{aligned} &\{(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon)\}_{\varepsilon\in(0,1)} \text{ is bounded in} \\ &\qquad \mathbf{V} \cap \mathbf{H}^2(\Omega) \times H^s(\Omega) \times L_0^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N) \end{aligned} \tag{4.8}$$

for all $s \in [1, \frac{3}{2})$. Thus, from (4.5)–(4.8), for any $\hat{\varepsilon} > 0$ taking parameter N sufficiently large we obtain

$$\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} + \|\hat{T}_\varepsilon - \hat{T}\|_{L^2(\Omega)} + \|\hat{g}_\varepsilon - \hat{g}\|^2_{L^2(\Gamma_N)} \leq \frac{\hat{\varepsilon}}{2}. \tag{4.9}$$

Denoting

$$\hat{\mathbf{f}}_\varepsilon = -\nu\,\Delta\hat{\mathbf{u}}_\varepsilon + (\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\hat{\mathbf{u}}_\varepsilon - \alpha\hat{T}_\varepsilon\mathbf{g} + \nabla\hat{p}_\varepsilon - \mathbf{f},$$

we easily have

$$\hat{\mathbf{f}}_\varepsilon \to \mathbf{0} \qquad \text{in } \mathbf{L}^2(\Omega). \tag{4.10}$$

By the Sobolev embedding theorem and interpolation theorem

$$\begin{aligned} &\|(\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\hat{\mathbf{u}}_\varepsilon - (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} \\ &\quad \leq \|((\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}) \cdot \nabla)\hat{\mathbf{u}}_\varepsilon + (\hat{\mathbf{u}} \cdot \nabla)(\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}})\|_{\mathbf{L}^2(\Omega)} \\ &\quad \leq C\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{H}^{4/3}(\Omega)}(\|\hat{\mathbf{u}}_\varepsilon\|_{\mathbf{H}^2(\Omega)} + \|\hat{\mathbf{u}}\|_{\mathbf{H}^2(\Omega)}) \\ &\quad \leq C\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{V}}^{1/4}\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|^{3/4}_{\mathbf{H}^2(\Omega)}(\|\hat{\mathbf{u}}_\varepsilon\|_{\mathbf{H}^2(\Omega)} + \|\hat{\mathbf{u}}\|_{\mathbf{H}^2(\Omega)}) \\ &\quad \leq C\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{V}}^{1/4}. \end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned} &-\nu\,\Delta(\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}) + (\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\hat{\mathbf{u}}_\varepsilon - (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} \\ &\qquad -\alpha(\hat{T}_\varepsilon - \hat{T})\mathbf{g} + \nabla(\hat{p}_\varepsilon - \hat{p}) = \hat{\mathbf{f}}_\varepsilon \qquad \text{in } \Omega \end{aligned} \tag{4.12}$$

and

$$\nabla \cdot (\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}) = 0, \qquad (\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}})|_{\partial\Omega} = 0. \tag{4.13}$$

By (4.11) and an *a priori* estimate for the Stokes problem (see [24]) we have

$$\begin{aligned} &\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{H}^2(\Omega)} + \|\hat{p}_\varepsilon - \hat{p}\|_{H^1(\Omega)} \\ &\qquad \leq C\big(\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{V}}^{1/4} + \|\hat{T}_\varepsilon - \hat{T}\|_{L^2(\Omega)} + \|\hat{\mathbf{f}}_\varepsilon\|_{\mathbf{L}^2(\Omega)}\big). \end{aligned} \tag{4.14}$$

Inequalities (4.9), (4.10), and (4.14) imply that for any $\hat{\varepsilon} > 0$ there exist an $N(\hat{\varepsilon}) > 0$ and $\varepsilon_0 > 0$ such that

$$\|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} + \|\hat{T}_\varepsilon - \hat{T}\|_{H^1(\Omega)} + \|\hat{p}_\varepsilon - \hat{p}\|_{L^2(\Omega)} + \|\hat{Q}_\varepsilon - \hat{Q}\|_{L^2(\Omega)} + \|\hat{g}_\varepsilon - \hat{g}\|_{L^2(\Gamma_N)} \le \hat{\varepsilon} \tag{4.15}$$

$\forall \varepsilon \in (0, \varepsilon_0)$. Therefore, without loss of generality, taking, if necessary, a subsequence one can prove that

$$(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) \rightharpoonup (\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}, \tilde{Q}, \tilde{g}) \quad \text{in } \mathbf{V} \cap \mathbf{H}^2(\Omega) \times H^s(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N)$$

for all $s \in [1, \frac{3}{2})$. In the same way, as was done in the proof of Theorem 3.1, one can show that $(\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}, \tilde{Q}, \tilde{g}) \in \mathcal{U}_{\rm ad}$. Moreover, inequality (4.15) and the weak lower semicontinuity of norms in Hilbert spaces imply

$$\|\tilde{\mathbf{u}} - \hat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} + \|\tilde{T} - \hat{T}\|_{H^1(\Omega)} + \|\tilde{p} - \hat{p}\|_{L^2(\Omega)} + \|\tilde{Q} - \hat{Q}\|_{L^2(\Omega)} + \|\tilde{g} - \hat{g}\|_{L^2(\Gamma_N)} \le \hat{\varepsilon}. \tag{4.16}$$

By the definition of J_ε, the inequality

$$J_\varepsilon(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) \le \mathcal{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})$$

yields

$$\mathcal{J}_1(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) \le \mathcal{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}). \tag{4.17}$$

Since the functional $\mathcal{J}_1$ is weakly lower semicontinuous we obtain

$$\mathcal{J}_1(\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}, \tilde{Q}, \tilde{g}) \le \mathcal{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}). \tag{4.18}$$

From the facts (4.16) and (4.18), we have that $(\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}, \tilde{Q}, \tilde{g})$ is a solution to the optimal control problem (3.8). Now, if we assume that $(\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}, \tilde{Q}, \tilde{g}) \neq (\hat{\mathbf{u}}, \hat{T}, \hat{p}, \tilde{Q}, \hat{g})$, then

$$\mathcal{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) - \mathcal{J}_1(\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}, \tilde{Q}, \tilde{g}) \ge \frac{1}{2}\|\tilde{\mathbf{u}} - \hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\tilde{T} - \hat{T}\|^2_{L^2(\Omega)} > 0,$$

which contradicts the fact that $(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})$ is a solution to the problem (3.8). Thus, $(\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}, \tilde{Q}, \tilde{g}) = (\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})$ and we have

$$(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) \rightharpoonup (\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}) \quad \text{in } \mathbf{H}^2(\Omega) \times H^s(\Omega) \times L^2_0(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N) \tag{4.19}$$

for all $s \in [1, \frac{3}{2})$. Moreover, by (4.18)–(4.19), we have

$$\lim_{\varepsilon \to 0+} \mathscr{J}_1(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) = \mathscr{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g}). \tag{4.20}$$

Hence, by (4.19) and (4.20),

$$\hat{g}_\varepsilon \to \hat{g} \qquad \text{in } L^2(\Gamma_N). \tag{4.21}$$

On the other hand, the facts (4.10), (4.14), (4.19), and (4.21) imply

$$\begin{aligned}(\hat{\mathbf{u}}_\varepsilon, \hat{T}_\varepsilon, \hat{p}_\varepsilon, \hat{Q}_\varepsilon, \hat{g}_\varepsilon) &\to (\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})\\ &\text{in } \mathbf{H}^2(\Omega) \times H^s(\Omega) \times L_0^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N).\end{aligned} \tag{4.22}$$

and

$$(\hat{\mathbf{u}}_\varepsilon, \hat{p}_\varepsilon, \hat{T}_\varepsilon) \to (\hat{\mathbf{u}}, \hat{p}, \hat{T}) \qquad \text{in } \mathbf{V} \times H^1(\Omega) \cap L_0^2(\Omega) \times H^s(\Omega) \tag{4.23}$$

for all $s \in [1, \frac{3}{2})$.

Taking first variations to problem (4.5)–(4.7) we obtain the optimality system

$$\Phi_\varepsilon = \gamma \hat{Q}_\varepsilon, \qquad -\nabla \cdot (\kappa \nabla \hat{T}_\varepsilon) + (\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\hat{T}_\varepsilon = \hat{Q}_\varepsilon \text{ in } \Omega, \tag{4.24}$$

$$\boldsymbol{\xi}_\varepsilon = \frac{1}{\varepsilon}(-\nu \Delta \hat{\mathbf{u}}_\varepsilon + (\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\hat{\mathbf{u}}_\varepsilon + \nabla \hat{p}_\varepsilon - \alpha \hat{T}_\varepsilon \mathbf{g} - \mathbf{f}), \tag{4.25}$$

$$-\kappa \Delta \Phi_\varepsilon - (\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\Phi_\varepsilon - \alpha(\boldsymbol{\xi}_\varepsilon, \mathbf{g}) + N(\hat{T}_\varepsilon - \hat{T}) = 0 \qquad \text{in } \Omega, \tag{4.26}$$

$$\begin{gathered}\left.\frac{\partial \Phi_\varepsilon}{\partial \mathbf{n}}\right|_{\Gamma_N} = 0, \qquad \Phi_\varepsilon|_{\Gamma_D} = 0,\\ (\delta \hat{g}_\varepsilon + N(\hat{g}_\varepsilon - \hat{g}) - \kappa \Phi_\varepsilon, g - \hat{g}_\varepsilon)_{L^2(\Gamma_N)} \geq 0 \qquad \forall g \in \mathscr{V},\end{gathered} \tag{4.27}$$

$$\begin{aligned}-\nu \Delta \boldsymbol{\xi}_\varepsilon - (\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\boldsymbol{\xi}_\varepsilon + B(\hat{\mathbf{u}}_\varepsilon, \boldsymbol{\xi}_\varepsilon) + \Phi_\varepsilon \nabla \hat{T}_\varepsilon&\\ + N(\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}) + \mathbf{curl}^2 \hat{\mathbf{u}}_\varepsilon = \nabla \sigma_\varepsilon \qquad \text{in } \Omega,&\end{aligned} \tag{4.28}$$

and

$$\nabla \cdot \boldsymbol{\xi}_\varepsilon = 0, \qquad \boldsymbol{\xi}_\varepsilon|_{\partial\Omega} = 0, \tag{4.29}$$

where the first equality in (4.27) makes sense due to the estimate

$$\left\| \frac{\partial \Phi_\varepsilon}{\partial \mathbf{n}} \right\|_{H^s(\partial\Omega)} \leq C(s)(\|\kappa \Delta \Phi_\varepsilon\|_{L^2(\Omega)} + \|\Phi_\varepsilon\|_{H^1(\Omega)})$$

$$\text{whenever } \Phi_\varepsilon \in H^1(\Omega),\ s < 0.$$

Now, setting $I_\varepsilon = \|\boldsymbol{\xi}_\varepsilon\|_{\mathbf{L}^2(\Omega)}$, we prove that $\liminf_{\varepsilon\to 0+} I_\varepsilon < +\infty$ by the contradiction. Let us assume that $\liminf_{\varepsilon\to 0+} I_\varepsilon = +\infty$ and denote

$$\tilde{\Phi}_\varepsilon = \frac{\Phi_\varepsilon}{I_\varepsilon}, \qquad \tilde{\boldsymbol{\xi}}_\varepsilon = \frac{\boldsymbol{\xi}_\varepsilon}{I_\varepsilon}, \qquad \text{and} \qquad \tilde{\sigma} = \frac{\sigma_\varepsilon}{I_\varepsilon}.$$

From the equations (4.24)–(4.29) the triple $(\tilde{\boldsymbol{\xi}}_\varepsilon, \tilde{\Phi}_\varepsilon, \tilde{\sigma}_\varepsilon)$ satisfies the equations

$$-\kappa\,\Delta\tilde{\Phi}_\varepsilon - (\hat{\mathbf{u}}_\varepsilon\cdot\nabla)\tilde{\Phi}_\varepsilon - \alpha\,(\tilde{\boldsymbol{\xi}}_\varepsilon, \mathbf{g}) + N\frac{\hat{T}_\varepsilon - \hat{T}}{I_\varepsilon} = 0 \qquad \text{in } \Omega, \tag{4.30}$$

$$\frac{\partial\tilde{\Phi}_\varepsilon}{\partial\mathbf{n}}\bigg|_{\Gamma_N} = 0, \qquad \tilde{\Phi}_\varepsilon|_{\Gamma_D} = 0,$$
$$\left(\frac{\delta\hat{g}_\varepsilon + N(\hat{g}_\varepsilon - \hat{g})}{I_\varepsilon} - \kappa\tilde{\Phi}_\varepsilon, g - \hat{g}_\varepsilon\right)_{L^2(\Gamma_N)} \geq 0 \qquad \forall g \in \mathcal{V}, \tag{4.31}$$

$$\begin{aligned} -\nu\,\Delta\tilde{\boldsymbol{\xi}}_\varepsilon - (\hat{\mathbf{u}}_\varepsilon\cdot\nabla)\tilde{\boldsymbol{\xi}}_\varepsilon + B(\hat{\mathbf{u}}_\varepsilon, \tilde{\boldsymbol{\xi}}_\varepsilon) + \tilde{\Phi}_\varepsilon\nabla\hat{T}_\varepsilon& \\ + N\frac{\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}}{I_\varepsilon} + \frac{\mathbf{curl}^2\hat{\mathbf{u}}_\varepsilon}{I_\varepsilon} = \nabla\tilde{\sigma}_\varepsilon \qquad \text{in } \Omega,& \end{aligned} \tag{4.32}$$

$$\nabla\cdot\tilde{\boldsymbol{\xi}}_\varepsilon = 0, \qquad \tilde{\boldsymbol{\xi}}_\varepsilon|_{\partial\Omega} = 0. \tag{4.33}$$

Since, by definition, $\|\tilde{\boldsymbol{\xi}}_\varepsilon\|_{\mathbf{L}^2(\Omega)} \leq 1$ and $\|\tilde{\Phi}_\varepsilon\|_{L^2(\Omega)} \leq \mathcal{J}_1(\hat{\mathbf{u}}, \hat{T}, \hat{p}, \hat{Q}, \hat{g})/I_\varepsilon$, taking, if necessary, a subsequence, one can show that

$$(\tilde{\boldsymbol{\xi}}_\varepsilon, \tilde{\Phi}_\varepsilon) \rightharpoonup (\tilde{\boldsymbol{\xi}}, 0) \qquad \text{in } \mathbf{L}^2(\Omega)\times L^2(\Omega). \tag{4.34}$$

Taking the inner product of (4.30) with Φ_ε in $L^2(\Omega)$ and integrating by parts we have

$$\int_\Omega \kappa|\nabla\tilde{\Phi}_\varepsilon|^2\,d\mathbf{x} \leq \|\tilde{\Phi}_\varepsilon\|^2_{L^2(\Omega)} + C\left(\frac{\|\hat{T}_\varepsilon - \hat{T}\|^2_{L^2(\Omega)}}{I_\varepsilon^2} + \|\tilde{\boldsymbol{\xi}}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)}\right). \tag{4.35}$$

By the definition of $\|\tilde{\Phi}_\varepsilon\|_{L^2(\Omega)} \leq C$ and (4.35) we can assume, without loss of generality, that

$$\|\tilde{\Phi}_\varepsilon\|_{H^1(\Omega)} \leq C, \tag{4.36}$$

where the constant C is independent of ε. Then again taking the inner product of (4.32) with $\tilde{\boldsymbol{\xi}}_\varepsilon$ in $\mathbf{L}^2(\Omega)$ and integrating by parts we obtain

$$\begin{aligned}
\nu \int_\Omega |\nabla \tilde{\boldsymbol{\xi}}_\varepsilon|^2\, d\mathbf{x}
&= -\int_\Omega \big(\tilde{\Phi}_\varepsilon(\nabla \hat{T}_\varepsilon, \tilde{\boldsymbol{\xi}}_\varepsilon) + N(\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}, \tilde{\boldsymbol{\xi}}_\varepsilon) + (B(\hat{\mathbf{u}}_\varepsilon, \tilde{\boldsymbol{\xi}}_\varepsilon), \tilde{\boldsymbol{\xi}}_\varepsilon)\big)\, d\mathbf{x} \\
&\quad - \int_\Omega (\mathbf{curl}^2 \hat{\mathbf{u}}_\varepsilon, \tilde{\boldsymbol{\xi}}_\varepsilon)/I_\varepsilon\, d\mathbf{x} \\
&\le C\Big(\|\nabla \hat{T}_\varepsilon\|_{L^2(\Omega)} \|\tilde{\Phi}_\varepsilon\|_{H^1(\Omega)} \|\tilde{\boldsymbol{\xi}}_\varepsilon\|_{\mathbf{V}} + \|\hat{\mathbf{u}}_\varepsilon\|_{\mathbf{V}} \|\tilde{\boldsymbol{\xi}}_\varepsilon\|^{1/2}_{\mathbf{L}^2(\Omega)} \|\tilde{\boldsymbol{\xi}}_\varepsilon\|^{3/2}_{\mathbf{V}} \\
&\qquad + (\|\hat{\mathbf{u}}_\varepsilon\|_{\mathbf{H}^2(\Omega)} \|\tilde{\boldsymbol{\xi}}_\varepsilon\|_{\mathbf{L}^2(\Omega)} + \|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} \|\tilde{\boldsymbol{\xi}}_\varepsilon\|_{\mathbf{L}^2(\Omega)})/I_\varepsilon\Big).
\end{aligned} \tag{4.37}$$

By (4.22), (4.34), and (4.36) the above estimate implies immediately

$$\|\tilde{\boldsymbol{\xi}}_\varepsilon\|_{\mathbf{V}} \le C, \tag{4.38}$$

where the constant C is independent of ε. From the facts (4.34), (4.36), and (4.38), again taking, if necessary, a subsequence, we obtain

$$\begin{aligned}
(\tilde{\boldsymbol{\xi}}_\varepsilon, \tilde{\Phi}_\varepsilon) &\rightharpoonup (\tilde{\boldsymbol{\xi}}, 0) \qquad \text{in } \mathbf{V} \times H^1(\Omega), \\
(\tilde{\boldsymbol{\xi}}_\varepsilon, \tilde{\Phi}_\varepsilon) &\to (\tilde{\boldsymbol{\xi}}, 0) \qquad \text{in } \mathbf{L}^2(\Omega) \times L^2(\Omega).
\end{aligned} \tag{4.39}$$

Furthermore, by (4.8) and (4.39) the sequence

$$\left\{ -(\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\tilde{\boldsymbol{\xi}}_\varepsilon + B(\hat{\mathbf{u}}_\varepsilon, \tilde{\boldsymbol{\xi}}_\varepsilon) + \tilde{\Phi}_\varepsilon \nabla \hat{T}_\varepsilon + N\frac{\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}}{I_\varepsilon} + \frac{\mathbf{curl}^2 \hat{\mathbf{u}}_\varepsilon}{I_\varepsilon} \right\}_{\varepsilon \in (0,1)}$$

is bounded in $\mathbf{L}^2(\Omega)$. Thus, we have (see [24])

$$\tilde{\boldsymbol{\xi}}_\varepsilon \rightharpoonup \tilde{\boldsymbol{\xi}} \qquad \text{in } \mathbf{H}^2(\Omega). \tag{4.40}$$

Since $\|\tilde{\boldsymbol{\xi}}_\varepsilon\|_{\mathbf{L}^2(\Omega)} = 1$ it follows from (4.39) that

$$\|\tilde{\boldsymbol{\xi}}\|_{\mathbf{L}^2(\Omega)} = 1. \tag{4.41}$$

Thus, passing to the limit in (4.30)–(4.33) as $\varepsilon \to 0+$, keeping in mind (4.19), (4.39), and (4.40), we obtain the optimality system (4.1)–(4.4) with $\tilde{\Phi} \equiv 0$:

$$-\alpha(\tilde{\boldsymbol{\xi}}, \mathbf{g}) = 0 \qquad \text{in } \Omega, \tag{4.42}$$

$$-\nu \Delta \tilde{\boldsymbol{\xi}} - (\hat{\mathbf{u}} \cdot \nabla)\tilde{\boldsymbol{\xi}} + B(\hat{\mathbf{u}}, \tilde{\boldsymbol{\xi}}) = \nabla \tilde{\sigma} \qquad \text{in } \Omega, \tag{4.43}$$

$$\nabla \cdot \tilde{\boldsymbol{\xi}} = 0, \qquad \tilde{\boldsymbol{\xi}}|_{\partial\Omega} = 0. \tag{4.44}$$

By (4.42) and (4.44) there exist vectors $\mathbf{a}_i$ $(i = 1, 2)$ such that

$$\frac{\partial \tilde{\boldsymbol{\xi}}_i}{\partial \mathbf{a}_i} = 0 \qquad \text{in } \Omega \qquad \forall i \in \{1, 2\}. \tag{4.45}$$

Hence, by (4.44) and (4.45), $\tilde{\boldsymbol{\xi}} \equiv 0$. But this contradicts (4.41). Thus, we have $\liminf_{\varepsilon\to 0+} I_\varepsilon < +\infty$ or, in other words,

$$\|\Phi_\varepsilon\|_{L^2(\Omega)} + \|\boldsymbol{\xi}_\varepsilon\|_{\mathbf{L}^2(\Omega)} \le C. \tag{4.46}$$

Now, taking the inner product of (4.26) with Φ_ε in $L^2(\Omega)$ and integrating by parts we have

$$\int_\Omega \kappa |\nabla \Phi_\varepsilon|^2 \, d\mathbf{x} \le \|\Phi_\varepsilon\|^2_{L^2(\Omega)} + C\big(\|\hat{T}_\varepsilon - \hat{T}\|^2_{L^2(\Omega)} + \|\boldsymbol{\xi}_\varepsilon\|^2_{\mathbf{L}^2(\Omega)}\big). \tag{4.47}$$

By (4.46) and (4.47) we can assume, without loss of generality, that

$$\Phi_\varepsilon \rightharpoonup \Phi \qquad \text{in } H^1(\Omega). \tag{4.48}$$

Then, again taking the inner product of (4.28) with $\boldsymbol{\xi}_\varepsilon$ in $\mathbf{L}^2(\Omega)$ and integrating by parts, we obtain

$$\begin{aligned} \nu \int_\Omega |\nabla \boldsymbol{\xi}_\varepsilon|^2 \, d\mathbf{x} &= -\int_\Omega \big(\Phi_\varepsilon(\nabla \hat{T}_\varepsilon, \boldsymbol{\xi}_\varepsilon) + (\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}, \boldsymbol{\xi}_\varepsilon) \\ &\qquad + (B(\hat{\mathbf{u}}_\varepsilon, \boldsymbol{\xi}_\varepsilon), \boldsymbol{\xi}_\varepsilon) + (\mathbf{curl}^2 \hat{\mathbf{u}}_\varepsilon, \boldsymbol{\xi}_\varepsilon)\big) \, d\mathbf{x} \\ &\le C\Big(\|\nabla \hat{T}_\varepsilon\|_{L^2(\Omega)} \|\Phi_\varepsilon\|_{H^1(\Omega)} \|\boldsymbol{\xi}_\varepsilon\|_{\mathbf{V}} \\ &\qquad + \|\hat{\mathbf{u}}_\varepsilon\|_{\mathbf{V}} \|\boldsymbol{\xi}_\varepsilon\|^{1/2}_{\mathbf{L}^2(\Omega)} \|\boldsymbol{\xi}_\varepsilon\|^{3/2}_{\mathbf{V}} + \|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{\xi}_\varepsilon\|_{\mathbf{L}^2(\Omega)}\Big). \end{aligned} \tag{4.49}$$

By (4.22), (4.46), and (4.48) the above estimate implies immediately

$$\boldsymbol{\xi}_\varepsilon \rightharpoonup \boldsymbol{\xi} \qquad \text{in } \mathbf{V}. \tag{4.50}$$

Furthermore, by (4.8), (4.48), and (4.50) the sequence

$$\Big\{-(\hat{\mathbf{u}}_\varepsilon \cdot \nabla)\boldsymbol{\xi}_\varepsilon + B(\hat{\mathbf{u}}_\varepsilon, \boldsymbol{\xi}_\varepsilon) + \Phi_\varepsilon \nabla \hat{T}_\varepsilon + N(\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}) + \mathbf{curl}^2 \hat{\mathbf{u}}_\varepsilon\Big\}_{\varepsilon\in(0,1)}$$

is bounded in $\mathbf{L}^2(\Omega)$. Thus, we have (see [24])

$$\boldsymbol{\xi}_\varepsilon \rightharpoonup \boldsymbol{\xi} \qquad \text{in } \mathbf{H}^2(\Omega). \tag{4.51}$$

Once again, since (4.8) and (4.48) imply boundedness of the sequence

$$\{-(\hat{\mathbf{u}} \cdot \nabla)\Phi_\varepsilon - \alpha(\mathbf{w}_\varepsilon, \mathbf{g}) + \hat{T}_\varepsilon - \hat{T}\}_{\varepsilon\in(0,1)}$$

in the space $L^2(\Omega)$ by regularity results for elliptic equations (see [22]) we have

$$\Phi_\varepsilon \rightharpoonup \Phi \qquad \text{in } H^s(\Omega) \text{ as } \varepsilon \to 0+ \tag{4.52}$$

for all $s \in [1, \frac{3}{2})$.

By (4.51) and (4.52) we obtain

$$(\boldsymbol{\xi}_\varepsilon, \Phi_\varepsilon) \rightharpoonup (\boldsymbol{\xi}, \Phi) \qquad \text{in } \mathbf{H}^2(\Omega) \times H^s(\Omega) \tag{4.53}$$

for all $s \in [1, \frac{3}{2})$. By (4.22), (4.50), and (4.53) passing to the limit in the equations (4.24)–(4.29) we obtain optimality system (4.1)–(4.4). Relations (4.22), (4.50), and (4.53) and Eq. (4.4) imply the necessary regularity of Lagrange multipliers. ∎

We note that if we take $\mathscr{V} = L^2(\Gamma_N)$, then the boundary condition $(\delta\hat{g} - \kappa\Phi, g - \hat{g})_{L^2(\Gamma_N)} \geq 0$ in (4.2) becomes $\delta\hat{g} - \kappa\Phi = 0$ on Γ_N.

4.2. *The Optimality System*

Using the optimality condition $Q = \frac{\Phi}{\gamma}$, we obtain the following optimality system: find $(\mathbf{u}, T, p, g, \boldsymbol{\xi}, \Phi, \sigma) \in \mathbf{H}_0^1(\Omega) \times H^1(\Omega) \times L_0^2(\Omega) \times \mathscr{V} \times \mathbf{H}_0^1(\Omega) \times H_D^1(\Omega) \times L_0^2(\Omega)$ such that

$$-\nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \alpha T\mathbf{g} + \mathbf{f} \qquad \text{in } \Omega,$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega,$$

$$-\kappa\,\Delta T + (\mathbf{u}\cdot\nabla)T = \frac{\Phi}{\gamma} \qquad \text{in } \Omega$$

$$-\nu\,\Delta\boldsymbol{\xi} - (\mathbf{u}\cdot\nabla)\boldsymbol{\xi} + \left(\left(\boldsymbol{\xi}, \frac{\partial\mathbf{u}}{\partial x_1}\right), \left(\boldsymbol{\xi}, \frac{\partial\mathbf{u}}{\partial x_2}\right)\right)^T + \Phi\nabla T + \mathbf{curl}^2\mathbf{u} = \nabla\sigma \quad \text{in } \Omega,$$

$$\nabla\cdot\boldsymbol{\xi} = 0$$

$$-\kappa\,\Delta\Phi - (\mathbf{u}\cdot\nabla)\Phi - \alpha\,(\boldsymbol{\xi}, \mathbf{g}) = 0 \qquad \text{in } \Omega,$$

$$T = h \text{ on } \Gamma_D, \qquad \frac{\partial T}{\partial\mathbf{n}} = g \text{ on } \Gamma_N,$$

$$\left.\frac{\partial\Phi}{\partial\mathbf{n}}\right|_{\Gamma_N} = 0, \qquad (\delta g - \kappa\Phi, \check{g} - g)_{L^2(\Gamma_N)} \geq 0 \qquad \forall\check{g} \in \mathscr{V}. \tag{4.54}$$

For the case $\mathscr{V} = L^2(\Gamma_N)$, using the optimality conditions $Q = \frac{\Phi}{\gamma}$ and $g = \frac{\kappa\Phi}{\delta}$, we obtain the following optimality system: find $(\mathbf{u}, T, p, \boldsymbol{\xi}, \Phi, \sigma) \in$

$\mathbf{H}_0^1(\Omega) \times H^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times H_D^1(\Omega) \times L_0^2(\Omega)$ such that

$$
\begin{aligned}
&-\nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \alpha T\mathbf{g} + \mathbf{f} \qquad \text{in } \Omega,\\
&\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega,\\
&-\kappa\,\Delta T + (\mathbf{u}\cdot\nabla)T = \frac{\Phi}{\gamma} \qquad \text{in } \Omega\\
&-\nu\,\Delta\boldsymbol{\xi} - (\mathbf{u}\cdot\nabla)\boldsymbol{\xi} + \left(\left(\boldsymbol{\xi}, \frac{\partial\mathbf{u}}{\partial x_1}\right), \left(\boldsymbol{\xi}, \frac{\partial\mathbf{u}}{\partial x_2}\right)\right)^T + \Phi\nabla T + \mathbf{curl}^2\mathbf{u} = \nabla\sigma \quad \text{in } \Omega,\\
&\nabla\cdot\boldsymbol{\xi} = 0,\\
&-\kappa\,\Delta\Phi - (\mathbf{u}\cdot\nabla)\Phi - \alpha\,(\boldsymbol{\xi}, \mathbf{g}) = 0 \qquad \text{in } \Omega,\\
&T = h \qquad \text{on } \Gamma_D,\\
&\frac{\partial T}{\partial\mathbf{n}} = \frac{\kappa\Phi}{\delta} \qquad \text{and} \qquad \frac{\partial\Phi}{\partial\mathbf{n}} = 0 \qquad \text{on } \Gamma_N.
\end{aligned}
\tag{4.55}
$$

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