The Picard number of certain algebraic surfaces

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Abstract

We consider families of complex algebraic surfaces for which we have a good knowledge of the Néron–Severi group of one of the fibres. Using the theory of ‘Variation of Hodge Structure’ we study which algebraic cycles on this special fibre can deform within the family. For surfaces in $\mathbb{P}^3$ all this can be made very explicit. We combine this with Shioda’s results on the structure of the Néron–Severi group of certain special varieties (e.g. Fermat varieties). As an application we give examples of families of algebraic surfaces for which the generic Picard number can be determined.

Introduction

The Néron–Severi group $\text{NS}(X)$ of a nonsingular projective variety $X$ is defined as the group of divisors on $X$ modulo algebraic equivalence; it is a finitely generated group. The rank $\rho(X)$ of $\text{NS}(X)$ is called the Picard number of $X$. There are no general methods by which one can compute the Picard number of a given variety, even if we work over the field of complex numbers in which case the Néron–Severi group can be described as the space of rational cohomology classes that are of type (1, 1). Because $\rho(X)$ is not a birational invariant it can be useful to consider the Lefschetz number $\lambda(X) := b_2(X) - \rho(X)$; this is a birational invariant and therefore its definition extends to singular surfaces.

In the first section of this article we review some results of Shioda concerning Fermat varieties and Delsarte surfaces. First we discuss a theorem from [8] from
which, among other things, the Picard number of a Fermat variety can be computed. Using this result, Shioda has given in [12] an explicit algorithm to compute the Lefschetz number of a Delsarte surface. For nonsingular Delsarte surfaces the Picard number can be computed directly from this. We will give a slight correction to a formula that Shioda gives as a special case of his algorithm.

In the rest of this article we will consider families of algebraic varieties. In such a family one can try to find an upper bound for the generic Picard number by looking at the classes that are infinitesimally fixed. This becomes particularly effective if we have a good knowledge of the Néron–Severi group of one of the fibres. In Section 3 we start with an example in which one of the fibres is the Fermat surface of degree 5; the result confirms a guess stated by Shioda in [9]. The idea for treating this example in this way comes from A. de Jong and J. Steenbrink—in their paper [4] a similar approach is used in a somewhat different context. From the first example we derive two other examples of families of algebraic surfaces for which the generic Picard number can be computed.

1. A review of some results of Shioda

We begin by recalling from [6] the explicit description of the cohomology of a nonsingular hypersurface. Let $X \subset \mathbb{P}^{n+1}_C$ be a nonsingular hypersurface of degree $d$, defined by a homogeneous equation $F(T_0, \ldots, T_{n+1}) = 0$. Then

\[ H^i(X, \mathbb{C}) \cong \begin{cases} 0 & \text{if } i \text{ is odd and } i \neq n, \\ \mathbb{C} & \text{if } i \text{ is even, } i \neq n \text{ and } 0 < i < 2n \end{cases} \]

and

\[ H^n(X, \mathbb{C}) \cong \begin{cases} H^n_0(X, \mathbb{C}) & \text{if } n \text{ is odd}, \\ H^n_0(X, \mathbb{C}) \oplus \mathbb{C} \cdot u^p & \text{if } n \text{ is even, } n = 2p \end{cases} \]

where the subscript ‘0’ denotes primitive cohomology and $u^p$ is the cohomology class of the intersection of $X$ with a general $\mathbb{P}^{p+1}_C \subset \mathbb{P}^{2p+1}_C$ if $n = 2p$. Now $H^n_0(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}_0(X, \mathbb{C})$ and using the residue map $\text{Res}_X : H^{n+1}_0(\mathbb{P}^{n+1}_C \setminus X, \mathbb{C}) \to H^n(X, \mathbb{C})$ we can describe the $H^{p,q}_0(X, \mathbb{C})$ as follows: Let $\text{Jac}(F) \subset \mathbb{C}[T_0, \ldots, T_{n+1}]$ be the ideal generated by $\frac{\partial F}{\partial T_0}, \ldots, \frac{\partial F}{\partial T_{n+1}}$ and let $\mathfrak{A} := \mathbb{C}[T_0, \ldots, T_{n+1}] / \text{Jac}(F)$. Let

\[ \Omega := \sum_{i=0}^{n+1} (-1)^i \cdot T_i dT_0 \wedge \cdots \wedge \hat{dT}_i \wedge \cdots \wedge dT_{n+1} \in \Gamma(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(n+2)) \]

and for $A \in \mathbb{C}[T_0, \ldots, T_{n+1}]$ define

\[ \Omega_A := A \cdot \Omega / F^{p+1} \in \Gamma(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}((p+1)X)) \].
Denote by Res$_X(\Omega_A)$ the image under Res$_X$ of the class in $H^{n+1}(\mathbb{P}^{n+1}_\mathbb{C} \setminus X, \mathbb{C})$ defined by $\Omega_A$ and let $\iota(p) := d(p + 1) - (n + 2)$. Then there are isomorphisms

$$\lambda_{\iota(p)} : H^{\iota(p)} \to H^{n-p-p}_0(X, \mathbb{C})$$

defined by $\lambda \mod \text{Jac}(F) \mapsto p!(-1)^{p} \cdot \text{Res}_{X}(\Omega_A)$.

We will describe some of Shioda’s results on Fermat varieties and Delsarte surfaces. Let $X' \subseteq \mathbb{P}^{n+1}_\mathbb{C}$ be defined by the equation $T_0^m + \cdots + T_{n+1}^m = 0$—this is called the Fermat variety of dimension $n$ and degree $m$. Let $\mu_m$ be the group of $m$th roots of unity in $\mathbb{C}$ and let $G_m^n$ denote the quotient of the $(n + 2)$-fold product of $\mu_m$ by its subgroup of diagonal elements. Then $G_m^n$ acts faithfully on $X'$ by

$$[\zeta_0 : \ldots : \zeta_{n+1}] : (x_0 : \ldots : x_{n+1}) \mapsto (\zeta_0 x_0 : \ldots : \zeta_{n+1} x_{n+1})$$

and this action makes $H^n_0(X, \mathbb{Q})$ into a $G_m^n$-module. The character group of $G_m^n$ can be identified with the group

$$\chi_m^n = \{ \alpha = (a_0, \ldots, a_{n+1}) \in G_m^n \mid a_i \in (\mathbb{Z}/m), a_0 + \cdots + a_{n+1} = 0 \}$$

and for $\alpha \in \chi_m^n$ we define

$$V(\alpha) := \{ \xi \in H_0^n(X, \mathbb{C}) \mid g^\alpha(\xi) = \alpha(g) \cdot \xi \text{ for all } g \in G_m^n \}.$$ 

Let

$$\mathcal{A}_m^n := \{ \alpha = (a_0, \ldots, a_{n+1}) \in \chi_m^n \mid a_i \neq 0 \text{ for all } i \}$$

and for $\alpha \in \mathcal{A}_m^n$ let $|\alpha| := \sum_{i=0}^{n+1} a_i/m \in \mathbb{Z}$, where for $x \in \mathbb{Q}$ we write $\langle x \rangle := x - \lfloor x \rfloor \in [0, 1)$.

The following theorem seems to have been more or less known to several people.

**Theorem 1** (Shioda [8]). With the above notations we have:

(i) $\dim(V(\alpha)) = 0$ or 1 for all $\alpha \in \chi_m^n$ and $\dim(V(\alpha)) \neq 0$ if and only if $\alpha \in \mathcal{A}_m^n$.

(ii) $H_p^{p,q}(X^n, \mathbb{C}) = \bigoplus_{|\alpha|=q+1} V(\alpha)$ if $p + q = n$.

(iii) Suppose $n$ is even, $n = 2p$. Then

$$(H_0^n(X, \mathbb{Q}) \cap H^{p,p}(X^n, \mathbb{C})) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\alpha \in \mathcal{A}_m^n} V(\alpha).$$

where $\mathcal{A}_m^n := \{ \alpha \in \mathcal{A}_m^n \mid |t \cdot \alpha| - p + 1 \text{ for all } t \in (\mathbb{Z}/m)^* \}$.

(iv) Suppose $n$ is odd, $n = 2p + 1$. Then

$$(H_0^n(X, \mathbb{Q}) \cap (H^{p,p+1}(X^n, \mathbb{C}) + H^{p+1,p}(X^n, \mathbb{C}))) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\alpha \in \mathcal{A}_m^n} V(\alpha),$$

where $\mathcal{A}_m^n := \{ \alpha \in \mathcal{A}_m^n \mid t \cdot \alpha \in \{p + 1, p + 2\} \text{ for all } t \in (\mathbb{Z}/m)^* \}$. 

Proof. For \( \alpha \in \mathcal{A}_m \) write \( \alpha = (a_0 \pmod{m}, \ldots, a_{n+1} \pmod{m}) \) with \( a_0, \ldots, a_{n+1} \in \{1, \ldots, m-1\} \) and let

\[
A_{\alpha} := T_a^{n+1-1} \cdot \ldots \cdot T_a^{n+1-1} \pmod{\text{Jac}(F)} \in \mathbb{H}^{1, q}_{(n+1)(q+1)}(X^m, \mathbb{C})
\]

\[
W(\alpha) := \mathbb{C} \cdot \text{Res}_n(\Omega_{A_{\alpha}}) \subseteq H^{n-|\alpha|+1, |\alpha|-1}(X^m, \mathbb{C})
\]

Since \( \text{Jac}(F) \) is the ideal generated by \( T_a^{n+1-1} \cdot \ldots \cdot T_a^{n+1-1} \) in \( \mathbb{C}[T_0, \ldots, T_{n+1}] \) we see that the \( A_{\alpha} \) with \( \alpha \in \mathcal{A}_m, |\alpha| = q+1 \) form a \( \mathbb{C} \)-basis for \( H_{(n+1)(q+1)}^{1, q}(X^m, \mathbb{C}) \) and therefore

\[
H^{p, q}_\alpha(X_m, \mathbb{C}) = \bigoplus_{\alpha \in \mathcal{A}_m, |\alpha| = q+1} W(\alpha)
\]

Statements (i) and (ii) now follow from the remark that \( W(\alpha) \subseteq V(\alpha) \) for \( \alpha \in \mathcal{A}_m \).

Now let \( \zeta \) be a primitive \( m \)-th root of unity and let any \( \varphi \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\zeta)) \) act on \( H^n(X_m, \mathbb{C}) = H^n(X_m, \mathbb{Q}(\zeta)) \otimes \mathbb{Q}(\zeta) \otimes \mathbb{C} \); then \( \varphi \) maps each space \( V(\alpha) \) onto itself. From (ii) it follows that the Hodge decomposition and the decomposition into spaces \( V(\alpha) \) are already defined over \( \mathbb{Q}(\zeta) \); let

\[
H_{\alpha}^{p, q}(X_m, \mathbb{Q}(\zeta)) := H_{\alpha}^p(X_m, \mathbb{Q}(\zeta)) \cap H^{p, q}(X_m, \mathbb{C})
\]

and let

\[
V_{\mathbb{Q}(\zeta)}(\alpha) := V(\alpha) \cap H^n(X_m, \mathbb{Q}(\zeta))
\]

then we have

\[
H^n(X_m, \mathbb{Q}(\zeta)) = \bigoplus_{p+q=n} H_{\alpha}^{p, q}(X_m, \mathbb{Q}(\zeta))
\]

\[
V(\alpha) = V_{\mathbb{Q}(\zeta)}(\alpha) \otimes \mathbb{Q}(\zeta) \otimes \mathbb{C}
\]

Statements (iii) and (iv) now follow from [11, Section 3, Lemma]. \( \square \)

Using this theorem we can calculate for instance the dimension of the space of Hodge cycles \( H^2(X_m, \mathbb{Q}) \cap H^1,1(X_m, \mathbb{C}) \)—this dimension is the Picard number \( \rho(X^m) \) of \( X^m \). In fact, Aoki and Shioda have proved a formula for \( \rho(X^m) \), cf. [1] and [10].

As an example let us compute the Picard number of the Fermat surface of degree 5; using (iii) we have: \( \rho(X_5) = \# \mathcal{B}_S + 1 \). Now the 44 elements of \( \mathcal{A}_5 \) having \( |\alpha| = 2 \) are represented by the 4-tuples \((a_0, a_1, a_2, a_3)\) with \( a_i \in \{1, 2, 3, 4\} \) and \( a_0 + a_1 + a_2 + a_3 = 10 \). Then \( 4 \cdot \alpha \) is represented by \((5 - a_0, 5 - a_1, 5 - a_2, 5 - a_3)\), so \( |4 \cdot \alpha| = 2 \) and to check whether \( \alpha \in \mathcal{B}_S \) it suffices to calculate \( |5 \cdot \alpha| \).
Moreover, we do not have to treat the different permutations of a 4-tuple separately, so the next calculations suffice:

| $\alpha$          | number of permutations | representative for $\bar{2} \cdot \alpha$ | $| \bar{2} \cdot \alpha |$ | $\alpha \in B_5^2$? |
|------------------|------------------------|-------------------------------------------|-----------------|-----------------|
| $(4, 4, 1, 1)$    | 6                      | $(3, 3, 2, 2)$                            | 2               | yes             |
| $(4, 3, 2, 1)$    | 24                     | $(3, 1, 4, 2)$                            | 2               | yes             |
| $(4, 2, 2, 2)$    | 4                      | $(3, 4, 4, 4)$                            | 3               | no              |
| $(3, 3, 3, 1)$    | 4                      | $(1, 1, 1, 2)$                            | 1               | no              |
| $(3, 3, 2, 2)$    | 6                      | $(1, 1, 4, 4)$                            | 2               | yes             |

From this table we conclude that $\rho(X_4^2) = 37$.

Next we consider a surface $X_A \subset \mathbb{P}_C^3$ defined by an equation

$$T_0^{a_{10}} \cdot T_1^{a_{11}} \cdot T_2^{a_{20}} \cdot T_3^{a_{30}} + \cdots + T_0^{a_{30}} \cdot T_1^{a_{31}} \cdot T_2^{a_{22}} \cdot T_3^{a_{12}} = 0 ,$$

where the coefficients $a_{ij}$, put together in a $4 \times 4$ matrix $A = (a_{ij})$ satisfy

(i) $\det(A) \neq 0$,

(ii) $\sum_{j=0}^3 a_{ij}$ is independent of $i$ (the equation is homogenous),

(iii) for each $j \in \{1, 2, 3, 4\}$ there is an $i \in \{1, 2, 3, 4\}$ such that $a_{ij} = 0$.

Such a surface $X_A$ is called the Delsarte surface with matrix $A$. For these kind of surfaces Shioda has given in [12] an algorithm for computing the Lefschetz number $\lambda(X_A)$. Recall that this number is defined as $b_2(X_A) - \rho(X_A)$, where $X_A$ is a nonsingular birational model of $X_A$. This algorithm can be stated as follows:

**Theorem 2** (Shioda [12]). Let $A^*$ be the cofactor matrix of $A$, $\delta$ the G.C.D. of all the coefficients of $A^*$ and let $d := |\det(A)| / \delta$. Then

$$\lambda(X_A) = \# \{ \alpha = (a_0, a_1, a_2, a_3) \in \mathbb{Z}_d^4 - B_d^2 \} .$$

For the proof we refer to [12]—it is not hard to see that our formulation is equivalent to the formulation that Shioda gives.

**Remark.** Shioda states his theorem for Delsarte surfaces over any field $k$—we do not need the result in that generality here.

In some cases we can give a more explicit formula for $\lambda(X_A)$. However, the formula that Shioda states [12, Corollary 2] is only correct under the assumption that $\alpha = (a_0, a_1, a_2, a_3)$ with $\text{g.c.d.}(a_0, a_1, a_2, a_3, d) = 1$. Let us give a version that works well without this assumption:
Corollary 3. Suppose the \( (\mathbb{Z}/d) \)-module

\[
\text{Ker}(\lambda) = \{ a = (a_0, a_1, a_2, a_3) \in \mathcal{A}_d^2 - \mathcal{B}_d^2 \mid (a_0, a_1, a_2, a_3) A = (0, 0, 0, 0) \}
\]

is generated by a single element \( \alpha \), hence is cyclic of some order \( d' \), with \( d'|d \). Let \( d'' = d/d' \) and choose \( \beta \in \mathcal{G}_d^2 \) with \( \alpha = d'' \beta \). Then

\[
\lambda(X_A) = \sum_{e|d''} \varphi(e) = \sum_{\beta \equiv (\bmod e) \in \mathcal{A}_e^2 - \mathcal{B}_e^2} \alpha \varphi(e).
\]

Proof. For \( e|d' \) there are exactly \( \varphi(e) \) elements in \( \text{Ker}(\lambda) \) that are of order \( e \), these can be written as \( c_i \cdot \alpha \) with \( d''/(c_i, d'') = e \) for \( i = 1, \ldots, \varphi(e) \). Now for any \( \gamma \in \mathcal{G}_d^2 \) and \( t \in (\mathbb{Z}/d) \) we have \( t \cdot \gamma \in \mathcal{A}_d^2 - \mathcal{B}_d^2 \) iff \( \gamma \equiv \gamma (\bmod f) \) where \( f := d/(d', t) \). It follows that the \( \varphi(e) \) elements \( c_i \cdot \alpha \) are either all in \( \mathcal{A}_d^2 - \mathcal{B}_d^2 \) or none of them is, the first possibility being the case precisely if \( \beta \equiv (\bmod e) \in \mathcal{A}_e^2 - \mathcal{B}_e^2 \) or equivalent: \( \alpha \equiv (\bmod d''e) \in \mathcal{A}_{d''e}^2 - \mathcal{B}_{d''e}^2 \). \( \square \)

Remark. In [13] the notion of capacity of the matrix \( A \) is introduced. One can verify that in the situation of the corollary the capacity of \( A \) equals \( d' \).

As an example we consider the Delsarte surface \( X_A \) with matrix

\[
A = \begin{pmatrix}
6 & 0 & 0 & 0 \\
1 & 5 & 0 & 0 \\
0 & 1 & 5 & 0 \\
0 & 0 & 1 & 5
\end{pmatrix}.
\]

For this matrix we have \( d = 750 \) and it is easy to see that the \( (\mathbb{Z}/750) \)-module \( \text{Ker}(\lambda) \subset \mathcal{G}_{750}^2 \) is generated by the element \( \alpha = (6, -36, 180, -900) \), hence \( \text{Ker}(\lambda) \) is cyclic of order 125. According to the corollary we have

\[
\lambda(X_A) = \sum_{e|125} \varphi(e).
\]

By looking at the last coefficient, which must be nonzero modulo \( 6e \), we see that \( \alpha \equiv (\bmod 6e) \in \mathcal{A}_{6e}^2 - \mathcal{B}_{6e}^2 \) can only be if \( e = 125 \). For \( e = 125 \) we have \( \alpha \equiv (\bmod 6e) = (6, 714, 180, 600) (\bmod 750) \) so \( |\alpha| = 2 \), but \( 13 \in (\mathbb{Z}/750)^* \) and \( 13 \cdot \alpha = (78, 282, 90, 300) \), so \( |13 \cdot \alpha| = 1 \) and we conclude that \( \alpha \equiv (\bmod 750) \in \mathcal{A}_{750}^2 - \mathcal{B}_{750}^2 \). The result is that \( \lambda(X_A) = \varphi(125) = 100 \) hence \( \rho(X_A) - b_2(X_A) = 106 - 100 = 6 \).
2. Variations of Hodge structure

Let $X$ and $S$ be connected complex analytic varieties and $f : X \to S$ a smooth morphism with connected fibres that factorises via a closed embedding $i : X \hookrightarrow \mathbb{P}_S^r$ for some $r$. As explained in [5] we have, for fixed $k$, a family of Hodge structures $(R^i f_* \mathbb{Z}_X, \mathcal{F})$ over $S$ with as fibres the cohomology groups $H^k(X_s, \mathbb{Z})$. Also, the primitive cohomology groups $H^k_p(X_s, \mathbb{Z})$ glue together to give a variation of Hodge structure $(\mathcal{H}_x, \mathcal{F}, Q)$, where $Q$ is a polarization form. For $p + q = k$ and $s \in S$ there is a map

$$\sigma^{p,q} : T_s(S) \to \text{Hom}(H^{p,q}(X_s, \mathbb{C}), H^{p-1,q+1}(X_s, \mathbb{C})),$$

which is defined by letting $\sigma^{p,q}(\xi)$ be the map induced by taking cup-product with the Kodaira–Spencer class $\kappa(\xi) \in H^1(X_s, \Theta_{X_s})$. This map can also be described as follows:

Since the Gauss–Manin connection $\nabla$ satisfies the Leibniz rule and satisfies Griffiths’ transversality,

$$\nabla(\mathcal{F}^p(R^k f_* \mathbb{Z}_X \otimes \mathcal{O}_S)) \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}(R^k f_* \mathbb{Z}_X \otimes \mathcal{O}_S),$$

it induces a $\mathcal{O}_S$-linear map

$$\text{def}(\nabla) : \text{Gr}_S^p(R^k f_* \mathbb{Z}_X \otimes \mathcal{O}_S) \to \Omega^1_S \otimes \mathcal{O}_S \text{Gr}_S^{p-1}(R^k f_* \mathbb{Z}_X \otimes \mathcal{O}_S),$$

which can be identified with a map

$$\text{def}(\nabla) : R^q f_* \Omega^{p,q}_{X/S} \to \Omega^1_S \otimes \mathcal{O}_S R^{p-1}f_* \Omega^{p-1}_{X/S}.$$

For $s \in S$ we take the induced map on the fibres and using the fact that $\Omega^1_{S,s} \otimes \mathbb{C} \cong T_S(S)^\vee$ we obtain the map $\sigma^{p,q}$. That this indeed describes the same map is proved in [5].

As in [2, Section 3(a)], we define

$$H_{i,i}^{p,q}(X_s, \mathbb{C}) := \{ h \in H^{p,q}(X_s, \mathbb{C}) \mid \sigma^{p,q}(\xi)(h) = 0 \text{ for all } \xi \in T_S(S) \}.$$

The elements in $H_{i,i}^{p,q}(X_s, \mathbb{C})$ are called the infinitesimally fixed classes. We are interested in the spaces of Hodge classes of the fibres $X_s$, so let us denote

$$\mathcal{B}^{p}_0 := H^{2p}(X_s, \mathbb{Q}) \cap H^{p,p}(X_s, \mathbb{C}).$$

Now there exists an at most countable collection of irreducible subvarieties $\Sigma_i \subseteq S$ such that $\dim_{\mathbb{Q}}(\mathcal{B}^{p}_0(X_s))$ is constant and minimal on $S - \bigcup \Sigma_i$. If $h_s \in \mathcal{B}^{p}_0(X_s)$ for some $s \in S - \bigcup \Sigma_i$ and if $h$ is a horizontal extension of $h_s$ on some open
neighbourhood $U$ of $s$, then $h_t \in \mathcal{B}_Q^p(X_t)$ for all $t \in U$. In particular, if we define $\mathcal{D}_Q^p$ to be the largest local subsystem of $R^2f_*Q$ having the property that $\mathcal{D}_Q^p \subset \mathcal{B}_Q^p(X_s)$ for all $s \in S$, then in fact we have that $\mathcal{D}_Q^p = \mathcal{B}_Q^p(X_s)$ for a point $s \in S - \bigcup_j \Sigma_j$.

The examples in the following section are mainly based on the following proposition:

**Proposition 4.** In the above situation, let $s \in S - \bigcup_j \Sigma_j$ and let $s_0 \in S$ be any point. Then

$$\dim_Q(\mathcal{B}_Q^p(X_s)) \leq \dim(\mathcal{B}_Q^p(X_{s_0}) \cap H_1^{p,p}(X_{s_0}, \mathbb{C})) .$$

**Proof.** To see this, let $\tilde{S} \to S$ be the universal covering of $S$, then it suffices to prove the statement for the pullback-family $X \times_S \tilde{S} \to \tilde{S}$, so we can assume that $S$ is simply connected. In this case the sheaf $R^2f_*\mathbb{Z}$ is globally constant on $S$ so if we have a Hodge class $h_t \in \mathcal{B}_Q^p(X_t)$ we can extend it to a global section $h$ of $R^2f_*\mathbb{Q}_e$. Since $s$ is in $S - \bigcup_j \Sigma_j$ the class $h_t \in H^{2p}(X_t, \mathbb{Q})$ is a Hodge class for all $t \in S$ so it remains to see that $h_{s_0}$ is infinitesimally fixed. This follows immediately from the second description of the map $\sigma^{p,q}$, since $h$ is a horizontal section. $\square$

Now suppose that our family $f : X \to S$ is a family of nonsingular hypersurfaces of degree $d$ in $\mathbb{P}_{\mathbb{C}}^{n+1}$. Let $s \in S$ and let $X_s$ be defined by the equation $F(T_0, \ldots, T_{n+1}) = 0$. For $G \in \mathbb{C}[T_0, \ldots, T_{n+1}]^d$ there is an infinitesimal deformation of $X$ given by the equation $F + \varepsilon G = 0$; sending $G$ to the class of this infinitesimal deformation in $H^1(X_s, \Theta_X)$ gives a linear map $\mathbb{C}[T_0, \ldots, T_{n+1}]^d \to H^1(X_s, \Theta_X)$. The image of this map will be denoted $H^1(X_s, \Theta_X)_o$, the kernel is precisely $\text{Jac}(F)$ so there is an isomorphism $\mu : H^1(\mathbb{P}_{\mathbb{C}}^{n+1}) \to H^1(X_s, \Theta_X)_o$. For the next lemma we refer to [3, especially p. 70].

**Lemma 5.** The following diagram commutes:

\[
\begin{array}{ccc}
\mathfrak{N}^d \times \mathfrak{N}^{(p)} & \xrightarrow{\text{multiplication}} & \mathfrak{N}^{(p+1)} \\
\mu_n \times \kappa_{(p)} \downarrow & & \downarrow \lambda_{(p+1)} \\
H^1(X, \Theta_X)_o \times H^{n-p-p}_o & \xrightarrow{\text{cup-product}} & H^{n-p-1-p+1}_o
\end{array}
\]

3. **Examples**

3.1. Fix a primitive 5th root of unity $\zeta_5 \in \mathbb{C}$. Define $\phi : \mathbb{P}_{\mathbb{C}}^3 \to \mathbb{P}_{\mathbb{C}}^3$ by

$$\phi : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \zeta_5 x_1 : \zeta_5^2 x_2 : \zeta_5^3 x_3) ;$$

then $\phi$, hence also $\Gamma := \{1, \phi, \phi^2, \phi^3, \phi^4\} \cong (\mathbb{Z}/5)$, acts proper and free on $X_2$. 

The quotient \( Y := \Gamma \backslash X^3 \) is a nonsingular projective surface; it is an example of a Godeaux surface with \( \pi_1(Y) \cong \Gamma \).

Let us consider the family of all nonsingular hypersurfaces of degree 5 in \( \mathbb{P}^3 \) for which \( \varphi \) is an automorphism. Any such surface is isomorphic to a surface defined by an equation

\[
F_{s_1, \ldots, s_8}(T_0, \ldots, T_3) = 0,
\]

with

\[
F_{s_1, \ldots, s_8}(T_0, \ldots, T_3) = T_0^5 + T_1^5 + T_2^5 + T_3^5 + s_1 \cdot T_0 T_1 T_2 T_3 + s_2 \cdot T_0 T_1^2 T_2 T_3 + s_3 \cdot T_0 T_1 T_2^2 + s_4 \cdot T_0 T_1 T_3^2 + s_5 \cdot T_0 T_2 T_3^2 + s_6 \cdot T_0^2 T_1 T_2^2 + s_7 \cdot T_0^2 T_1 T_3^2 + s_8 \cdot T_0^2 T_2 T_3^2,
\]

and where \( s = (s_1, \ldots, s_8) \) can be taken in a Zariski-open subset \( S = \mathbb{A}^8 - \Sigma \) of \( \mathbb{A}^8 \). In this way we obtain a family of nonsingular surfaces \( f : X \rightarrow S \) satisfying the conditions mentioned in Section 2. For any \( s \in S \) the group \( \Gamma \) acts fixedpoint-free on \( X_s \) and therefore we have a nonsingular projective quotient \( Y_s := \Gamma \backslash X_s \).

**Lemma 6.** The Hodge numbers of \( X_s \) and \( Y_s \) are given by:

\[
h^{0,0}(X_s) = h^{2,2}(X_s) = 1;\]

\[
h^{2,0}(X_s) = h^{0,2}(X_s) = 4;\]

\[
h^{1,1}(X_s) = 45;\]

\[
h^{2,0}(Y_s) = h^{0,2}(Y_s) = 0;\]

\[
h^{1,1}(Y_s) = 9;\]

all other Hodge numbers are zero.

**Proof.** The Hodge numbers do not depend on \( s \) so we can take \( s = 0 \). The computation of the Hodge numbers of \( X_0 \) is more or less standard, we can use for example the description of the \( H'(X, \mathbb{C}) \) in Section 1 and the remark that \( \dim(\Omega^1) = 4 \) and \( \dim(\Omega^6) = 44 \). For the computation of the \( h^{p,q}(Y_0) \) we use the fact that \( H'(Y_0, \mathbb{Q}) \) is isomorphic as a \( \mathbb{Q} \)-Hodge structure to \( H'(X_0, \mathbb{Q})' \) and if \( p + q = 2 \) we also use Theorem 1, (i) and (ii). Thus, \( H^{2-q,q}(Y_0, \mathbb{Q})' \) is isomorphic to the direct sum of the spaces \( V(\alpha) \), taken over the set of all \( \alpha = (a_0, a_1, a_2, a_3) \in \mathbb{A}_S^4 \) for which \( |a| = q + 1 \) and \( a_1 + 2a_2 + 3a_3 \equiv 0 \pmod{5} \). It is an easy verification that this set is empty if \( q = 0 \) or \( q = 2 \) and has eight elements if \( q = 1 \), hence the lemma.  \( \Box \)

From this lemma we immediately obtain a lower bound for the Picard number of the surfaces \( X_s \): since the \( \mathbb{Q} \)-Hodge structure \( H^2(X_s, \mathbb{Q}) \) can be regarded as a \( \mathbb{Q} \)-sub-Hodge structure of \( H^2(X, \mathbb{Q}) \) we have \( \rho(X_s) \geq 9 \) for any \( s \in S \) and therefore also \( \rho_{\text{gen}} \geq 9 \) if \( \rho_{\text{gen}} \) is the generic Picard number of the family we are considering.

We are going to prove that the generic Picard number in the family \( X/S \) equals 9. To do this we apply Proposition 4 where we take for the point \( s_0 \) the point \( s_0 = (0, \ldots, 0) \in S \), corresponding to the fibre \( X_0 = X_0^3 \). We know that the decomposition into spaces \( V(\alpha) \) is already defined over \( \mathbb{Q}(\xi) \), so to determine \( \dim_{\mathbb{Q}}(\mathcal{H}_Q \cap H^1_{\text{et}}(X_0, \mathbb{C})) \) we have to take the action of the Galois group
Gal(Q(ξ₅)/Q) into consideration. We choose an isomorphism (Z/5)* \rightarrow Gal(Q(ξ₅)/Q) by sending \( t \in (\mathbb{Z}/5)* \) to the element \( \gamma_t \) of Gal(Q(ξ₅)/Q) for which \( \gamma_t(ξ₅) = ξ₅_t \). The group (Z/5)* acts on \( \mathcal{O}_5^2 \) by coordinate-wise multiplication and the action of (Z/5)* on \( H^1(X_5^2, Q(ξ₅)) \) is such that \( t \in (\mathbb{Z}/5)* \) maps a space \( V_{\mathcal{O}(ξ₅)}(t \cdot α) \) into the space \( V_{\mathcal{O}(ξ₅)}(t \cdot α) \). Given \( α \in \mathcal{O}_5 \) there is a unique 4-tuple \( a \in \{1, 2, 3, 4\}^4 \) that represents \( α \) and if we put

\[
T^{(α)} := T_0^{a_0-1} \cdot T_1^{a_1-1} \cdot T_2^{a_2-1} \cdot T_3^{a_3-1},
\]

\[
\overline{T^{(α)}} := T^{(α)} \mod \text{Jac}(T_0^5 + T_1^5 + T_2^5 + T_3^5),
\]

then \( \overline{T^{(α)}} \) is a well-defined element of \( H^1(\{α\}^{-1}) \) and \( ω_α := \text{Res}_{X_5}(Ω_{T_0}) \) is an element of \( V(α) \).

The claim that the generic Picard number in the family \( X/S \) equals 9 will follow from the next statement: Let

\[
\mathcal{I} := \{ α \in \mathcal{O}_5^2 \mid \text{ω}_{mα} \text{ is infinitesimally fixed for all } t \in (\mathbb{Z}/5)* \};
\]

then

\[
\dim_{\mathbb{Q}}(\mathcal{O}_{\mathcal{O}_5}^1 \cap H_{1,1}^1(X_5, \mathbb{C})) = \#\mathcal{I}.
\] (1)

In fact, since the class \( u^p \) (cf. Section 1) clearly is infinitesimally fixed it follows from (1) that

\[
\dim_{\mathbb{Q}}(\mathcal{O}_{\mathcal{O}_5}^1 \cap H_{1,1}^1(X_5, \mathbb{C})) = 1 + \#\mathcal{I}.
\]

Now, given \( α \in \mathcal{O}_5^2 \), it is very easy to calculate whether \( ω_α \) is infinitesimally fixed or not (compare the definition of \( F_{s_1, \ldots, s_8} \) and Lemma 5): First we remark that under the isomorphism \( H^1(\{α\}^{-1}) \rightarrow H_{1,1}^1(X^2_5, \mathbb{C}) \) the class \( ω_α \) corresponds to a multiple of the monomial \( T^{(α)} \). Then compute the eight monomials \( T^{(α)} \cdot T_0^5T_1^5T_2^5 \), \( T^{(α)} \cdot T_0T_1T_2 \), \( T^{(α)} \cdot T_0T_1^5T_3 \), \( T^{(α)} \cdot T_0^5T_1T_3 \), \( T^{(α)} \cdot T_0T_1^5T_3 \), \( T^{(α)} \cdot T_0^5T_1^5T_3 \). If the eight induced elements of \( H^1 \) all are zero (that is, if in each of these eight monomials there is a \( T_i \) with exponential \( \geq 4 \)) then \( ω_α \) is infinitesimally fixed, otherwise it is not.

In the first example of Section 1 we already described the set \( \mathcal{O}_5^2 \); the remaining calculations are very simple and we find that

\[
\mathcal{I} = \{(1, 4, 4, 1), (2, 3, 3, 2), (3, 2, 2, 3), (4, 1, 1, 4),
(1, 2, 3, 4), (2, 4, 1, 3), (3, 1, 4, 2), (4, 3, 2, 1)\}.
\]

Together with Proposition 4 and the fact that \( \rho_{\text{gen}} \geq 9 \) it follows that the generic Picard number equals 9. This confirms Shioda’s guess stated in [9, p. 316].

Now let us prove (1): Because the spaces \( V(α) \) for \( α \in \mathcal{O}_5 \) are one-dimensional
we can assume that the classes $\omega_a$ are normalised in such a way that $\omega_a \in H^2(X_5^2, \mathbb{Q}(\zeta_5))$ and that $t \cdot \omega_a = \omega_{ta}$ for all $t \in (\mathbb{Z}/5)^*$. We can split the set $\mathcal{B}_5$ into orbits under the action of $(\mathbb{Z}/5)^*$. Any orbit consists of four elements and we will take a fixed order in each orbit. Now take an orbit $\mathcal{O} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The elements

$$\eta_j(\mathcal{O}) := \zeta_5^j \cdot \omega_{\alpha_1} + \zeta_5^{2j} \cdot \omega_{\alpha_2} + \zeta_5^{3j} \cdot \omega_{\alpha_3} + \zeta_5^{4j} \cdot \omega_{\alpha_4}$$

are in $\mathcal{D}_{\mathbb{Q},0}(X_5^2)$ and the collection

$$\cup_{\text{orbit of } (\mathbb{Z}/5)^*} \{\eta_0(\mathcal{O}), \eta_1(\mathcal{O}), \eta_2(\mathcal{O}), \eta_3(\mathcal{O})\}$$

is a $\mathbb{Q}$-basis for $\mathcal{D}^1_{\mathbb{Q},0}(X_5^2)$. Statement (1) now follows if we remark that a linear combination $\sum_{i=1}^N \lambda_i \omega_{\alpha_i}$ (all $\alpha_i$ different elements of $\mathcal{B}_5$, all $\lambda_i \in \mathbb{C}^*$) is infinitesimally fixed if and only if all $\omega_{\alpha_i}$ are infinitesimally fixed. In fact, if $M$ is one of the eight monomials $T_0^5 T_1 T_2, \ldots, T_5^5 T_0 T_1 T_2$ then the monomials $T^{(a)} M$ that we have to compute are linear independent in $\mathbb{C}[T_0, \ldots, T_{n+1}]$. This completes the proof that the generic Picard number in the family $X/S$ equals 9.

3.2. For the second example we take the restriction of the family $X/S$ to the base space $R \subseteq S$ defined by $s_1 = s_3 = s_4 = s_5 = s_6 = s_7 = s_8 = 0$. This gives a 2-parameter family $g : X_R \rightarrow R$ of nonsingular surfaces defined by the equation

$$T_0^5 + T_1^5 + T_2^5 + T_3^5 + r_0 T_0 T_1 T_2 + r_1 T_0^2 T_1 T_2 = 0.$$ 

There is an involution $\iota$ defined by $(T_0 : T_1 : T_2 : T_3) \mapsto (T_2 : T_1 : T_0 : T_3)$ and the restriction of $\iota$ to a fibre $X_0$ will be denoted by $\iota_0$. Let $\mathcal{D}^1_{\mathbb{Q}}(X/S)$ be the largest local subsystem of $\mathcal{R}^2T_X \mathbb{Q}_X$ that is of type $(1, 1)$ at every point of $S$ and similarly let $\mathcal{D}^1_{\mathbb{Q}}(X_R/R)$ be the largest local subsystem of $\mathcal{R}^2T_{X_R} \mathbb{Q}_{X_R}$ that is of type $(1, 1)$ at every point of $R$. Clearly $\mathcal{D}^1_{\mathbb{Q}}(X/S)|_R \subseteq \mathcal{D}^1_{\mathbb{Q}}(X_R/R)$. On the other hand, it is clear that $\mathcal{D}^1_{\mathbb{Q}}(X_R/R)$ is stable under the action of $\iota_0$ on the local system $\mathcal{R}^2T_{X_R} \mathbb{Q}_{X_R}$.

Again we consider the fibre $X_0^2 = X_0$, we proved in (3.1) that

$$\mathcal{D}^1_{\mathbb{Q}}(X/S)|_0 \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C} \cdot u_0 \otimes \bigoplus_{\alpha \in \mathcal{J}} V(\alpha)$$

where $u_0$ is the class of a hyperplane section of $X_0^2$. The action of $\iota$ on $H^2(X_0^2, \mathbb{C})$ is described by $u_0 \mapsto u_0$ and $\omega(\omega_{u_0, a_1, a_2, a_3}) \mapsto -\omega(\omega_{u_0, a_1, a_2, a_3})$ (remark that $\omega_0(\Omega) = -\Omega$). From this it follows that

$$\mathcal{D}^1_{\mathbb{Q}}(X_R/R)|_0 \otimes_{\mathbb{Q}} \mathbb{C} \subseteq \mathbb{C} \cdot u_0 \otimes \bigoplus_{\alpha \in \mathcal{J}} V(\alpha)$$

(2)
We claim that there actually holds equality in (2); this would show that the
generic Picard number in the family $X_n/R$ is 17. As in 3.1 we see that to prove
this statement it suffices to show that

$$\mathcal{J} = \{ \alpha \in B_5^+ : \omega_\alpha \text{ is infinitesimally fixed for all } t \in (\mathbb{Z}/5)^* \}.$$  

'infinitesimally fixed' in this case being relative to the family $X_n/R$ of course.
Again it is very easy to calculate whether a class $\omega_\alpha$ is infinitesimally fixed or not;
in this case we only have to calculate the two polynomials $T^{(n)}_1 T^{(n)}_2$ and
$T^{(n)}_1 T^{(n)}_5$. Sorted into orbits of $(\mathbb{Z}/5)^*$ in $B_5^+$ we find the following:

<table>
<thead>
<tr>
<th>Orbit</th>
<th>$\alpha$</th>
<th>Infinitesimally Fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 1, 4, 4), (2, 2, 3, 3), (3, 3, 2, 2), (4, 4, 1, 1)</td>
<td>i.f.</td>
</tr>
<tr>
<td>2</td>
<td>(4, 1, 4, 1), (3, 2, 2, 3), (2, 3, 3, 2), (1, 4, 1, 1)</td>
<td>i.f.</td>
</tr>
<tr>
<td>3</td>
<td>(1, 2, 3, 4), (2, 4, 1, 3), (3, 1, 4, 2), (4, 3, 2, 1)</td>
<td>i.f.</td>
</tr>
<tr>
<td>4</td>
<td>(3, 2, 1, 4), (1, 4, 2, 3), (4, 1, 3, 2), (2, 3, 4, 1)</td>
<td>i.f.</td>
</tr>
<tr>
<td>5</td>
<td>(1, 2, 4, 3), (2, 4, 3, 1), (3, 1, 2, 4), (4, 3, 1, 2)</td>
<td>i.f.</td>
</tr>
<tr>
<td>6</td>
<td>(4, 2, 1, 3), (3, 4, 2, 1), (2, 1, 3, 4), (1, 3, 4, 2)</td>
<td>not i.f.</td>
</tr>
<tr>
<td>7</td>
<td>(1, 3, 2, 4), (2, 1, 4, 3), (3, 4, 1, 2), (4, 2, 3, 1)</td>
<td>not i.f.</td>
</tr>
<tr>
<td>8</td>
<td>(2, 3, 1, 4), (4, 1, 2, 3), (1, 4, 3, 2), (3, 2, 4, 1)</td>
<td>i.f.</td>
</tr>
<tr>
<td>9</td>
<td>(1, 4, 1, 4), (2, 3, 2, 3), (3, 2, 3, 2), (4, 1, 4, 1)</td>
<td>i.f.</td>
</tr>
</tbody>
</table>

This proves that

$$(\mathcal{B}_2^+) = C \cdot \omega_\alpha \oplus \bigoplus_{\alpha \in \mathcal{J}} V(\alpha)$$

and therefore the generic Picard number $\rho_{gen}(X_n/R)$ in this family is 17, as
claimed.

3.3. The locus of fixed points of the involution $\iota$ on $X_n$ is the union of the
hyperplane section $H_\iota$ of $X_n$ defined by the equation $T_1 = T_2$ and the single point
$P_\iota = (1 : 0 : -1 : 0)$. If we blow up the points $P_\iota$ then we obtain a family
$g : \tilde{X}_n \to R$ and the involution $\iota$ lifts to an involution $\tilde{\iota}$ on $\tilde{X}_n$. The fixedpoint-locus
of $\tilde{\iota}$ is $\tilde{H}_\iota \cup E_\iota$, here $\tilde{H}_\iota$ is the strict transform of $H_\iota$, and $E_\iota$ is the exceptional
divisor on $\tilde{X}_n$. For all $r \in R$ for which $H_\iota$ is nonsingular the quotient $Y_r := X_n/\tilde{\iota}$ is again a nonsingular projective surface. Remark that the divisor class $[\tilde{H}_\iota + E_\iota]$ in
$\text{Pic}(\tilde{X}_n)$ is equal to the canonical divisor class $[\mathcal{K}_{\tilde{X}_n}]$. If $\pi : \tilde{X}_n \to Y_r$ is the
projection, $\mathcal{R}$ the ramification divisor and $[\mathcal{K}_{Y_r}]$ the canonical class on $Y_r$,
then it follows from the Hurwitz formula $\pi^*[\mathcal{K}_{Y_r}] = [\mathcal{K}_{\tilde{X}_n} - \mathcal{R}]$, that $\pi^*[\mathcal{K}_{Y_r}] = 0$, hence
$[2\mathcal{K}_{Y_r}] = \pi_* \pi^*[\mathcal{K}_{Y_r}] = 0$. We conclude that the surface $Y_r$ is either a K3-surface or
The Picard number of certain algebraic surfaces

...Marchuk's surface. Because the class Res_\chi(\Omega_Y) - Res_\chi(\Omega_X) is an \epsilon_i-invariant in H^{2,0}(X, \mathbb{C}) it follows that H^{2,0}(Y, \mathbb{C}) \neq 0 so Y must be a K3-surface. If U \subset R is the open part of R such that H_r is nonsingular for all r \in U then we obtain a 2-parameter family h : Y \to U of K3-surfaces.

For r \in R there is an isomorphism of \mathbb{Q}-Hodge structures

\[ \pi^* : H^2(Y, \mathbb{Q}) \cong H^2(\tilde{X}, \mathbb{Q}) \subset H^2(\tilde{X}, \mathbb{Q}) \]

and this isomorphism induces a bijection between the spaces of rational infinitesimally fixed classes H^2_{i,t}(Y, \mathbb{Q}) and H^2_{i,t}(\tilde{X}, \mathbb{Q}). Let \epsilon_r be the class of \text{E}_r. Since \epsilon_r and u_\epsilon are clearly \nu-invariant we find that

\[ H^2_{i,t}(Y, \mathbb{Q}) = \mathbb{Q} \cdot u_\epsilon \oplus \mathbb{Q} \cdot \epsilon_r \oplus H^2_{i,t}(X, \mathbb{Q})^* \]

In a similar way as before we define local systems \mathcal{L}_0(\tilde{X}/U) = \mathbb{Q} \cdot \nu \oplus \mathcal{L}_0(X/R) and \mathcal{L}_0(Y/U). Then \{1, \iota\} acts on \mathcal{L}_0(\tilde{X}/U) and \mathcal{L}_0(Y/U) = \mathcal{L}_0(\tilde{X}/U)^*.

We already know that

\[ (\mathcal{L}_0^1(X/R)_0 \otimes \mathbb{Q}) = \mathbb{C} \cdot u_0 \oplus \bigoplus_{\alpha \in \mathcal{V}} V(\alpha). \]

Let us look again at the table in 3.2. Corresponding to an orbit \text{C}_i = \text{orbit } i - \alpha_1, \alpha_2, \alpha_3, \alpha_4 there is a space \text{V}(i) := V(\alpha_i) \oplus V(\alpha_2) \oplus V(\alpha_3) \oplus V(\alpha_4) and in 3.1 we described a \mathbb{Q}-basis \{\eta_0, \eta_1, \eta_2, \eta_3\} for \text{V}(i). The numbering in the table is chosen in such a way that \iota interchanges \text{V}(1) and \text{V}(2), \text{V}(3) and \text{V}(4), \text{V}(5) and \text{V}(6), \text{V}(7) and \text{V}(8) and acts as -1 on \text{V}(9). We can assume that the basis elements \eta_j(i) are numbered and chosen such that \iota(\eta_j(i)) = -\eta_j(i + 1) for \j \in \{0, 1, 2, 3\} and \i \in \{1, 3, 5, 7\}. Then a \mathbb{Q}-basis of \mathcal{L}_0^1(\tilde{X}_0)^* is formed by \text{u}_0 and \text{e}_0 together with the 16 elements \eta_j(i) - \eta_j(i + 1), i \in \{1, 3, 5, 7\}.

It follows that the K3-surface \text{Y}_0 has Picard number 18. Furthermore, since

\[ \mathcal{L}_0^1(X/U)_0 \otimes \mathbb{Q} \mathbb{C} = \mathbb{C} \cdot u_0 \oplus \bigoplus \text{V}(1) \oplus \bigoplus \text{V}(2) \oplus \bigoplus \text{V}(3) \oplus \bigoplus \text{V}(4), \]

it follows that a \mathbb{Q}-basis of \mathcal{L}_0^1(Y/U)_0 is given by \text{u}_0 and \text{e}_0 together with the eight elements \eta_0(1) - \eta_0(2), \eta_3(1) - \eta_3(2), \eta_0(3) - \eta_0(4), \eta_3(3) - \eta_3(4), so the dimension of the local system \mathcal{L}_0^1(Y/U) is 10. This proves that the generic Picard number in the 2-parameter family of K3-surfaces \text{Y}/\text{U} is equal to 10.

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References


