On 2-extendable abelian Cayley graphs

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Abstract

A graph $G$ is 2-extendable if any two independent edges of $G$ are contained in a perfect matching of $G$. A Cayley graph of even order over an abelian group is 2-extendable if and only if it is not isomorphic to any of the following circulant graphs:

(I) $\mathbb{Z}_{2n}(1,2n-1)$, $n \geq 3$;

(II) $\mathbb{Z}_{2n}(1,2,2n-1,2n-2)$, $n \geq 3$;

(III) $\mathbb{Z}_{4n}(1,4n-1,2n)$, $n \geq 2$;

(IV) $\mathbb{Z}_{4n+2}(2,4n,2n+1)$, $n \geq 1$; and

(V) $\mathbb{Z}_{4n+2}(1,4n+1,2n,2n+2)$, $n \geq 1$.

1. Introduction

Let $G$ be an (additive) group and $S$ a generating set of $G$ such that the identity element $0 \notin S$ and $-x \in S$ for each $x \in S$. The Cayley graph $G(S)$ over $G$ is defined by setting its vertex and edge sets to be, respectively,

$$V(G(S)) = G \quad \text{and} \quad E(G(S)) = \{xy | x, y \in G, -x + y \in S\}.$$

An edge $xy$ in $G(S)$ is said to be of type $a$ (or an $a$-edge) if $-x + y = a$ or $-a$. Hence if $xy$ is of type $a$, then either $y = x + a$ or $x = y + a$. For convenience, if $S = \{a_1, \ldots, a_n\}$, we shall denote $G(S)$ also by $G(a_1, \ldots, a_n)$. For each positive integer $n$ we shall denote by $\mathbb{Z}_n$ the additive group modulo $n$ under the binary operation $\oplus$ (i.e. for $a, b \in \mathbb{Z}_n$, $a \oplus b$ denotes the residue of $a + b$ modulo $n$). The cycle and the path of order $n$ will be denoted by $C_n$ and $P_n$, respectively. The product of two graphs $G$ and $H$ is defined to be the graph $G \square H$ with

$$V(G \square H) = V(G) \times V(H)$$
and

\[ E(G \sqcup H) = \{(g_1, h_1)(g_2, h_2) | g_1 = g_2, h_1h_2 \in E(H) \text{ or } h_1 = h_2, g_1g_2 \in E(G)\}. \]

If \( T \) is a nonempty subset of a group \( G \), then \( \langle T \rangle \) will denote the subgroup of \( G \) generated by \( T \).

For each \( a \in G \), we shall denote by \( \theta_a \) the automorphism of \( G(S) \) defined by \( \theta_a(x) = a + x \). It is well known that every Cayley graph is vertex transitive (\( x \) is mapped to \( y \) by the automorphism \( \theta_{y-x} \)) and, thus, regular of degree equal to \( |S| \).

For each positive integer \( k \), a graph \( \Gamma \) is said to be \( k \)-extendable if it contains \( k \) independent edges and any \( k \) independent edges of \( \Gamma \) can be extended to a perfect matching (i.e. a 1-factor) of \( \Gamma \). The concept of \( k \)-extendability seems to have its early roots in a paper of Hetyei [3] who studied it for bipartite graphs, and papers of Kotzig (see [7]) who used it to develop a decomposition theory for graphs with perfect matchings. A graph \( \Gamma \) is said to be bicritical if \( \Gamma - x - y \) has a perfect matching for any distinct vertices \( x \) and \( y \) of \( V(\Gamma) \), and to be a brick if \( \Gamma \) is a 3-connected bicritical graph. In 1972, Lovász [5] introduced the brick decomposition to give a clearer structure for all elementary graphs (a graph is called elementary if the set of its edges which lie in at least one perfect matching forms a connected subgraph). This decomposition has also turned out to be very useful in the study of the matching polyhedra ([6]).

Because of the connection between extendability and bricks (see [8]), much attention today continues to focus on the properties of \( k \)-extendable graphs. Little et al. [4] gave a characterization of 1-extendable graphs. Plummer (see, for example, [8–10]) studied the relations between \( k \)-extendability and other graph parameters (e.g. degree, connectivity, genus, etc.). Recently, Schrag and Cammack [11] and Yu [12] classified the 2-extendable generalized Petersen graphs. As every edge of a Cayley graph over an abelian group is contained in a Hamiltonian cycle (see, for example, [1] or [2]), every Cayley graph over an abelian group of even order is 1-extendable. Characterizing 2-extendability of Cayley graphs over abelian groups is the main object of this paper.

Throughout this paper, all Cayley graphs are assumed to be defined over abelian groups.

**Main Theorem.** Let \( \Gamma = G(S) \) be a Cayley graph over the abelian group \( G \) of even order. Then \( \Gamma \) is 2-extendable if and only if it is not isomorphic to any of the following graphs.

(I) \( Z_{2n}(1, 2n - 1), n \geq 3 \);

(II) \( Z_{2n}(1, 2, 2n - 1, 2n - 2), n \geq 3 \);

(III) \( Z_{4n}(1, 4n - 1, 2n), n \geq 2 \);

(IV) \( Z_{4n+2}(2, 4n, 2n + 1), n \geq 1 \); and

(V) \( Z_{4n+2}(1, 4n + 1, 2n, 2n + 2), n \geq 1 \).

Note that the graph in (I) is just an even cycle of length \( 2n \), whereas that in (IV) is isomorphic to \( C_{2n+1} \square P_2 \).
2. Basic lemmas

We need the following lemmas in the proof of the main theorem.

Lemma 1 (Chen and Quimpo [1]). Every Cayley graph of even order is 1-extendable.

Lemma 2. $C_{2n}$ is 2-extendable if and only if $n = 2$.

Proof. Clearly, $C_4$ is 2-extendable. On the other hand, assume that $n \geq 3$. Let $C_{2n} = v_1v_2 \ldots v_{2n}v_1$. Then it is easy to see that there is no perfect matching in $C_{2n}$ containing the edges $v_1v_2$ and $v_4v_5$. □

Let $m \geq 3$ and $n \geq 2$ be integers. Let $e_1 = (a,b)(c,d)$ and $e_2 = (u,v)(w,x)$ be independent edges in $C_m \boxplus P_n$. We say that $e_1$ and $e_2$ are perpendicular if either $(a = c$ and $v = x)$, or $(u = w$ and $b = d)$. Otherwise, $e_1$ and $e_2$ are said to be parallel.

Lemma 3. Let $m$ and $n$ be any positive integers with $mn$ even, $m \geq 4$ and $n \geq 2$. Then any two perpendicular (independent) edges $e_1$ and $e_2$ of $C_m \boxplus P_n$ can be extended to a perfect matching of $C_m \boxplus P_n$.

Proof. Let $C_m = 12 \ldots m1$ and $P_n = 12 \ldots n$. Let $e_1 = (a,b)(c,d)$ and $e_2 = (u,v)(w,x)$ be two perpendicular edges of $C_m \boxplus P_n$. Without loss of generality, assume that $a = c$, $v = x$, $d = b + 1$ and $w = u + 1$. Clearly $\{(a,b),(c,d)\} \cap \{(u,v),(w,x)\} = \emptyset$.

We first consider the case when $m$ is even. If $v = b$ or $d$ (say $d$), then let $M = \{e_1,e_2,(u,b)(w,b)\} \cup \{(g,b)(g,d) \mid g \in V(C_m) \setminus \{a,u,w\}\}$. Then $M$ is a set of independent edges containing $e_1$ and $e_2$ that can be extended to a perfect matching of $C_m \boxplus P_n$, since the subgraph of $C_m \boxplus P_n$ induced by the set of vertices not in $M$ can be partitioned into disjoint even cycles. On the other hand, if $v \neq b$ and $d$, then let $M = \{(g,b)(g,d) \mid g \in V(C_m)\} \cup N$, where $N$ is a perfect matching of the subgraph induced by $\{(g,v) \mid g \in V(C_m)\}$ containing $e_2$. Then $M$ is a set of independent edges containing $e_1$ and $e_2$ that can be extended to a perfect matching of $C_m \boxplus P_n$, for the same reason as above.

We next consider the case when $m$ is odd. Then $n$ must be even. Assume that $v = b$ or $d$ (say $d$). If $d$ is even, then let $M = \{e_1,e_2,(u,b)(w,b)\} \cup \{(g,b)(g,d) \mid g \in V(C_m) \setminus \{a,u,w\}\}$. Then $M$ is a set of independent edges containing $e_1$ and $e_2$ which can be extended to a perfect matching of $C_m \boxplus P_n$, since the subgraph of $C_m \boxplus P_n$ induced by the set (if $\neq \emptyset$) of vertices not in $M$ can be partitioned into subgraphs of the form $C_m \boxplus P_2$, each of which obviously has a perfect matching. On the other hand, if $d$ is odd, then choose a vertex $y \in V(C_m) \setminus \{a,u,w\}$ and let $M = \{e_1,e_2,(u,b)(w,b), (y,b-1)(y,b),(y,d)(y,d+1)\} \cup \{(g,b)(g,d) \mid g \neq a,u,w,y\}$. Then $M$ is a set of independent edges containing $e_1$ and $e_2$, which can be extended to a perfect matching of $C_m \boxplus P_n$, as the subgraph of $C_m \boxplus P_n$ induced by the set of vertices not in $M$ can be
partitioned into two even paths and subgraphs of the form $C_m \square P_2$, each of which obviously has a perfect matching.

Hence we need only consider the case that $v \neq b$ and $d$. Without loss of generality we may assume that $v > d$, if $v$ is odd, then let $H = \{(g,h) \in V(C_m \square P_n) \mid h < v\}$ and $K = \{(g,h) \in V(C_m \square P_n) \mid h \geq v\}$. Then, by an argument similar to that in the preceding paragraph, $H$ has a perfect matching $M_1$ containing $e_1$, whereas $K$ has a perfect matching $M_2$ containing $e_2$. So $M_1 \cup M_2$ will be a perfect matching of $C_m \square P_n$ containing $e_1$ and $e_2$. On the other hand, if $v$ is even and $v > d + 1$, then let $H = \{(g,h) \in V(C_m \square P_n) \mid h < v - 1\}$ and $K = \{(g,h) \in V(C_m \square P_n) \mid h \geq v - 1\}$. Then, $H$ has a perfect matching $M_1$ containing $e_1$, whereas $K$ has a perfect matching $M_2$ containing $e_2$. So $M_1 \cup M_2$ will be a perfect matching of $C_m \square P_n$ containing $e_1$ and $e_2$. Finally, if $v$ is even and $v = d + 1$, then we choose a vertex $y \in V(C_m) \{a,u,w\}$ adjacent to $u$ or $w$ (say $w$) and let $M = \{e_1,e_2,(y,b-1)(y,b),(y,d)(y,d+1)\} \cup \{(g,h)(g,d) \mid g \in V(C_m) \{a,y\}\}$. Then $M$ is a set of independent edges containing $e_1$ and $e_2$ which can be extended to a perfect matching of $C_m \square P_n$, since the subgraph of $C_m \square P_n$ induced by the set of vertices not in $M$ can be partitioned into two even paths and subgraphs of the form $C_m \square P_2$, each of which obviously has a perfect matching. 

**Lemma 4.** Let $G(S)$ be an abelian Cayley graph and $T$ be a nonempty subset of $S$ with $-T = T$. Then any perfect matching of $\langle T \rangle \langle T \rangle$ can be extended to a perfect matching of $G(S)$.

**Proof.** Since the subgraph induced by any coset of $\langle T \rangle$ in $G$ contains a spanning subgraph isomorphic to $\langle T \rangle \langle T \rangle$, $G(S)$ also contains a spanning subgraph which can be partitioned into a finite number of copies of $\langle T \rangle \langle T \rangle$. The result thus follows. 

**Lemma 5.** Let $\Gamma$ be a Cayley graph of even order. Then every two adjacent edges of $\Gamma$ lie on an even cycle of $\Gamma$.

**Proof.** Let $\Gamma = G(S)$ be a Cayley graph of even order. If $\Gamma$ is an even cycle then the result is trivially true. Hence, we may assume that $\Gamma$ is of degree $k$, where $k \geq 3$. Let $e_1$ and $e_2$ be two adjacent edges in $\Gamma$. We have the following cases to consider.

*Case 1:* $e_1$ and $e_2$ are of different types (say of type $a$ and type $b$, respectively).

In this case, we may assume without loss generality that $e_1 = 0a$ and $e_2 = 0b$. Then $e_1$ and $e_2$ are in the 4-cycle $0a(a + b)b0$.

*Case 2:* $e_1$ and $e_2$ are of the same type (say $a$).

In this case, we may assume without loss of generality that $e_1 = 0a$ and $e_2 = a(2a)$. If $a$ is of even order, then the two given edges will be in the even cycle generated by $a$. Hence we may assume that $a$ is of odd order. As $G$ is of even order, there must be another element $b \in S$, $b \notin \langle a \rangle$. Then, $e_1,e_2$ are in the 6-cycle $0a(2a)(2a + b)(a + b)b0$. 

□
Lemma 6. Let $\Gamma$ be a 1-extendable graph or a graph in which every two adjacent edges lie on an even cycle of $\Gamma$. Then $\Gamma \Box P_2$ is 2-extendable.

Proof. Let $P_2 = 01$. Let $e_1 = (a,b)(c,d)$ and $e_2 = (u,v)(w,x)$ be any two independent edges of $\Gamma \Box P_2$. We have the following cases to consider.

Case 1: $b \neq d$ and $v \neq x$.

In this case, $e_1$ and $e_2$ are contained in the following perfect matching of $\Gamma \Box P_2$:

$\{(g,0)(g,1) \mid g \in V(\Gamma)\}$.

Case 2: $b = d = v = x$.

In this case, we may assume without loss of generality that $b = d = v = x = 0$. Then $e_1$ and $e_2$ are contained in the following perfect matching of $\Gamma \Box P_2$:

$\{e_1, e_2, (a,0)(c,1)(u,1)(w,1)\} \cup \{(g,0)(g,1) \mid g \in V(\Gamma) \setminus \{a, u, c, w\}\}$.

Case 3: $b = d$ and $v \neq x$ (or vice versa).

In this case, assume that $b = d = 0$, say. Then $e_1$ and $e_2$ are contained in the following perfect matching of $\Gamma \Box P_2$:

$\{(e_1, (a,1)(c,1)) \cup \{(g,0)(g,1) \mid g \in V(\Gamma) \setminus \{a, c\}\}$.

Case 4: $b = d = v = x$.

Assume that $b = d = 0$ and $v = x = 1$, say. Then $ac, uw \in E(\Gamma)$. Let $e_3 = ac$ and $e_4 = uw$. We consider the following subcases.

Case 4.1: $\Gamma$ is a 1-extendable graph.

In this case, the edges $e_3$ and $e_4$ can each be extended to perfect matchings $M_1$ and $M_2$ of $\Gamma$, respectively. Let

$M = \{(g,0)(h,0) \mid gh \in M_1\} \cup \{(g,1)(h,1) \mid gh \in M_2\}$.

Then $M$ is a perfect matching in $\Gamma \Box P_2$ which contains $e_1$ and $e_2$.

Case 4.2: $\Gamma$ is a graph in which every two adjacent edges lie on an even cycle of $\Gamma$.

If $e_3$ and $e_4$ are independent or the same, then $e_1$ and $e_2$ are contained in the perfect matching $\{e_1, e_2, (a,1)(c,1)(u,0)(w,0)\} \cup \{(g,0)(g,1) \mid g \in V(\Gamma) \setminus \{a,c,u,w\}\}$ or $\{e_1, e_2\} \cup \{(g,0)(g,1) \mid g \in V(\Gamma) \setminus \{a,c\}\}$, respectively.

If $e_3$ and $e_4$ are adjacent edges of $\Gamma$, then, by hypothesis, they are contained in an even cycle $C$ of $\Gamma$. Let $M_1$ and $M_2$ be perfect matchings of $C$ containing $e_3$ and $e_4$ respectively. Then $e_1$ and $e_2$ are contained in the following perfect matching of $\Gamma \Box P_2$:

$\{(g,0)(h,0) \mid gh \in M_1\} \cup \{(g,1)(h,1) \mid gh \in M_2\} \cup \{(g,0)(g,1) \mid g \in V(\Gamma) \setminus V(C)\}$.

The proof of Lemma 6 is now complete. \qed

Corollary 1. Let $e_1 = (a,b)(c,d)$ and $e_2 = (u,v)(w,x)$ be two independent edges of $\Gamma = C_n \Box P_2$, where $n \geq 3$. Then, except when $n$ is odd, $b = d \neq v = x$ and $|\{a,c\} \cap \{u,w\}| = 1$, there exists a perfect matching in $\Gamma$ containing $e_1$ and $e_2$. 
Proof. If \( n \) is even, then every two adjacent edges of \( C_n \) lie on the even cycle \( C_n \). So we are done. If \( n \) is odd, then by the argument in the proof of Lemma 6, it is easy to see that \( e_1 \) and \( e_2 \) can be extended to a perfect matching of \( \Gamma \) except in the last case of Case 4.2, i.e. when \( b = d = v = x \) and \( |\{a,c\} \cap \{u,w\}| = 1 \). \( \square \)

From this corollary, a second corollary follows immediately.

**Corollary 2.** \( C_n \square P_2 \) is 2-extendable if and only if \( n \) is even.

Combining Lemmas 5 and 6, we obtain a third corollary.

**Corollary 3.** If \( \Gamma \) is a Cayley graph of even order, then \( \Gamma \square P_2 \) is 2-extendable.

**Lemma 7.** \( C_{2n} \square P_m \) is 2-extendable, for \( n \geq 2 \) and \( m \geq 2 \).

**Proof.** Let \( C_{2n} = 12\ldots(2n)1 \) and \( P_m = 12\ldots m \). Let \( e_1 = (a,b)(c,d) \) and \( e_2 = (u,v)(w,x) \) be any two independent edges of \( C_{2n} \square P_m \). By Lemma 3, we may assume that \( e_1 \) and \( e_2 \) are parallel. We have the following cases to consider.

**Case 1:** \( b = d \) (and hence \( v = x \)).

If \( b \neq v \), then \( e_1 \) and \( e_2 \) lie in two distinct 2n-cycle \( C_{2n} \square \{b\} \) and \( C_{2n} \square \{v\} \), each of which has a perfect matching (say \( M_1, M_2 \)) containing \( e_1 \) and \( e_2 \), respectively. Clearly, \( M_1 \cup M_2 \) can be extended to a perfect matching of \( C_{2n} \square P_m \), since the subgraph induced by vertices not in \( M_1 \cup M_2 \) can be partitioned into disjoint even cycles and so contains a perfect matching. On the other hand, if \( b = v \), then we choose a vertex \( y \) in \( P_m \) adjacent to \( b \). Let \( M = \{e_1, e_2, (a,y)(c,y), (u,y)(w,y)\} \cup \{(g,b)(g,y) \mid g \in C_{2m}\{a,c,u,w\}\} \). Then \( M \) is a set of independent edges containing \( e_1 \) and \( e_2 \). As the subgraph induced by vertices not in \( M \) can be partitioned into disjoint even cycles, \( M \) can be extended to a perfect matching of \( C_{2n} \square P_m \).

**Case 2:** \( a = c \) (and hence \( u = w \)).

If \( \{b,d\} \cap \{v,x\} = \emptyset \) then let \( M = \{(g,b)(g,d) \mid g \in C_{2n}\} \cup \{(g,v)(g,x) \mid g \in C_{2n}\} \). Then \( M \) is a set of independent edges containing \( e_1 \) and \( e_2 \). As the subgraph induced by vertices not in \( M \) can be partitioned into disjoint even cycles, \( M \) can be extended to a perfect matching of \( C_{2n} \square P_m \). On the other hand, if \( \{b,d\} \cap \{v,x\} \neq \emptyset \), then either \( \{b,d\} = \{v,x\} \) or \( \{|\{b,d\} \cap \{v,x\}| = 1 \). In the first case, \( M = \{(g,b)(g,d) \mid g \in C_{2n}\} \) is obviously a set of independent edges containing \( e_1 \) and \( e_2 \) which can be extended to a perfect matching of \( C_{2n} \square P_m \). For the second case, without loss of generality, assume that \( v = d = b + 1 \) and \( x = v + 1 = b + 2 \). As \( C_{2n} \) is of even order at least 4, there exist distinct \( k, m \in V(C_{2n})\{a,u\} \) such that \( ak \) and \( um \) are independent edges in \( C_{2n} \) and \( V(C_{2n})\{a,u,k,m\} \) is a disjoint union of two even paths (possibly empty), and so has a perfect matching \( M \). Then \( e_1 \) and \( e_2 \) are contained in the following perfect
matching of the subgraph induced by \( V(C_{2n}) \setminus \{b, b + 1, b + 2\} \):

\[
\{e_1, e_2, (k, b)(k, b + 1), (u, b)(m, b), (m, b + 1)(m, b + 2), (a, b + 2)(k, b + 2)\}
\]

\[\cup \{(p, r)(q, r) | pq \in M, r = b, b + 1, b + 2\}.
\]

We can extend this to a perfect matching of \( C_{2n} \setminus P_m \).

Now that we know enough about the 2-extendability of the 'skeleton', \( C_n \setminus P_m \), of Cayley graphs, we shall proceed to study the 2-extendability of the Cayley graphs themselves.

**Lemma 8.** Let \( \Gamma \) be a Cayley graph of even order. Then any two independent edges of \( \Gamma \) of different types are contained in a perfect matching of \( \Gamma \).

**Proof.** Let \( \Gamma = G(S) \) where \( G \) is a finite abelian group of even order. Let \( e_1 = ab \) and \( e_2 = cd \) be edges of \( \Gamma \) of type \( s \) and \( t \), respectively, where \( s, t \in S \) and \( s \neq t \) or \( -t \). As \( \Gamma \) is vertex-transitive, we may assume that \( a = 0 \) and \( b = s \). We shall consider the following cases.

**Case 1:** \( s \) is even order \( 2n \) and \( t \notin \langle s \rangle \).

Suppose \( n \geq 2 \). Let \( H \) be the Cayley graph \( \langle s, t \rangle \langle s, -s, -t \rangle \). Then \( H \) has a spanning subgraph \( K \) isomorphic to \( C_{2n} \setminus P_m, m \geq 2 \), whose edge-set contains \( e_1 \). If \( e_2 \) is an edge of \( H \), then we may choose \( K \) so that \( e_1, e_2 \in E(K) \) and hence by Lemma 7, there is a perfect matching in \( E(H) \) containing \( e_1 \) and \( e_2 \), which can then be extended to a perfect matching of \( \Gamma \), by Lemma 4. On the other hand, if \( e_2 \) is not an edge of \( H \), then \( e_2 \in E(\langle 0 \rangle \langle H \rangle) \). There is a perfect matching \( M \) in \( H \) containing \( e_1 \) and a perfect matching \( M' \) in \( E(\langle 0 \rangle \langle H \rangle) \) containing \( e_2 \). Then \( M \cup M' \) can then be extended to a perfect matching of \( \Gamma \), as the set of vertices of \( \Gamma \) not in \( M \cup M' \) can be decomposed into a finite number of copies of \( H \). The case where \( n = 1 \) may be handled in a similar fashion.

**Case 2:** \( s \) is of even order \( 2n \) and \( t \in \langle s \rangle \), (say \( t = ks \)).

Let \( H \) be the Cayley graph \( \langle s \rangle \langle s, -s, -t \rangle \). In this case \( n \geq 2 \). If \( e_2 \) is not an edge of \( H \), then we can settle this case as in Case 1. Hence assume that \( e_2 \) is an edge of \( H \) with \( c = c's, d = d's \) and \( c' < d' \), say. We then have the following four subcases to consider. For each case, we find a matching \( M \) containing \( e_1 \) and \( e_2 \) such that the set of vertices of \( \Gamma \) not in \( M \cup M' \) can be decomposed into even paths and so \( e_1 \) and \( e_2 \) can be extended to a perfect matching of \( \Gamma \).

**Case 2.1:** If \( k \) is odd and \( c' \) is even, then let \( M = \{e_1, e_2\} \).

**Case 2.2:** If \( k \) is odd and \( c' \) is odd, then let \( M = \{e_1, e_2, ((c' + 1)s)((d' - 1)s), (c' + 1)s)((d' + 1)s))\).

**Case 2.3:** If \( k \) is even and \( c' \) is even, then let \( M = \{e_1, e_2, ((c' + 1)s)((d' + 1)s))\).

**Case 2.4:** If \( k \) is even and \( c' \) is odd, then let \( M = \{e_1, e_2, ((c' - 1)s)((d' - 1)s))\).

**Case 3:** \( s \) and \( t \) are of odd order.

Since \( S \) generates \( G \) which is of even order, one of the generating elements, say \( r \in S \), must have been even order. Hence \( r \notin V(H) \), where \( H \) is the Cayley graph
\( \langle s, t \rangle (s, t, -s, -t) \). Let \( K \) be the Cayley graph \( \langle r, s, t \rangle (r, -r, s, -s, t, -t) \). If \( e_2 \) is not an edge of \( K \), then as in Case 1, \( e_1 \) and \( e_2 \) can be extended to a perfect matching of \( \Gamma \), as \( \Gamma \) can be decomposed into a finite number of copies of \( K \), each of which is 1-extendable by Lemma 1. Hence, assume that \( e_2 \in E(K) \). Note that \( K \) can be decomposed into a finite number of copies of \( L \cong H \Box P_2 \), where \( V(L) = V(H) \cup \delta(H) \). If \( e_2 \) is not an edge of \( L \), then we can settle the case as before. Hence assume that \( e_2 \) is an edge of \( L \). We note that every two adjacent edges in \( H \) lie on an even cycle. By Lemma 6, there exists a perfect matching \( M \) in \( L \) which contains \( e_1 \) and \( e_2 \) and so can be extended to a perfect matching of \( \Gamma \). □

**Lemma 9.** The Cayley graph \( Z_{2n}(1, 2n - 1, n) \), \( n > 2 \), is 2-extendable if and only if \( n \) is odd.

**Proof.** If \( n \) is even, we let \( e_1 = 01 \) and \( e_2 = (n - 1)n \). Then there is not perfect matching in \( \Gamma \) containing \( e_1 \) and \( e_2 \).

On the other hand, assume that \( n \) is odd. Let \( e_1 = ab \) and \( e_2 = cd \) be two independent edges of \( Z_{2n}(1, 2n - 1, n) \). By Lemma 8, we may assume that they are of the same type. If they are of type \( n \), then they are contained in the perfect matching consisting of all edges of type \( n \). Thus, we need only to consider the case when they are of type 1. Without loss of generality, we may assume that \( a = 0, b = 1 \) and \( d = c + 1 \). If \( c \) is even, \( e_1 \) and \( e_2 \) are contained in the perfect matching consisting of all edges \( x(x + 1) \), where \( x = 0, 2, 4, \ldots, 2(n - 1) \). So, let \( c \) be odd. Then there exists an even integer \( y \in \{2, 3, \ldots, c - 1\} \), such that \( e_1 \) lies on the cycle \( C = 012 \ldots y (y + n)(y + n + 1) \ldots (2n - 1)0 \), whereas, \( e_2 \) lies on the path \( P = (y + 1)(y + 2) \ldots (y + n - 1) \). As \( C \) is an even cycle, it has a perfect matching \( M_1 \) containing \( e_1 \). Also, as \( P \) is of even order and \( y + 1 \) is odd, \( P \) also has a perfect matching \( M_2 \) containing \( e_2 \). Then \( M_1 \cup M_2 \) will be a perfect matching of \( Z_{2n}(1, 2n - 1, n) \), containing \( e_1 \) and \( e_2 \). □

**Lemma 10.** The Cayley graph \( Z_{4n+2}(1, 4n + 1, k, 4n + 2 - k) \), \( n \geq 1 \) and \( k < 2n + 1 \), is 2-extendable if and only if \( k \neq 1, 2, 2n \).

**Proof.** Let \( \Gamma = Z_{4n+2}(1, 4n + 1, k, 4n + 2 - k) \). If \( k = 1 \), then \( \Gamma \) is not 2-extendable, by Lemma 2. If \( k = 2 \), we let \( e_1 = 01 \) and \( e_2 = 34 \). Then there is no perfect matching in \( \Gamma \) containing \( e_1 \) and \( e_2 \). If \( k = 2n \), we let \( e_1 = 01 \) and \( e_2 = (2n + 1)(2n + 2) \). Then there is no perfect matching in \( \Gamma \) containing \( e_1 \) and \( e_2 \).

Conversely, assume that \( k \neq 1, 2, 2n \). Let \( e_1 = ab \) and \( e_2 = cd \) be two independent edges of \( \Gamma \), with \( a < b \) and \( c < d \). As \( \Gamma \) is vertex-transitive, we may assume that \( a = 0 \). By Lemma 8, we may also assume that \( e_1 \) and \( e_2 \) are of the same type. We consider the following two cases:

**Case 1:** \( k \) is odd.

If \( e_1 = 01 \), then by symmetry we may assume that \( d \leq 2n + 2 \). Assume first there \( c \) is even, the \( M = \{(2i, 2i + 1) | i = 0, 1, \ldots, 2n\} \) is a required perfect matching.
Assume next that \( c \) is odd, then \( e_2 \) is contained in the even cycle \( C = (c - 1)c(c + 1) \ldots (c - 1 + k)(c - 1) \) which has a perfect matching \( M_2 \) containing \( e_2 \) and \( e_1 \) is contained in the even path \( P = (c + k)(c + k + 1) \ldots (4n + 1)1012 \ldots (c - 2) \) which has a perfect matching \( M_1 \) containing \( e_1 \). Then \( M_1 \cup M_2 \) will be a perfect matching of \( \Gamma \) containing \( e_1 \) and \( e_2 \).

Next, let \( e_1 = 0k \). Then we have the following cases to consider.

Case 1.1: \( c \) is odd and \( c < b \).

Let \( M = \{e_1, e_2,(c + 1)(d + 1)\} \). The vertices of \( \Gamma \) not in \( M \) can be partitioned into even paths and so \( M \) can be extended to a perfect matching of \( \Gamma \).

Case 1.2: \( c \) is odd and \( c > b \).

Let \( M = \{e_1, e_2,(c - 1)(d - 1),(c + 1)(d + 1)\} \). Then as before \( M \) can be extended to a perfect matching of \( \Gamma \).

Case 1.3: \( c \) is even and \( c < b \).

Let \( M = \{e_1, e_2,(c - 1)(d - 1)\} \). Then \( M \) can be extended to a perfect matching of \( \Gamma \).

Case 1.4: \( c \) is even and \( c > b \).

Let \( M = \{e_1, e_2\} \). Then \( M \) can be extended to a perfect matching of \( \Gamma \). Thus Case 1 is dealt with. We now have Case 2.

Case 2: \( k \) is even.

If \( e_1 = 01 \), then \( d = c + 1 \) and by symmetry we may assume that \( d \leq 2n + 2 \). Assume first that \( c \) is even, then \( M = \{(2i,2i + 1) | i = 0, 1, \ldots, 2n\} \) is a required perfect matching. Assume next that \( c \) is odd, then let \( M = \{(c - 1)(c - 1 + k), cd, (d + 1)(d + 1 + k),(c + k)(d + k)\} \). Then \( M \) is a set of independent edges containing \( e_2 \), whereas the set of vertices of \( \Gamma \) not in \( M \) induces a subgraph which is the disjoint union of even paths with a perfect matching \( M' \) containing \( e_1 \). Then \( M \cup M' \) is a required perfect matching of \( \Gamma \).

Finally let \( e_1 = 0k \). We then have the following subcases to consider.

Case 2.1: \( c \) is odd and \( c < b \).

In this case, the set of vertices of \( \Gamma \setminus \{a, b, c, d\} \) can be partitioned into even paths and so \( e_1 \) and \( e_2 \) can be extended to a perfect matching of \( \Gamma \).

Case 2.2: \( c \) is odd and \( c > b \).

Then we have either \( c - b \geq 2 \) or \( 4n + 2 - d \geq 2 \), say the former. Let \( M = \{e_1, e_2,(c - 1)(d - 1), (b + 1)1\} \). Then the set of vertices of \( \Gamma \) not in \( M \) can be extended to a perfect matching of \( \Gamma \), for the same reason as that for Case 2.1.

Case 2.3: \( c \) is even and \( c < b \).

Let \( M = \{e_1, e_2,(c - 1)(d - 1), (c + 1)(d + 1)\} \). Then \( M \) can be extended to a perfect matching of \( \Gamma \), as in Case 2.1.

Case 2.4: \( c \) is even and \( c > b \).

Let \( M = \{e_1, e_2,(c + 1)(d + 1), 1(b + 1)\} \). Then \( M \) can be extended to a perfect matching of \( \Gamma \). \( \Box \)

**Lemma 11.** The Cayley graph \( \Gamma = Z_{4n}(1,4n - 1,k,4n-k) \), \( 1 \leq k \leq 2n \), \( n \geq 2 \), is 2-extendable if and only if \( k \neq 1,2 \) and \( 2n \).
**Proof.** If \( k = 1 \), then \( \Gamma \) is not 2-extendable, by Lemma 2. If \( k = 2 \), then \( \Gamma \) is not 2-extendable, as there is not perfect matching containing 01 and 34. If \( k = 2n \), then \( \Gamma \) is not 2-extendable, by Lemma 9.

Conversely, assume that \( k \) is different from 1, 2 and 2n. Let \( e_1 = ab \) and \( e_2 = cd \) be two independent edges of \( \Gamma \) with \( a < b \) and \( c < d \). As \( \Gamma \) is vertex-transitive, we may let \( a = 0 \). By Lemma 8, we may assume that they are of the same type \( t \). We have the following cases to consider.

**Case 1:** \( t = 1 \).

By symmetry, we may assume that \( d \leq 2n + 1 \). If \( c \) is even, then \( \{01, 23, \ldots, (4n - 2)(4n - 1)\} \) will be a required perfect matching. Assume that \( c \) is odd. If \( k \) is odd, let \( M = \{e_1, e_2, (c - 1)(c - 1 + k)\} \). Then vertices of \( \Gamma \) not in \( M \) can be partitioned into even paths and so \( M \) can be extended to a perfect matching of \( \Gamma \). On the other hand, if \( k \) is even, let \( M = \{e_1, e_2, (c - 1)(c - 1 + k), (d + 1)(d + 1 + k)\} \). Then, as before, \( M \) can be extended to a perfect matching of \( \Gamma \).

**Case 2:** \( t = k \).

In this case, we construct a matching \( M \) containing \( e_1 \) and \( e_2 \) for each of the following subcases, such that the vertices of \( \Gamma \) not in \( M \) can be partitioned into even paths, and therefore \( M \) can be extended to a perfect matching of \( \Gamma \). Let

\[
M = \begin{cases} 
\{e_1, e_2, (c + 1)(d + 1)\} & \text{if } k \text{ is odd, } c \text{ is odd and } c < b; \\
\{e_1, e_2, (c - 1)(d - 1)(c + 1)(d + 1)\} & \text{if } k \text{ is odd, } c \text{ is odd and } c > b; \\
\{e_1, e_2, (c - 1)(d - 1)\} & \text{if } k \text{ is odd, } c \text{ is even and } c < b; \\
\{e_1, e_2\} & \text{if } k \text{ is even, } c \text{ is odd and } c < b; \\
\{e_1, e_2, (c - 1)(d - 1)(b + 1)\} & \text{if } k \text{ is even, } c \text{ is odd and } c - b \geq 2; \\
\{e_1, e_2, (c + 1)(d + 1)(4n - 1)(b - 1)\} & \text{if } k \text{ is even, } c \text{ is odd and } c - b = 1; \\
\{e_1, e_2, (c - 1)(d - 1)(c + 1)(d + 1)\} & \text{if } k \text{ is even, } c \text{ is even and } c < b; \\
\{e_1, e_2, (c + 1)(d + 1)(b + 1)\} & \text{if } k \text{ is even, } c \text{ is even and } c > b.
\end{cases}
\]

It is easy to see that \( M \) is as required. The proof is now complete. \( \square \)

**Lemma 12.** The Cayley graph \( \Gamma = Z_{2\eta}(1, 2n - 1, 2, 2n - 2, n - 1, n + 1), n \geq 2 \) and \( n \neq 3 \), is 2-extendable.

**Proof.** Let \( e_1 = ab \) and \( e_2 = cd \) be two independent edges of \( \Gamma \) with \( a < b \) and \( c < d \). As \( \Gamma \) is vertex-transitive, we may assume that \( a = 0 \) and by Lemma 8, we may assume that \( e_1 \) and \( e_2 \) are of the same type \( t \).

**Case 1:** \( t = 1 \).

By symmetry, we may assume that \( d \leq n + 1 \). If \( n \) is even, then by Lemma 11, the spanning subgraph \( Z_{2\eta}(1, 2n - 1, n - 1, n + 1) \) of \( \Gamma \) is \( \epsilon \)-extendable and so \( e_1 \) and \( e_2 \) can be extended to perfect matching of \( \Gamma \). Hence we let \( n \geq 5 \) be odd. If \( c \) is even,
then $e_1$ and $e_2$ are contained in the perfect matching $\{01, 23, \ldots, (2n-2)(2n-1)\}$ of $\Gamma$. On the other hand, if $c$ is odd, let $M = \{e_1, e_2, (c+1)(c+3)\}$. Then $M$ can be extended to a perfect matching of $\Gamma$, as vertices of $\Gamma$ not in $M$ can be partitioned into even paths.

**Case 2: $t = 2$.**

By symmetry, we may assume that $d \leq n + 2$. If $c = 1$, then vertices of $\Gamma$ not on $e_1$ and $e_2$ induce an even path and so $\Gamma$ has a perfect matching containing $e_1$ and $e_2$. Let $c > 1$. If $c$ is odd, let $M = \{e_1, e_2, (2n-1)(c-1)(c+1)\}$. Then vertices of $\Gamma$ not in $M$ induce even paths and so $M$ can be extended to a perfect matching of $\Gamma$. On the other hand, if $c$ is even, let $M = \{e_1, e_2, (2n-1)(c-1)(c+1)\}$. Then, as above, $M$ can be extended to a perfect matching of $\Gamma$.

**Case 3: $t = n - 1$.**

If $n$ is even, then by Lemma 11, the spanning subgraph $Z_{2n}(1, 2n-1, n-1, n+1)$ of $\Gamma$ is 2-extendable and so $\Gamma$ has a perfect matching containing $e_1$ and $e_2$. Hence assume that $n \geq 5$ is odd. In this case, we then have several subcases to discuss. For each subcase, as in the proof of Lemma 11, we construct a matching $M$ containing $e_1$ and $e_2$ such that the vertices of $\Gamma$ which are not in $M$ can be partitioned into even paths, and hence $M$ can be extended to a perfect matching of $\Gamma$. The desired matching $M$ is as follows:

$$M = \begin{cases} 
\{e_1, e_2, (c-1)(c+1), (d-1)(d+1)\} & \text{if } c \text{ is even and } c < b; \\
\{e_1, e_2\} & \text{if } c \text{ is odd and } c < b; \\
\{e_1, e_2, (n-2)(2n-3), (2n-4)(2n-2)\} & \text{if } c \text{ is odd and } c > b. 
\end{cases}$$

We are now ready to prove our main theorem.

**3. Proof of the main theorem**

We first assume that $\Gamma$ is isomorphic to one of the given graphs. If $\Gamma$ is of type (I), then it is not 2-extendable, by Lemma 2.

If $\Gamma$ is of type (II), then the edges 01 and 34 cannot be extended to a perfect matching of $\Gamma$ and hence it is not 2-extendable.

If $\Gamma$ is of type (III), then it is not 2-extendable, by Lemma 9.

If $\Gamma$ is of type (IV), then $\Gamma$ is isomorphic to $C_{2n+1} \square P_2$ and so it is not 2-extendable, by Corollary 2 of Lemma 6.

Finally, if $\Gamma$ is of type (V), then $\Gamma$ is not 2-extendable, by Lemma 10.

Conversely, assume that $\Gamma$ is not isomorphic to any graph of the given types. We shall prove that $\Gamma$ is 2-extendable.

If $\Gamma$ is regular of degree 2, then it must be a 4-cycle and is so 2-extendable.

If $\Gamma$ is regular of degree 3, let $S = \{a, b, c\}$. If $a, b$ and $c$ are of order 2, then $\Gamma$ is isomorphic to the complete graph $K_4$ or the cube $C_4 \square P_2$ and is so 2-extendable. Otherwise, we may assume that $a + b = 0$ and $c + c = 0$. If $c \notin \langle a \rangle$, then $\Gamma \cong C_m \square P_2$.
where \( m = o(a) \). By hypothesis, \( m \) must be even and so by Lemma 7, \( \Gamma \) is 2-extendable. On the other hand, if \( c \in \langle a \rangle \), then \( a \) must be of even order \( 2n \) and \( c = na \). Hence \( \Gamma \cong Z_{2n}(1, 2n - 1, n) \) and so by hypothesis, \( n \) must be odd. Hence \( \Gamma \) is 2-extendable, by Lemma 9.

If \( \Gamma \) is regular of degree 4, let \( S = \{a, b, c, d\} \). Again if \( a, b, c \) and \( d \) are of order 2, then \( \Gamma \) is isomorphic to \( K_4 \sqcup P_2 \), or the complete bipartite graph \( K_{4,4} \), or \( C_4 \sqcup C_4 \) and is so 2-extendable. If \( a, b \) are of order 2 and \( c + d = 0 \), let \( e_1 \) and \( e_2 \) be independent edges of \( \Gamma \). By Lemma 8, we may assume that \( e_1 \) and \( e_2 \) are of the same type. If they are of type \( a \) or \( b \) (say \( a \)), then the set of all \( a \)-edges will be required perfect matching. Hence, assume that they are of type \( c \). If \( a \in \langle c \rangle \), then \( c \) must be of even order \( 2n \) and \( b \notin \langle c \rangle \). Hence \( \Gamma \) has a spanning subgraph which is isomorphic to \( C_{2n} \sqcup P_2 \) and contains the edges \( e_1 \) and \( e_2 \). By Lemma 7, \( \Gamma \) has a required perfect matching. Assume that both \( a \) and \( b \) are not in \( \langle c \rangle \). If \( c \) is of even order \( 2n \), then \( \langle c, a \rangle \) is isomorphic to \( k_{2n} \sqcup P_2 \) which is 2-extendable. It follows from Lemma 4 that there is a perfect matching of \( \Gamma \) containing \( e_1 \) and \( e_2 \). If \( c \) is of odd order \( k \), then \( \Gamma \) is isomorphic to \( C_k \sqcup P_2 \sqcup P_2 \), which is 2-extendable by Lemma 6, since \( C_k \sqcup P_2 \) is 1-extendable. Finally, assume that \( a + b = 0 \) and \( c + d = 0 \). At least one of \( a \) or \( c \) (say \( a \)) is of even order \( 2n \). If \( c \notin \langle a \rangle \), then for any two independent edges \( e_1 \) and \( e_2 \) of \( \Gamma \), there exists a spanning subgraph of \( \Gamma \) which is isomorphic to \( C_{2n} \sqcup P_m \), \( m \geq 2 \), and contains \( e_1 \) and \( e_2 \). So by Lemma 7, \( \Gamma \) has a perfect matching containing \( e_1 \) and \( e_2 \). If \( c \in \langle a \rangle \), then \( c \neq a, na, 2a \) and \( c \neq (n - 1)a \) if \( n \) is odd. By Lemmas 10 and 11, \( \Gamma \) is 2-extendable.

Hence, we may assume \( \Gamma \) is regular of degree at least 5. Let \( e_1 \) and \( e_2 \) be any two independent edges of \( \Gamma \). By Lemma 8, we need only to consider the case when \( e_1 \) and \( e_2 \) are of the same type (say \( a \)). As \( \Gamma \) is vertex-transitive, we may assume that \( e_1 = 0a \).

Now, we need only to consider the following two cases:

Case 1: \( a \) is of even order \( 2n \).

If \( n = 1 \), then all the \( a \)-edges of \( \Gamma \) will be a perfect matching of \( \Gamma \). Hence, we may let \( n \geq 2 \). The set of all \( a \)-edges forms a spanning subgraph of \( G \) which is the disjoint union of \( 2n \)-cycles. If \( e_1 \) and \( e_2 \) are in two distinct cycles, then clearly they can be extended to a perfect matching of \( \Gamma \). Hence, \( e_2 = (ta)((t + 1)a) \), \( 1 < t < 2n - 1 \). If \( \langle a \rangle \neq G \), then there exists \( b \in S \) such that \( b \notin \langle a \rangle \). Then \( \langle a, b \rangle (a, -a, b, -b) \) has a spanning subgraph \( H \) isomorphic to \( C_{2n} \sqcup P_m \) containing \( e_1 \) and \( e_2 \), for some \( m \geq 2 \). By Lemma 7, \( e_1 \) and \( e_2 \) can be extended to a perfect matching of \( H \), which in turn can be extended to a perfect matching of \( \Gamma \), by Lemma 4. Finally, we may assume that \( b \notin \langle a \rangle \) for all \( b \in S \). As \( \Gamma \) is of degree at least 5, by Lemma 12, we may assume that there exists \( b \in S \) with \( b \notin a,(2n - 1)a, 2a,(2n - 2)a,(n - 1)a \) and \( (n - 1)a \). If \( n \) is odd, then by Lemmas 9 and 10, the subgraph induced by the set of all \( a \)-edges and \( b \)-edges is a 2-extendable spanning subgraph of \( \Gamma \). Hence \( e_1 \) and \( e_2 \) can be extended to a perfect matching of \( \Gamma \). Suppose \( n \) is even. If there is an element \( c \) of \( S \) other than \( \pm 2a \) and \( na \), then by Lemma 11, the subgraph induced by the set of all \( a \)-edges and \( c \)-edges is a 2-extendable spanning subgraph of \( \Gamma \). Hence \( e_1 \) and \( e_2 \) can be extended to a perfect matching of \( \Gamma \). If such a \( c \) does not exist, then we have \( \Gamma \cong Z_{2n}(1, 2n - 1, 2, 2n - 2, n) \). If \( e_2 = (ta)((t + 1)a) \) with \( t \) even, then
$M = \{(ia)((i+1)a) \mid i = 0, 2, \ldots, 2n-2\}$ is a perfect matching of $\Gamma$ that contains $e_1$ and $e_2$. It remains to consider $e_2 = (ta)((t+1)a)$ with $t$ odd. By symmetry we may take $3 \leq t \leq n-1$. Now $M = \{e_1, e_2, (2a)((n+2)a), (n+1)a((n+3)a)\}$ may be extended to a perfect matching of $\Gamma$ since the set of vertices of $\Gamma$ not in $M$ may be partitioned into even paths.

Case 2: $a$ is of odd order $2n+1$.

As $G$ is of even order, there exists an element $b \in S$ of even order and so $b \notin \langle a \rangle$. Let $m$ be the smallest positive integer such that $mb \in \langle a \rangle$. Then $m$ is even. Let $H$ be the subgraph of $\Gamma$ induced by $\bigcup (ib + \langle a \rangle) \mid i = 0, 1, \ldots, m-1\}$. Then $H_i \cong C_{2n+1} \square P_2$ and is 1-extendable. Hence if $e_1$ and $e_2$ are in different $H_i$, there is clearly a perfect matching of $\Gamma$ (by Lemma 4) containing $e_1$ and $e_2$. Hence we may assume that $e_2$ is an edge in $H_0$. By Corollary 1 of Lemma 6, we need only to consider the case

$$e_2 = (b + sa)(b + (s + 1)a) \text{ for some } 1 \leq s \leq 2n \text{ and } |\{0, 1\} \cap \{s, s + 1\}| = 1 \text{ (say } s = 1\).$$

In this case, let $(m-1)b + ta$ be the vertex in $H_{m-2}$ such that $mb + ta = 2a$. Let $M = \{e_1, e_2, (2a)((m-1)b + ta), b(b + 2a)\} \cup \{(ka)(b + ka) \mid k \neq 0, 1, 2\}$. Then $M$ can be extended to a perfect matching of $\Gamma$, since the subgraphs induced by $(2b + \langle a \rangle) \setminus \{2b\}$ and $(m-1)b + \langle a \rangle) \setminus \{(m-1)b + ta\}$ are even paths and the subgraph induced by $\bigcup \{ib + \langle a \rangle \mid i = 3, 4, \ldots, m-2\}$ is isomorphic to $C_{2n+1} \square P_2$, where $r = (m-4)/2$.

Finally, we need only consider the case when $m = 2$. If $e_2 = (sa)((s+1)a)$, or $(b + sa)(b + (s + 1)a)$ (for some $s$ with $|\{0, 1\} \cap \{s, s + 1\}| = 0$), or $b(b + a)$, the as in the previous case, $H$ has a perfect matching containing $e_1$ and $e_2$. Hence we may assume that $e_2 = (b + a)(b + 2a)$. If $\langle a, b \rangle \neq G$, then there exists $c \in S \setminus V(H)$. Let $K = V(H) \cup \{c \in V(H)\}$. Then $K$ has the following perfect matching containing $e_1$ and $e_2$:

$$M = \{e_1, e_2, c(c + a), (c + b + a)(c + b + 2a), (2a)(2a + c), b(b + c)\} \cup \{g(g + c) \mid g \in V(H) \setminus \{0, a, 2a, b, a + b, 2a + b\}\}.$$

Hence $M$ can be extended to a perfect matching of $\Gamma$, as the subgraph of $\Gamma$ induced by the set of all vertices of $\Gamma$ not in $K$ contains a spanning subgraph which is the disjoint union of a finite copies of $H$. Therefore it remains to consider the case that $\langle a, b \rangle = G$.

As $G$ is regular of degree at least 5, there exists $c \in S \setminus \{a, -a, b, -b\}$. We may let $c = ta$ or $b + ta$ for some $1 \leq t \leq n$. If $c = ta$ with $t \geq 3$, let $M = \{e_1, e_2, (2a)\ \ (2t + 1)a), b(b + ta), (ta)(t + 1)a), (b + (t + 1)a)(b + (t + 2)a)\} \cup \{(sa)(b + sa) \mid s \neq 0, 1, 2, t, t + 1, t + 2\}$. Then $M$ is a perfect matching of $\Gamma$ containing $e_1$ and $e_2$. If $c = 2a$, then let $M = \{e_1, e_2, (2a)(4a), (3a)(b + 3a)\}$. Then $M$ is a set of independent edges containing $e_1$ and $e_2$ and the set of all vertices of $\Gamma$ not in $M$ can be decomposed into the disjoint union of even paths and so has a perfect matching $M'$. Hence $M \cup M'$ will be a required perfect matching of $\Gamma$. Now, we need only to consider the case where $c = b + ta$. As $c \neq b$ and $-b$, we have $t \neq 0$ and $2b \neq 2t_a$, respectively.
If \( t \geq 3 \), then let \( M = \{e_1, e_2, c(a + c), (b + a - c)(b + 2a - c)\} \cup \{x(x + c) | x \in \langle a \rangle \setminus \{0, a, b + a - c, b + 2a - c\}\}. \) Then \( M \) is a perfect matching of \( \Gamma \) containing \( e_1 \) and \( e_2 \). If \( t = 1 \), then let \( M = \{e_1, e_2\} \cup \{x(x + c) | x \in \langle a \rangle \setminus \{0, a\}\}. \) Then \( M \) is a perfect matching of \( \Gamma \) containing \( e_1 \) and \( e_2 \). Finally, we may assume that \( t = 2 \). Then \( 2b \neq 4na = (2n - 1)a \). If \( 2b \neq 0 \), then let \( M = \{e_1, e_2, (2b + a)(2b + 2a), (-b)(a - b)\} \cup \{(y + b)y | y \in (b + \langle a \rangle) \setminus \{b + a, b + 2a, -b, a - b\}\}. \) Then \( M \) is a perfect matching of \( \Gamma \) containing \( e_1 \) and \( e_2 \). On the other hand, if \( 2b = 0 \), then let \( M = \{e_1, e_2, (3a)(4a), (-c)(a - c)\} \cup \{(y + c)y | y \in (b + \langle a \rangle) \setminus \{b + a, b + 2a, -c, a - c\}\}. \) Then \( M \) is a perfect matching of \( \Gamma \) containing \( e_1 \) and \( e_2 \). \( \square \)

To end the paper, we shall like to raise the following problems.

**Problem 1.** Characterize 3-extendable abelian Cayley graphs and, in general, \( k \)-extendable abelian Cayley Graphs.

**Problem 2.** Characterize 1-extendable and 2-extendable Cayley graphs.

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**References**


