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Groups whose real irreducible characters have degrees coprime to p [☆]

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ABSTRACT

In this paper we study groups for which every real irreducible character has degree not divisible by some given odd prime p.

1. Introduction

A classic result of Noboru Itô is that for a solvable group G, a Sylow p-subgroup is normal and abelian if and only if no irreducible character of G has degree divisible by p. Part of this theorem appears as Proposition 5 of Itô's 1951 paper [5], which asserts that the Sylow p-subgroup is normal if the character degree condition is satisfied. (Itô does not explicitly state that the Sylow subgroup is abelian in this case, but this is trivial to prove once normality is established.) For the converse, solvability is not relevant, and the result follows since irreducible character degrees always divide the index of an abelian normal subgroup. (This too is a result of Itô; it appears in [6], also published in 1951.)

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In the decades since the middle of the 20th century, a number of improvements and variations on Itô's theorem have been established. For example, the result holds for p-solvable groups as well as for solvable groups. (For a more modern proof of Itô's theorem, including the p-solvable case, see Corollary 12.34 of [2].) In order to extend Itô's theorem to groups with no solvability condition at all, it suffices to show that every nonabelian simple group of order divisible by p has an irreducible character of degree divisible by p. The sufficiency of this condition[1] follows by Theorem 12.33 of [2], and this property of simple groups was eventually established by Gerhard Michler [7] using the classification of simple groups.

Another variation on Itô's theorem would be to weaken the condition that *all* irreducible characters of G have degree not divisible by p, and to assume instead only that some specified subset of $\mathrm{Irr}(G)$ has this property. For example, one might consider only real-valued characters. This was the point of view of [1], where Dolfi, Navarro and Tiep show that if all real irreducible characters of a finite group have degree not divisible by 2, then a Sylow 2-subgroup is normal. (As might be expected, this theorem relies on the simple group classification, and most of the work in [1] is in the analysis of specific types of simple groups.)

What can one conclude if we assume that all real irreducible characters of G have p'-degree, where p is odd? Since groups of odd order have no nonprincipal real irreducible characters, it is clear that this "real-Itô" hypothesis says nothing about such groups, so there is no hope of proving, for example, that a Sylow p-subgroup is normal. Nevertheless, something can be said. Recently, for example, Tiep has proved the following.

Theorem *(Tiep). Let $p > 2$ be prime, and assume that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Then $\mathbf{O}^{p'}(G)$ is solvable.*

Here, we investigate what more can be said in Tiep's situation. In order to make this paper independent of Tiep's work, we will assume that the groups we consider satisfy the conclusion of his result. We show the following.

Theorem A. *Let p be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Let $K = \mathbf{O}^{2'}(G)$. Then K has a normal Sylow p-subgroup Q. Also, if $p > 2$, then Q' is central in K.*

Note that if $p = 2$ in Theorem A, then the Sylow 2-subgroup Q of K is actually a Sylow 2-subgroup of G, and we have $Q \triangleleft G$. Our theorem, therefore, includes the solvable case of the main result of [1]. Also, we mention that since Theorems B, C and D below are trivially true if G has a normal Sylow p-subgroup, these results are interesting only if $p > 2$.

As we observed, if $p \neq 2$, we cannot hope to prove in the situation of Theorem A that a Sylow p-subgroup of G is normal because we can say essentially nothing about the odd-order group G/K. In some sense, however, that is the only obstruction: if G/K has a normal Sylow p-subgroup, then G does too.

Theorem B. *Let p be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Let $K = \mathbf{O}^{2'}(G)$, and assume that G/K has a normal Sylow p-subgroup. Then G has a normal Sylow p-subgroup.*

Although in Theorem A, a Sylow p-subgroup of G need not be normal if $p > 2$, there is a sense in which the normalizer of a Sylow p-subgroup is "large". For the statement of the next result, we recall that a conjugacy class of a group G is a **real class** if it contains the inverse of each of its elements, or equivalently, if it contains the inverse of any one of its elements.

[1] In the early 1960s, Isaacs asked his teacher, Richard Brauer, if this property of simple groups might be true. Brauer's answer was that "the world is not yet ready for that conjecture". As it turned out, Brauer was exactly right: Gorenstein's program for classifying simple groups was first proposed about 15 years later.

Theorem C. *Let p be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Let $P \in \mathrm{Syl}_p(G)$. Then $\mathbf{N}_G(P)$ contains an element in each real class of G.*

Theorems B and C and the part of Theorem A that asserts that K has a normal Sylow p-subgroup can be viewed as approximations to the conclusion in Itô's theorem that a Sylow p-subgroup of G is normal. In the same vein, the assertion in Theorem A that Q' is central in K if $p > 2$ can be viewed as a rather weak approximation to the conclusion in Itô's theorem that the Sylow p-subgroup of G is abelian. This suggests the possibility that perhaps in Theorem A, the subgroup Q' must actually be trivial, so that Q is abelian. That is false, however, and in fact, Q' need not even be central in G. (It is easy to produce a counterexample of order $6 \cdot 7^3$, with $p = 7$.)

Recall that the McKay conjecture asserts that the numbers of irreducible characters of p'-degree of a group G and of the normalizer N of a Sylow p-subgroup of G are equal. A refinement of this conjecture due to the authors [4] is the degree-congruence condition, which proposes that for each integer k not divisible by p, the numbers of characters in $\mathrm{Irr}(G)$ and $\mathrm{Irr}(N)$ that are congruent mod p to $\pm k$ are equal. This suggests the possibility that if all real irreducible characters of G have p'-degree, there might be a natural bijection from this set of characters onto the set of real irreducible characters of p'-degree of a Sylow p-normalizer. In fact, this is true, and furthermore, this bijection realizes the degree-congruence condition.

Theorem D. *Let p be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Let $P \in \mathrm{Syl}_p(G)$ and $N = \mathbf{N}_G(P)$. Then there is a canonically defined bijection from the set of real characters in $\mathrm{Irr}(G)$ onto the set of real p'-degree characters in $\mathrm{Irr}(N)$. Also, if $\chi \mapsto \xi$ under this mapping, then $\xi(1) \equiv \pm\chi(1) \bmod p$.*

It is not true in general, however, that in the situation of Theorem D, every real irreducible character of $\mathbf{N}_G(P)$ has p'-degree. An example with $p = 3$ can be constructed by taking G to be the semidirect product of an elementary abelian group of order 27 with the normalizer in its automorphism group of a Sylow 13-subgroup. Then $|G| = 2 \cdot 3^4 \cdot 13$, and G has exactly three real irreducible characters. These have degrees 1, 1 and 26, so none of them has degree divisible by 3. The Sylow 3-normalizer N in G has order $2 \cdot 3^4$, and as predicted by Theorem D, it has exactly three real irreducible characters with $3'$-degree, and these have degrees 1, 1 and 2. But N has four additional real irreducible characters, each having degree 6.

2. Real characters and odd index normal subgroups

We begin with an easy observation.

Lemma 2.1. *Let $K \lhd G$, where G/K has odd order, and suppose that $\chi \in \mathrm{Irr}(G)$ is real. Then every irreducible constituent of χ_K is real.*

Proof. The odd-order group G/K permutes the set $\mathcal{S}$ of irreducible constituents of χ_K transitively, and hence $|\mathcal{S}|$ is odd. Also, since $\chi = \overline{\chi}$, complex conjugation permutes the set $\mathcal{S}$, and since this set has odd cardinality, some member of $\mathcal{S}$ is fixed, and hence is real. But conjugates of real characters are real, and since the action of G/K is transitive, it follows that all members of $\mathcal{S}$ are real. $\square$

In the more interesting reverse direction, we have the following, much of which is known. (See, for example, Corollaries 2.1 and 2.2 of [8], or Lemma 3.2 of [9].)

Lemma 2.2. *Let $K \lhd G$, where G/K has odd order.*

(a) *If $\theta \in \mathrm{Irr}(K)$ is real and G-invariant, then θ has a real extension to G.*
(b) *Let $K \subseteq H \subseteq G$, and suppose that $\psi \in \mathrm{Irr}(H)$ is real. Then ψ^G has a unique real irreducible constituent χ.*

(c) *If ψ and χ are as in* (b), *then $[\psi^G, \chi] = 1$ and $\chi(1)|T : S| = \psi(1)|G : H|$, where T and S are respectively the stabilizers in G and H of some irreducible constituent θ of ψ_K.*

We will prove (a) under the assumption that G/K is solvable, and of course, this is sufficient by the odd-order theorem. We mention, however, that at least two approaches are available that avoid the Feit–Thompson result. After one proves the solvable case as we do, extendibility in the general case can be established by appealing to Theorem 11.31 of [2], which shows that it suffices to prove it when G/K is a p-group. An easy argument then produces a real extension. (This is the argument used by Navarro and Tiep in [8].) In [9], Wilde took a completely different approach, using Frobenius–Schur theory.

Proof of Lemma 2.2. First, we prove that if θ extends to G, then there is a real extension $\chi \in \mathrm{Irr}(G)$, and χ is the unique real irreducible constituent of θ^G. To see this, observe first that by Gallagher's theorem (Corollary 6.17 of [2]) the number of extensions of θ to G is equal to the number of linear characters of G/K, and so is odd. Since $\bar{\theta} = \theta$, the set of extensions of θ to G is permuted by complex conjugation, and hence some extension χ satisfies $\overline{\chi} = \chi$, as wanted. Now if χ' is any real irreducible character of G lying over θ, then by Gallagher's theorem, we have $\chi' = \chi\beta$ for some character $\beta \in \mathrm{Irr}(G/K)$. Also, β is uniquely determined by the real characters χ and χ', and it follows that β is real. But $\beta \in \mathrm{Irr}(G/K)$ and $|G/K|$ is odd, and thus β is the principal character and $\chi' = \chi$, as wanted.

To prove (a), we use the solvability of G/K, and we proceed by induction on $|G : K|$. The result is trivial if $K = G$, so we can assume that $K < G$, and we choose M with $K \subseteq M \lhd G$ and $|G : M|$ prime. By the inductive hypothesis, θ has a real extension $\psi \in \mathrm{Irr}(M)$, and ψ is unique by the result of the previous paragraph. It follows that ψ is invariant in G, and since $|G : M|$ is prime, ψ extends to G by Corollary 6.20 of [2], and thus θ extends to G. By the result of the first paragraph, θ has a real extension to G.

For (b) and (c), let θ be an irreducible constituent of ψ_K, and note that θ is real by Lemma 2.1. Let S and T be the stabilizers of θ in H and G, and let μ and ν be the unique real extensions of θ to S and T, respectively, so that $S \subseteq T$ and $\nu_S = \mu$. The Clifford correspondent of ψ with respect to θ is some character $\gamma \in \mathrm{Irr}(S|\theta)$, and γ is uniquely determined by ψ and θ. Since ψ and θ are real, it follows that γ is a real irreducible character of S lying over θ, and by the first paragraph of the proof, we have $\gamma = \mu$. Then $\psi = \mu^H$, and thus $\psi^G = (\mu^H)^G = \mu^G = (\mu^T)^G$.

Since $[\mu^T, \nu] = [\mu, \nu_S] = [\mu, \mu] = 1$, we can write $\mu^T = \nu + \Delta$, where ν is not a constituent of Δ. Let $\chi = \nu^G$, so that χ is real, and it is irreducible by the Clifford correspondence. Then

$$\psi^G = \left(\mu^T\right)^G = \nu^G + \Delta^G = \chi + \Delta^G,$$

and we argue that Δ^G has no real irreducible constituent. It will then follow that χ is the unique real irreducible constituent of ψ^G and that $[\psi^G, \chi] = 1$. Suppose then that χ' is a real irreducible constituent of Δ^G. Then χ lies over some irreducible constituent δ of Δ, and $\delta \in \mathrm{Irr}(T)$ lies over μ and hence over θ. It follows that δ is the Clifford correspondent of χ' with respect to θ. Since χ' and θ are real, δ is real, and thus by the first paragraph, $\delta = \nu$. This is a contradiction, however, because ν is not a constituent of Δ, and it follows that χ' cannot exist.

To complete the proof of (c), we observe that

$$\frac{\chi(1)}{\psi(1)} = \frac{\nu(1)|G : T|}{\mu(1)|H : S|} = \frac{|G : T|}{|H : S|} = \frac{|G : H|}{|T : S|},$$

where the second equality follows since $\nu(1) = \theta(1) = \mu(1)$. It follows that $\chi(1)|T : S| = \psi(1)|G : H|$, as required. □

As an application of Lemma 2.2, we prove the following, which will be used in the proof of Theorem D.

Corollary 2.3. *Let $K \lhd G$, where G/K has odd order, and suppose that P/K is a Sylow p-subgroup of G/K. Let $H = \mathbf{N}_G(P)$, and suppose that $\psi \in \mathrm{Irr}(H)$ is real. Let χ be the unique real irreducible constituent of ψ^G. Then $\chi(1) \equiv \psi(1) \bmod p$.*

Proof. First, note that the existence and uniqueness of χ follow by Lemma 2.2(b), and by Lemma 2.2(c), we have $|G:H|\chi(1) = |T:S|\psi(1)$, where S and T are the stabilizers in H and G, respectively, of an irreducible constituent θ of ψ_K. Also $|G:H| \equiv 1 \bmod p$ by Sylow theory applied in the group G/K, and thus $\chi(1) \equiv |T:S|\psi(1) \bmod p$. If $\psi(1) \equiv 0 \bmod p$, then $\chi(1) \equiv 0 \bmod p$, and so we can assume that $\psi(1)$ is not divisible by p. Since ψ is induced from a character of S, we see that $|H:S|$ is not divisible by p, and since $P \lhd H$, we deduce that $P \subseteq S \subseteq T$. Now $S = T \cap H = \mathbf{N}_T(P)$, and it follows by Sylow theory in the group T/K that $|T:S| \equiv 1 \bmod p$, and thus $\chi(1) \equiv \psi(1) \bmod p$, as required. $\square$

3. Theorem A

We begin with an easy lemma.

Lemma 3.1. *Let T be a 2-group that acts nontrivially on some group X. Then there is an element $t \in T$ and a nonidentity element $x \in X$ such that $x^t = x^{-1}$.*

Proof. We can replace T by $T/\mathbf{C}_T(X)$, and thus we can assume that T acts faithfully on X. Since the action is nontrivial, $T > 1$, and we can choose an involution $t \in T$. Also, because the action is faithful and $t \neq 1$, we can choose $y \in X$ such that $y^t \neq y$, and thus $x = y^{-1}y^t \neq 1$. Then $x^t = (y^{-1})^t(y^t)^t = (y^t)^{-1}y = x^{-1}$. $\square$

In the following, we will use the notion of a "canonical extension" of a character. Let $N \lhd G$, and let $\theta \in \mathrm{Irr}(N)$ be invariant in G. If both $\theta(1)$ and the determinantal order $o(\theta)$ are coprime to $|G:N|$, then by Corollary 8.16 of [2], there is a unique extension $\chi \in \mathrm{Irr}(G)$ of θ such that $o(\chi) = o(\theta)$, and we refer to χ as the **canonical extension** of θ.

Theorem 3.2. *Let p be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Suppose also that $\mathbf{O}^{2'}(G) = G$. Then G has a normal Sylow p-subgroup.*

Proof. Let $N = \mathbf{O}^{p'}(G)$ and $M = \mathbf{O}^p(N)$. Our goal is to show that $M = 1$, so we suppose that $M > 1$, and we work to obtain a contradiction. Let M/L be a chief factor of G. By hypothesis, N is solvable, and thus M/L is an abelian q-group for some prime $q \neq p$. Also, M/L is not central in N/L since otherwise it is a direct factor of N/L, and thus $\mathbf{O}^q(N) < N$, which is not the case since $\mathbf{O}^{p'}(N) = N$. Then $[M/L, N] \neq 1$, and because M/L is a chief factor of G, we have $[M/L, N] = M/L$.

Let $C = \mathbf{C}_G(M/L)$. Then $C \lhd G$, and since $N \nsubseteq C$, we have $C < G$. By assumption, G has no proper normal subgroup of odd index, and it follows that $|G:C|$ is even. Now let $T \in \mathrm{Syl}_2(G)$, and observe that $T \nsubseteq C$, and thus T acts nontrivially on M/L. Since M/L is abelian, it follows that the action of T on the group of linear characters of M/L is nontrivial, and hence by Lemma 3.1, there is some element $t \in T$ and some nonprincipal linear character λ of M such that $\lambda^t = \lambda^{-1} = \bar{\lambda}$.

Let I be the stabilizer of λ in N, and note that $I < N$ since $[M/L, N] = M/L$ and thus N has no nontrivial fixed points in its action on $\mathrm{Irr}(M/L)$. Now N/M is a p-group, so $|I:M|$ is a power of p, and thus $|I:M|$ is coprime to $o(\lambda)$, which is a power of q because M/L is a q-group. Also, $|I:M|$ is coprime to $\lambda(1) = 1$ and it follows that λ has a canonical extension $\mu \in \mathrm{Irr}(I)$. Since I is the stabilizer in N of $\bar{\lambda} = \lambda^t$, we see that t normalizes I and μ^t is the canonical extension of λ^t to I. But $\overline{\mu}$ is the canonical extension of $\bar{\lambda} = \lambda^t$, and we deduce that $\mu^t = \overline{\mu}$.

By the Clifford correspondence, the character $\alpha = \mu^N$ is irreducible, and we have $\overline{\alpha} = (\overline{\mu})^N = (\mu^t)^N = \alpha^t$. Also, $\alpha(1) = |N:I|$ is a power of p, and since $N = \mathbf{O}^{p'}(N)$, we see that $o(\alpha)$ cannot be divisible by any prime different from p, and so $o(\alpha)$ is a power of p. But G/N is a p'-group, and it follows that α has a canonical extension $\beta \in \mathrm{Irr}(J)$, where J is the stabilizer of α in G. Since $\alpha^t = \overline{\alpha}$, we can reason as before, and we see that t normalizes J and that the canonical extension of $\alpha^t = \overline{\alpha}$

to J is $\beta^t = \overline{\beta}$. Now let $\chi = \beta^G$, so that χ is irreducible by the Clifford correspondence. We have $\overline{\chi} = (\overline{\beta})^G = (\beta^t)^G = \beta^G = \chi$, and thus χ is real valued. By hypothesis, therefore, χ has p'-degree, and thus α, which is an irreducible constituent of χ_N has p'-degree. This is a contradiction, however, since $\alpha(1) = |N : I|$ is a nontrivial power of p. This completes the proof. $\square$

The following is part of Theorem A.

Corollary 3.3. *Let p be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Let $K = \mathbf{O}^{2'}(G)$. Then K has a normal Sylow p-subgroup.*

Proof. It suffices to show that K satisfies the hypotheses of Theorem 3.2. We have $\mathbf{O}^{p'}(K) \subseteq \mathbf{O}^{p'}(G)$, and so $\mathbf{O}^{p'}(K)$ is solvable. If $\theta \in \mathrm{Irr}(K)$ is real, then by Lemma 2.2(b), there exists a real-valued character $\chi \in \mathrm{Irr}(G|\theta)$, and by hypothesis, χ has p'-degree. Also, $\theta(1)$ divides $\chi(1)$ since $K \lhd G$, and thus θ has p'-degree, as wanted. Finally, $\mathbf{O}^{2'}(K) = K$ since $K = \mathbf{O}^{2'}(G)$, and thus all hypotheses of Theorem 3.2 hold for K. $\square$

We complete the proof of Theorem A with a result that is slightly more general than needed.

Theorem 3.4. *Let $p > 2$ be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Let $K = \mathbf{O}^{2'}(G)$ and let $P \in \mathrm{Syl}_p(G)$. Suppose that $M \subseteq K \cap P'$, where $M \lhd G$. Then $M \subseteq \mathbf{Z}(K)$.*

In the situation of Theorem 3.4, Corollary 3.3 tells us that K has a normal Sylow p-subgroup Q, and thus $Q' \lhd G$ and $Q' \subseteq K \cap P'$. By Theorem 3.4, therefore, $Q' \subseteq \mathbf{Z}(K)$, and this will complete the proof of Theorem A.

Proof of Theorem 3.4. Suppose that M is not central in K. Then $C = \mathbf{C}_K(M)$ is a proper normal subgroup of K, and since $K = \mathbf{O}^{2'}(K)$, it follows that C does not contain a Sylow 2-subgroup T of K, and thus T acts nontrivially on M. Since M is a p-group and this is a coprime action, T acts nontrivially on M/M', and hence it also acts nontrivially on the group of linear characters of M. By Lemma 3.1, therefore, some nonprincipal linear character λ of M satisfies $\lambda^t = \bar{\lambda}$ for some element $t \in T$.

Now let σ be the operator $\gamma \mapsto (\overline{\gamma})^t$ for characters γ of normal subgroups of K. Then σ fixes λ, and hence it permutes the irreducible constituents of λ^Q, where $Q = K \cap P$ is the normal Sylow p-subgroup of K. Since $p > 2$, the degree of λ^Q is odd, and since the permutation σ has 2-power order, it fixes some irreducible constituent α of λ^Q, and hence we have $\alpha^t = \overline{\alpha}$.

Since Q is a normal Sylow p-subgroup of K, it follows that α has a canonical extension β to its stabilizer I in K. Reasoning as in the proof of Theorem 3.2, we see that t normalizes I and $\beta^t = \overline{\beta}$. By the Clifford correspondence, the character $\theta = \beta^K$ is irreducible, and $\bar{\theta} = (\overline{\beta})^K = (\beta^t)^K = \beta^K = \theta$, so θ is real. By Lemma 2.2(b), there exists a real character $\chi \in \mathrm{Irr}(G|\theta)$, and by hypothesis, χ has p'-degree. We conclude that χ_P has a linear constituent μ. But $M \subseteq P'$, and thus μ_M is the principal character of M, and since $M \lhd G$, it follows that $M \subseteq \ker(\chi)$. But λ is an irreducible constituent of χ_M and λ is nonprincipal, and this is a contradiction. $\square$

4. Real classes

By Brauer's permutation lemma, which is Theorem 6.32 of [2], we know that an arbitrary finite group K contains equal numbers of real irreducible characters and real classes. We need the following generalization for our proof of Theorem B. We will establish a deeper and even more general result later in order to prove Theorem C.

Lemma 4.1. *Suppose that a p-group P acts on a p'-group K. Then*:

(a) *K has equal numbers of P-invariant real irreducible characters and P-invariant real classes.*
(b) *If P fixes all real irreducible characters of K, then P centralizes some element in each real class of K.*

Proof. Let $C = \mathbf{C}_K(P)$, and recall that the Glauberman correspondence is a bijection from the set of P-invariant irreducible characters of K onto $\mathrm{Irr}(C)$. (See Theorem 13.1 of [2].) In fact, the image in $\mathrm{Irr}(C)$ of a P-invariant character $\chi \in \mathrm{Irr}(K)$ is the unique irreducible constituent ξ of χ_C such that $[\chi_C, \xi] \not\equiv 0 \bmod p$, and so it is clear that if $\chi \mapsto \xi$ under the Glauberman correspondence, then $\overline{\chi} \mapsto \overline{\xi}$. The Glauberman correspondence thus defines a bijection from the set of real P-invariant characters $\chi \in \mathrm{Irr}(K)$ onto the set of real characters $\xi \in \mathrm{Irr}(C)$, and in particular, the number of P-invariant real irreducible characters of K is equal to the number of real irreducible characters of C.

If X is a P-invariant class of K then $X \cap C$ is nonempty, and is a class of C. (See Corollary 13.10 of [2].) If X is a real class, then for every element $x \in X$, also $x^{-1} \in X$, and thus if $x \in X \cap C$, we see that $x^{-1} \in X \cap C$, and thus $X \cap C$ is a real class of C. Conversely, if $X \cap C$ is a real class of C, it contains an element and its inverse, and thus X is a real class of K. This establishes a bijection between the real P-invariant classes of K and the real classes of C, and in particular, the number of P-invariant real classes of K is equal to the number of real classes of C. Combining this with the result of the previous paragraph, we see that to prove (a), it suffices to show that C has equal numbers of real irreducible characters and real classes. This is true by Brauer's lemma, however.

Now suppose that P fixes all real irreducible characters of K. The total numbers of real irreducible characters and real classes of K are equal by Brauer's lemma, and by (a), the numbers of P-invariant real irreducible characters and P-invariant real classes of K are equal. Since we are assuming that every real irreducible character is P-invariant, it follows that the same must be true about real classes. Thus if X is any real class of K, then X is P-invariant, and so $X \cap C$ is nonempty, as wanted. $\square$

The following is a slightly strengthened version of Theorem B, and like Theorem B, it has no essential content in the case $p = 2$.

Theorem 4.2. *Let p be prime, and assume that $\mathbf{O}^{p'}(G)$ is solvable and that every real character $\chi \in \mathrm{Irr}(G)$ has p'-degree. Let $K = \mathbf{O}^{2'}(G)$ and let $U/K = \mathbf{O}_p(G/K)$. Then U has a normal Sylow p-subgroup.*

Proof. Let $P \in \mathrm{Syl}_p(U)$. The hypotheses of the theorem are inherited by factor groups, and thus working by induction on $|G|$, we can assume that whenever $1 < M \lhd G$ with $M \subseteq K$, we have $MP/M \lhd U/M$, and so $MP \lhd U$. In particular, if $Q > 1$, where Q is the normal Sylow p-subgroup of K, we have $P = PQ \lhd U$, and there is nothing further to prove. We can assume, therefore, that K is a p'-group.

Now suppose that $\theta \in \mathrm{Irr}(K)$ is real. By Lemma 2.2, there exists a real character $\chi \in \mathrm{Irr}(G|\theta)$, and by hypothesis, p does not divide $\chi(1)$. Since $U \lhd G$, the irreducible constituents of χ_U have p'-degree, and thus they restrict irreducibly to K. It follows that θ is the restriction of some irreducible character of U, and thus θ is invariant in U. This shows that P fixes each real irreducible character of K, and thus by Lemma 4.1, each real class of K contains a P-fixed element.

Since $KP = U \lhd G$, we see that to prove that P is normal in U, it suffices to show that P centralizes K, or equivalently, that $[K, P] = 1$. We have $[K, P] \lhd G$ since $G = U\mathbf{N}_G(P) = K\mathbf{N}_G(P)$, and both K and $\mathbf{N}_G(P)$ normalize $[K, P]$. Supposing that $[K, P] > 1$, we work to obtain a contradiction.

Since $MP \lhd U$ whenever $M \lhd G$ and $1 < M \subseteq K$, we see that $[K, P] \subseteq K \cap MP = M(K \cap P) = M$ for every such subgroup M. It follows that $[K, P]$ is the unique minimal normal subgroup of G contained in K. Also, $P \subseteq \mathbf{O}^{p'}(G)$, and thus $[K, P] \subseteq \mathbf{O}^{p'}(G)$, and it follows that $[K, P]$ is solvable. We conclude that $[K, P]$ is an abelian q-group for some prime $q \neq p$. Then $[K, P, P] = [K, P]$, and so by Fitting's lemma, $\mathbf{C}_{[K,P]}(P) = 1$, and it follows that none of the nonidentity classes of K contained in $[K, P]$ is real. In particular, $[K, P]$ contains no involution, and thus $q \neq 2$. Since $\mathbf{O}^{2'}(K) = K$, we see that K cannot be abelian, and thus since $[K, P]$ is the unique minimal normal subgroup of G contained in K, we have $[K, P] \subseteq K'$.

We argue next that $[K, P]$ is not central in K. Otherwise, $[K, P, K] = 1$, and we deduce by the three-subgroups lemma that $[K', P] = 1$. But then $[K, P, P] = 1$, and this is a contradiction since $[K, P, P] = [K, P]$. Let $C = \mathbf{C}_K([K, P])$. Then C is a proper normal subgroup of K, which must, therefore, have even index. A Sylow 2-subgroup T of K, therefore, fails to centralize $[K, P]$, so Lemma 3.1 guarantees that $[K, P]$ contains a nonidentity real element of K. This is our desired contradiction since we know that $[K, P]$ contains no nonidentity real class of K. $\square$

Recall that in the proof of Lemma 4.1, we considered an action of a p-group P on a p'-group K, and we counted P-invariant real classes and P-invariant real irreducible characters of K. In order to prove Theorem C, we need an analogous result, where the group acting on K is not necessarily a p-group.

We will appeal to some of the material in [3], which we review briefly. For this discussion, we fix a prime p, and we suppose that a group G acts on a set Ω, and that Λ is an orbit of this action. We say that D is a **defect group** for Λ if D is a Sylow p-subgroup of the stabilizer of a point in Λ, and we observe that Λ determines D up to conjugacy in G. If the trivial group is a defect group for Λ, we say that Λ has **defect zero**.

We consider two actions of G, and we seek conditions sufficient to guarantee that for each p-subgroup $D \subseteq G$, the numbers of orbits with defect group D in each action are equal. The two actions that were of primary interest in [3] were the actions of G on classes and irreducible characters of a p'-group K, where G acts via automorphisms on K. Theorem B of [3] asserts that in that case, the numbers of orbits on irreducible characters and classes with a given defect group are indeed equal. Here we are interested in the actions of G on the *real* irreducible characters and *real* classes of K, and we get equality if G has odd order. (As was pointed out in [3], an alternative proof of Theorem B of that paper is available, using the invertibility of the character table, viewed as a matrix. It seems impossible to construct an analogous proof for real characters and classes because in general, the real part of a character table is not an invertible matrix.)

Theorem 4.3. *Let $K \lhd G$, and assume that $|G : K|$ is odd and that p is a prime not dividing $|K|$. Then for each p-subgroup D of G, the numbers of G-orbits having defect group D in the actions of G on real irreducible characters of K and real classes of K are equal. In particular, the total numbers of G-orbits of real irreducible characters and G-orbits of real classes are equal.*

We need some preliminary observations and results. Continuing to suppose that G acts on Ω, and following [3], let $P \lhd M \subseteq G$, where P is a p-group. We refer to M/P as a p-normal section of G, and in particular, $G = G/1$ is a p-normal section. Note that via an obvious identification, a p-normal section of a p-normal section of G is itself a p-normal section of G. Still following [3], we define two integer-valued functions on p-normal sections. To do this, we observe that if $P \lhd M$, then M/P has a natural action on the possibly empty set Ω_0 of P-fixed points in Ω. We write $\mathrm{orb}(M/P)$ and $\mathrm{zer}(M/P)$ to denote the total number of orbits and the number of defect-zero orbits, respectively, of the p-normal section M/P on Ω_0. (Of course, if Ω_0 is empty, then both $\mathrm{orb}(M/P)$ and $\mathrm{zer}(M/P)$ are zero.) Note that the definitions of orb( ) and zer( ) respect the identification of a p-normal section of a p-normal section of G with a p-normal section of G.

According to Lemma 2.1 of [3], the number of G-orbits with defect group D on Ω is $\mathrm{zer}(\mathbf{N}_G(D)/D)$, so to prove Theorem 4.3, it suffices to show that for all p-normal sections N/D of G, the two quantities $\mathrm{zer}(N/D)$ computed for the actions on real classes and real irreducible characters of K are equal. By applying Theorem D of [3] to the group N/D, we see that for each of the two actions, $\mathrm{zer}(N/D)$ can be expressed in terms of $\mathrm{orb}(M/P)$ for p-normal sections M/P of N/D.

Proof of Theorem 4.3. Let $P \lhd M \subseteq G$, where P is a p-group. By the foregoing discussion, it suffices to show that M has equal numbers of orbits on the sets of P-invariant real irreducible characters of K and P-invariant real classes of K. To see this, consider the fixed-point subgroup $C = \mathbf{C}_K(P)$. As we saw in the proof of Lemma 4.1, the Glauberman correspondence yields a natural bijection from the set of P-invariant real irreducible characters of K onto the set of all real irreducible characters of C. Furthermore, because this bijection is natural, it respects the action of M on these sets. The numbers of orbits of M on these two sets are therefore equal. Similarly, we saw that intersection with C defines a natural bijection from the set of P-invariant real classes of K onto the set of real classes of C, and the numbers of M-orbits on these two sets are also equal. It suffices, therefore, to show that M has equal numbers of orbits on the sets of real irreducible characters and real classes of C.

To prove that these two permutation actions of M have equal numbers of orbits, it suffices to show that the corresponding permutation characters are equal. We want to show, therefore, that for each element $x \in M$, there are equal numbers of real irreducible characters of C and real classes of C that are fixed by x. (Note that x normalizes $C = \mathbf{C}_K(P)$ since $x \in M \subseteq \mathbf{N}_G(P)$.) First, observe that the permutations on irreducible characters and on classes of K induced by x have odd order since $|G/K|$ is odd. Because of the two natural bijections, it follows that the permutations induced by x on irreducible characters and on classes of C have odd order. We can thus replace x by a suitable power and assume that x has odd order, say r.

Now consider a cyclic group $U = \langle u \rangle$ of order $2r$, acting on characters $\chi \in \mathrm{Irr}(C)$ and on elements $c \in C$ according to the formulas

$$\chi^u = (\overline{\chi})^x \quad \text{and} \quad c^u = \left(c^{-1}\right)^x,$$

and note that the action of U on elements defines an action on conjugacy classes. Then $\chi^u(c^u) = \chi(c)$, and so by Brauer's permutation lemma, U fixes equal numbers of irreducible characters and classes of C. Since U is generated by u^r and u^2, we see that $\chi \in \mathrm{Irr}(C)$ is fixed by U if and only if it is fixed by u^r and u^2, or equivalently, by complex conjugation and by $\langle x^2 \rangle = \langle x \rangle$. Thus the U-invariant characters χ in $\mathrm{Irr}(C)$ are exactly the real irreducible characters fixed by x. Similarly, the U-invariant classes of C are exactly the real classes fixed by x. There are equal numbers of these objects, and this completes the proof of the first statement. The final assertion follows by summing over a set of representatives for all conjugacy classes of p-subgroups of G. $\square$

Proof of Theorem C. Given a Sylow subgroup $P \in \mathrm{Syl}_p(G)$, we must show that $\mathbf{N}_G(P)$ contains a representative of every real class of G, or equivalently, given a real element $x \in G$, we must show that x normalizes some Sylow p-subgroup of G.

As usual, let $K = \mathbf{O}^{2'}(G)$. Since the image of x in each homomorphic image of G is real and the odd-order group G/K has no nonidentity real elements, we conclude that $x \in K$. Let Q be the normal Sylow p-subgroup of K, which exists by Theorem A. Then $Q \lhd G$, and the image $\bar{x}$ of x in $\overline{G} = G/Q$ is real. If we can prove that $\bar{x}$ normalizes a Sylow subgroup $\overline{P}$ of $\overline{G}$, it would follow that x normalizes the preimage $\overline{P}$ in G. Since that preimage is a Sylow p-subgroup of G, we see that it suffices to prove the theorem in the case where K is a p'-group. We assume now that is the case.

If $\theta \in \mathrm{Irr}(K)$ is real, then by Lemma 2.2, there exists a real character $\chi \in \mathrm{Irr}(G|\theta)$, and by hypothesis χ has p'-degree. The size of the G-orbit of θ divides $\chi(1)$ and hence it is not divisible by p, and thus some Sylow p-subgroup of G stabilizes θ. This shows that a Sylow p-subgroup of G is a defect group for every G-orbit of real irreducible characters of K. Since no G-orbit of real irreducible characters of K has a defect group properly smaller than a Sylow subgroup, it follows by Theorem 4.3 that no G-orbit of real classes of K can have a defect group properly smaller than a Sylow subgroup. The G-orbit of the K-class X of x, therefore, has a Sylow p-subgroup of G as defect group, and thus there exists $P \in \mathrm{Syl}_p(G)$ such that P stabilizes X. Since K is a p'-group, P must centralize some element of X, and thus some conjugate of P centralizes x. The result now follows. $\square$

5. Theorem D

We used the Glauberman correspondence in the proof of Lemma 4.1, where we considered the action of a p-group on a p'-group. To prove Theorem D, we need a generalization of the Glauberman correspondence, in which we weaken the assumption that the group K being acted on is a p'-group. We assume only that K has a normal Sylow p-subgroup.

Theorem 5.1. *Suppose that a p-group P acts on a group K, where K has a normal Sylow p-subgroup Q. Let $C/Q = \mathbf{C}_{K/Q}(P)$. Then for each P-invariant character $\chi \in \mathrm{Irr}(K)$, the restriction χ_C has a unique P-invariant irreducible constituent ξ such that $[\chi_C, \xi] \not\equiv 0 \bmod p$, and in fact $[\chi_C, \xi] \equiv \pm 1 \bmod p$ and $\xi(1) \equiv \pm\chi(1) \bmod p$. Also, the map $\chi \mapsto \xi$ is a bijection from the set of P-invariant irreducible characters of K onto the set of P-invariant irreducible characters of C.*

Proof. Given a P-invariant character $\chi \in \mathrm{Irr}(K)$, we construct a P-invariant character $\xi \in \mathrm{Irr}(C)$ in several steps. First, since K/Q is a p'-group, we can apply Glauberman's lemma (which is Lemma 13.8

of [2]) to deduce that χ_Q has a P-invariant irreducible constituent θ. Also, all P-invariant constituents of χ_Q are conjugate in C, and so every P-invariant irreducible constitituent of χ_C lies over θ.

Let T be the stabilizer of θ in K and write $S = T \cap C$. Since Q is a Sylow subgroup of K, it follows that θ has canonical extensions $\mu \in \mathrm{Irr}(S)$ and $\nu \in \mathrm{Irr}(T)$, and we have $\nu_S = \mu$. Also, S, T, μ and ν are all P-invariant. Now $\chi \in \mathrm{Irr}(K|\theta)$, and by Gallagher's theorem, the Clifford correspondent of χ with respect to θ has the form $\nu\beta$ for some character $\beta \in \mathrm{Irr}(T/Q)$. Since β is uniquely determined by χ and θ, it follows that β is P-invariant.

Now P acts on the p'-group T/Q, and $S/Q = \mathbf{C}_{T/Q}(P)$, and thus the Glauberman correspondence applies. Let $\gamma \in \mathrm{Irr}(S/Q)$ be the Glauberman correspondent of β, so that γ is the unique irreducible constituent of β_S having p'-multiplicity, and in fact $[\beta_S, \gamma] \equiv \pm 1 \mod p$. (See Theorem 13.14 of [2].) By Gallagher's theorem, $\gamma\mu$ is irreducible, and thus $\xi = (\gamma\nu)^C$ is irreducible by the Clifford correspondence. Also, $\xi \in \mathrm{Irr}(C|\theta)$ and ξ is P-invariant. We will show ξ is the unique P-invariant irreducible constituent of χ_C with multiplicity not divisible by p.

We can write $\beta_S = a\gamma + p\Delta$, where $a \equiv \pm 1 \mod p$ and Δ is a possibly zero character of S. Then $(\beta\nu)_S = a\gamma\mu + p\Delta\mu$. Let $\eta \in \mathrm{Irr}(C|\theta)$ be arbitrary. Then the Clifford correspondent of η with respect to θ has the form $\delta\mu$, for some character $\delta \in \mathrm{Irr}(S/Q)$. We have

$$[\chi_C, \eta] = \left[\chi_C, (\delta\mu)^C\right] = [\chi_S, \delta\mu] = \left[(\beta\nu)_S, \delta\mu\right] = [\beta_S\mu, \delta\mu] = [\beta_S, \delta] = [a\gamma + p\Delta, \delta],$$

where the third equality holds because $\beta\nu$ is the Clifford correspondent of χ with respect to θ, and so χ lies over $\beta\nu$ and $[\chi_Q, \theta] = [(\beta\nu)_Q, \theta]$, and therefore every irreducible character of S that lies over θ has multiplicity in χ_S equal to its multiplicity in $(\beta\nu)_S$. The fifth equality holds by Gallagher's theorem.

It follows that the multiplicity of η in χ_C fails to be divisible by p if and only if $\delta = \gamma$, or equivalently, $\eta = \xi$, and in that case, the multiplicity is congruent to ± 1 modulo p. In particular, ξ is the unique irreducible constituent of χ_C that lies over θ and has multiplicity not divisible by p, and $[\chi_C, \xi] \equiv \pm 1 \mod p$, as claimed.

Since every irreducible constituent of χ_C different from ξ is either not P-invariant or has multiplicity divisible by p, we can write

$$\chi_C = [\chi_C, \xi]\xi + p\Xi + \Gamma,$$

where all irreducible constituents of Ξ are P-invariant and no irreducible constituent of Γ is P-invariant. Since χ is P-invariant and uniquely determines Γ, it follows that Γ is P-invariant, and thus it is a sum of nontrivial P-orbit sums of irreducible characters. In particular, $\Gamma(1)$ is divisible by p, and thus

$$\chi(1) \equiv [\chi_C, \xi]\xi(1) \equiv \pm\xi(1) \mod p.$$

We now have a well-defined map $\chi \mapsto \xi$ from the set of P-invariant members of $\mathrm{Irr}(K)$ to the set of P-invariant members of $\mathrm{Irr}(C)$, and our assertions about multiplicities and degrees hold. It remains to show that this map is a bijection. For injectivity, apply the above reasoning to a P-invariant character $\chi' \in \mathrm{Irr}(K)$, where χ' lies over a P-invariant character $\theta' \in \mathrm{Irr}(Q)$. This yields β' in place of β and γ' in place of γ, and from these, we obtain $\xi' = (\gamma'\mu')^C$ in place of ξ. If $\xi' = \xi$, we can replace θ' by a C-conjugate and assume that $\theta' = \theta$, and thus $\mu' = \mu$. It follows that $\gamma'\mu = \gamma\mu$, so $\gamma' = \gamma$ by Gallagher's theorem. We deduce that $\beta' = \beta$ since the Glauberman correspondence is injective. Also $\nu' = \nu$, so we have $\chi' = (\beta'\nu')^G = (\beta\nu)^G = \chi$, and this shows that our map is injective.

For surjectivity, let $\xi \in \mathrm{Irr}(C)$ be P-invariant, and choose a P-invariant irreducible constituent θ of ξ_Q. With μ and ν as before, let $\gamma\mu$ be the Clifford correspondent of ξ in $\mathrm{Irr}(S|\theta)$, where $\gamma \in \mathrm{Irr}(S/Q)$. Then γ is the Glauberman correspondent of some P-invariant character $\beta \in \mathrm{Irr}(T/Q)$, and we let $\chi = (\beta\nu)^G$. Then $\chi \in \mathrm{Irr}(K|\theta)$ is P-invariant, and we see that χ maps to ξ under our map. This completes the proof. $\square$

We mention that one can prove an analogous result for classes: there is a natural bijection from the set of P-invariant classes of K onto the set of P-invariant classes of C.

Proof of Theorem D. Let $K = \mathbf{O}^{2'}(G)$ and $Q = P \cap K$, so that $Q \lhd G$. Let $C/Q = \mathbf{C}_{K/Q}(P)$, and observe that $C = N \cap K$, and thus $|N : C|$ is odd. Given a real character $\chi \in \mathrm{Irr}(G)$, we will construct a uniquely determined character $\xi \in \mathrm{Irr}(N)$, and we shall see that ξ is real and has p'-degree, and that in fact, $\xi(1) \equiv \pm\chi(1) \bmod p$. First, observe that since p does not divide $\chi(1)$ by hypothesis, there exists a P-invariant irreducible constituent θ of χ_K, and θ is real by Lemma 2.1. Also, we argue that the N-conjugacy class of θ is uniquely determined by χ. To see this, suppose that θ' is also a P-invariant irreducible constituent of χ_K. Then $\theta' = \theta^g$ for some element $g \in G$, and thus P and P^g are contained in the stabilizer of θ'. We can thus write $P^{gx} = P$ for some element x stabilizing θ', and we have $gx \in N$ and $\theta^{gx} = \theta'$, as required.

By Theorem 5.1, we have a canonical bijection from the set of P-invariant irreducible characters of K onto the set of P-invariant irreducible characters of C. The image α of θ under this map is the unique P-invariant irreducible constituent of θ_C having multiplicity not divisible by p, and we know that $\alpha(1) \equiv \pm\theta(1) \bmod p$. Also, since the map $\theta \mapsto \alpha$ is canonically determined and θ is real, it follows that α is real. Furthermore, because θ is uniquely determined up to conjugacy by elements of N, the same is true of α, and finally, since α is real and N/C has odd order, it follows by Lemma 2.2(b) that α^C has a unique real irreducible constituent ξ. Since α is uniquely determined up to conjugacy by elements of N, we see that ξ is uniquely determined, and we have constructed a map $\chi \mapsto \xi$ from the real irreducible characters of G to the real irreducible characters of N.

Now let T be the stabilizer of α in N, and note that α extends to T by Lemma 2.2(a). Then $\xi(1) = |N : T|\alpha(1)$ by the Clifford correspondence. We show next that in fact $\xi(1) \equiv \pm\chi(1)$, and thus in particular, $\xi(1)$ is not divisible by p. To see this, let $H = NK$ and let ψ be the unique real irreducible constituent of θ^H. Since θ and α uniquely determine each other by Theorem 5.1, it follows that T stabilizes θ, and in fact, KT is the full stabilizer of θ in H. Since θ extends to its stabilizer in H by Lemma 2.2(b), we have $\psi(1) = |H : KT|\theta(1)$ by the Clifford correspondence. But $|H : KT| = |N : T|$, and so we have $\psi(1) = |N : T|\theta(1)$. Recall now that $\alpha(1) \equiv \pm\theta(1) \bmod p$. Then

$$\xi(1) = |N : T|\alpha(1) \equiv \pm|N : T|\theta(1) \equiv \pm\psi(1) \equiv \pm\chi(1) \mod p,$$

where the final congruence holds by Corollary 2.3.

At this point, we have established the existence of a map $\chi \mapsto \xi$ from the set of real characters in $\mathrm{Irr}(G)$ into the set of real characters with p'-degree in $\mathrm{Irr}(N)$, and we have shown that $\xi(1) \equiv \pm\chi(1) \bmod p$, as required. To see that this map is injective, suppose that also χ' maps to ξ, and let $\theta' \in \mathrm{Irr}(K)$ and $\alpha' \in \mathrm{Irr}(C)$ play the roles of θ and α in the construction of the image of χ' under our map. Then α' lies under ξ, so $\alpha' = \alpha^n$ for some element $n \in N$. Since the action of N respects the bijection from P-invariant irreducible characters of K to P-invariant irreducible characters of C, it follows that $\theta' = \theta^n$, and thus both χ and χ' lie over θ. By Lemma 2.2(b), however, only one real member of $\mathrm{Irr}(G)$ can lie over θ, and thus $\chi = \chi'$, and our map is injective.

Finally, for surjectivity, let ξ be any real character with p'-degree lying in $\mathrm{Irr}(N)$. We can then choose a P-invariant irreducible constituent α of ξ_C, and thus α is real and corresponds to some P-invariant character $\theta \in \mathrm{Irr}(K)$ under the bijection of Theorem 5.1. Also, θ is real because α is real, and thus there exists a real irreducible constituent χ of θ^G. Then $\chi \mapsto \xi$ under our map, and the proof is complete. $\square$

References

[1] S. Dolfi, G. Navarro, P.H. Tiep, Primes dividing the degrees of the real characters, Math. Z. 259 (2008) 755–774.
[2] I.M. Isaacs, Character Theory of Finite Groups, corrected reprint of 1976 original, AMS Chelsea Publishing, Providence, RI, 2006.
[3] I.M. Isaacs, Orbit sizes, local subgroups and chains of p-groups, J. Aust. Math. Soc. 71 (2001) 325–337.
[4] I.M. Isaacs, G. Navarro, New refinements of the McKay conjecture for arbitrary finite groups, Ann. of Math. (2) 156 (2002) 333–344.
[5] N. Itô, Some studies on group characters, Nagoya Math. J. 2 (1951) 17–28.

[6] N. Itô, On the degrees of irreducible representations of a finite group, Nagoya Math. J. 3 (1951) 5–6.
[7] G.O. Michler, Brauer's conjectures and the classification of finite simple groups, in: Lecture Notes in Math., vol. 1178, Springer, Berlin, 1986, pp. 129–142.
[8] G. Navarro, P.H. Tiep, Rational irreducible characters and rational conjugacy classes in finite groups, Trans. Amer. Math. Soc. 360 (2008) 2443–2465.
[9] T. Wilde, The real part of the character table of a finite group, Comm. Algebra 35 (2007) 4042–4056.