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# Hardy Inequalities and Some Critical Elliptic and Parabolic Problems

J. P. García Azorero and I. Peral Alonso*

*Departamento de Matemáticas, U.A.M., 28049 Madrid, Spain*

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To Eugene Fabes, in Memoriam

We study the behaviour of the nonlinear critical *p-heat equation* (and the related stationary *p-laplacian equation*)



$$\begin{cases} \cdots \dfrac{\ }{|x|^p} & \\ u(x,0)=f(x), & x \in \Omega \\ u(x,t)=0, & x \in \partial\Omega,\ t>0, \end{cases} \tag{1}$$

where $-\Delta_p u \equiv -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $f(x) \geqslant 0$ verifying convenient regularity assumptions, $\Omega$ is a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$, and $1<p<N$. The analysis reveals that the behaviour depends on $p$. The results depend in general on the relation between $\lambda$ and the best constant in Hardy's inequality. 

## 1. INTRODUCTION

In several *reaction-diffusion* problems involving the heat equation with *supercritical* reaction term, it appears a stationary singular solution. For instance, this is the case for

$$u_t - \Delta u = \lambda e^u, \qquad \text{and} \qquad u_t - \Delta u = \lambda u + u^{\alpha-1}, \qquad \text{where} \qquad \frac{2N}{N-2} < \alpha.$$

(See [19] and [23] respectively.) The linearization on this singular solution gives a linearized equation of the type

$$u_t - \Delta u = \frac{\lambda}{|x|^2}\, u. \tag{2}$$

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(See [24] for the details in the exponential case.) This linear equation is a borderline case with respect to the classical theory of parabolic equations, namely, the potential $\lambda/|x|^2$ belongs to $L^r_{loc}$ if and only if $1 \leqslant r < N/2$; therefore the standard uniqueness and regularity theories do not apply to this case. For this reason the study of this kind of equation is interesting. The linear equation (2) was studied by Baras–Goldstein in [5], where it was obtained the behaviour of the solutions depending on the values of the parameter $\lambda$. More precisely Baras–Goldstein prove that the critical value $\lambda_N = (N-2)^2/4$, determines the behaviour of the solutions to the equation (2). We point out that the constant $\lambda_N$ is the best constant in the classical Hardy inequality, see [21], and we remark that such a constant is not attained in the Sobolev space. This remark and some applications are pointed out in [24]. In this sense we can expect some lack of compactness.

We reformulate below a result by Baras and Goldstein in the particular case concerning with the problem that we will discuss in this paper. (See [5] for more details and extensions).

THEOREM. (*Baras–Goldstein*).
*Consider the initial value problem with Dirichlet boundary data,*

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda}{|x|^2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad N \geqslant 3, \quad t > 0, \quad \lambda \in \mathbb{R} \\ u(x,0) = f(x), & x \in \Omega, \quad f \in L^2, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{3}$$

*where $\Omega$ is a domain such that $0 \in \Omega$, then*

(i) *If $\lambda \leqslant \lambda_N$, problem* (3) *has a solution.*

(ii) *If $\lambda > \lambda_N$ problem* (3) *has no local solution for any $f > 0$.*

In this paper we study the same kind of problems for the nonlinear *p-heat equation*, namely, the nonlinear parabolic equation with diffusion term given by the *p-laplacian*. More precisely we study

$$\begin{cases} u_t - \Delta_p u = \dfrac{\lambda}{|x|^p}\, u^{p-1}, & x \in \Omega, \quad t > 0, \quad \lambda \in \mathbb{R} \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{4}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$, $1 < p < N$ and $f(x) \geqslant 0$, with convenient regularity assumptions. In this case we cannot see as a motivation the *linearization* in a strict sense, but nevertheless this equation is interesting by itself because it provides some new phenomena.

This parabolic problem and the associated elliptic problem are *critical* in a different way that the Sobolev critical exponent problems. In fact, the potential $\lambda/|x|^p$ belongs to $L^r_{loc}$ if and only if $r < N/p$, that corresponds to the same situation as in the linear case. Moreover some lack of compactness appears in the associated elliptic problem, and then, the results depend on some relation between the parameter $\lambda$ with $p$ and the dimension $N$. For the evolution problem (4) with $\lambda = 0$, it is well known that the behaviour depends strongly on $p$. More precisely: (i) If $p < 2$ the problem is singular and then for small $p$ there are extinction in finite time. (ii) If $p > 2$ there are *finite speed of propagation.* (See the details in the book [12]).

Assume now $\lambda > 0$. As one can expect, if $p > 2$, the behaviour is in some sense similar to the linear case, namely, in this case there are *good* $\lambda$, $\lambda \leqslant \lambda_{N,p}$, and *bad* $\lambda$, $\lambda > \lambda_{N,p}$, where $\lambda_{N,p} = ((N-p)/p)^p$ is the best constant in the corresponding Hardy's inequality in $W^{1,p}(\mathbb{R}^N)$. In fact we get *instantaneous blow-up* for $\lambda > \lambda_{N,p}$ in all the norm $L^r$ (even in a stronger sense that will be precise later). The biggest differences appear if $1 < p < 2$. In this case the behaviour of the problem is the same for *good* $\lambda$, namely there exists global solution in the classical sense, but for *bad* $\lambda$ is not so much bad, going from better to worst as $p$ goes from 1 to 2. We will prove that there exists strong global solution for $1 < p < 2N/(N+2)$ and there exists weak global solution if $2N/(N+2) \leqslant p < 2N/(N+1)$. We can say that in these cases the expected *blow-up* takes place in a weaker sense: some $L^r$ norms are finite for all time, while the $L^\infty$ norm can blow-up instantaneously.

Finally if $2 > p \geqslant 2N/(N+1)$ we obtain that there exists solution away from the origin, i.e., we have solution to the problem in distributions sense in $(\Omega - \{0\}) \times [0, T]$, but in general the solution is not in $L^1_{loc}(\Omega)$.

In some way the dependence on Hardy's inequality is weaker if $p$ is close to 1, while for $p \geqslant 2N/(N+1)$ Hardy's inequality plays an strong role in the behaviour of the problem (4).

We will organize the contents of the paper in the following way. For convenience of the reader, in the next section we will study the Hardy Inequality. The associated elliptic problem will be studied in Section 3. It is interesting to emphasize that it is in some sense a *critical problem* for which the classical variational approach does not apply. We give also a Pohozaev type nonexistence result for the critical potential $\lambda|x|^{-p}$ and an existence result for subcritical potentials that makes clear the meaning of *criticallity* in this context. Section 4 will be devoted to the positive result, namely the *good* $\lambda$ case, and Section 5 to the study of *instantaneous and complete blow up* in the case $p > 2$. To prove this blow up result we use the separation of variables method and we solve the related elliptic problem by using a *bifurcation from infinity* result that allow us to construct a convenient subsolution to the parabolic problem. The case of *bad* $\lambda$ and $1<p<2$ will be studied in Section 6. Section 7 deals with the positive result in the particular case $\lambda=\lambda_{N,p}$ for $p\geqslant 2N/(N+1)$, that is not solved by the methods in Sections 4 and 6. Finally, some remarks on uniqueness and nonuniqueness will be given in the last section. The main result in this last section shows that in the case $1<p<2$ the critical $\lambda_{N,p}$ is also critical for the uniqueness.

We point out that the blow-up discussed in this paper appears from *spectral properties* of the elliptic operator with singular potential $\lambda|x|^{-p}$. The blow-up depending on the relation between growth and diffusion has been widely studied in the past years. An interesting reference could be the recent paper [17].

## 2. ON A HARDY INEQUALITY

The main point of this section is to discuss the following classical result, essentially dues to Hardy. ( See [21]). By completeness we include the proof.

LEMMA 2.1. *Assume* $1<p<N$, *then if* $u\in W^{1,p}(\mathbb{R}^N)$

1. $u/|x|\in L^p(\mathbb{R}^N)$.
2. (*Hardy Inequality*)

$$\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}\,dx\leqslant C_{N,p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx$$

*with* $C_{N,p}=(p/(N-p))^p$.

3. *The constant* $C_{N,p}$ *is optimal.*

*Proof.*

*Step* 1. A density argument allows us to consider only smooth functions $u\in C_0^\infty(\mathbb{R}^N)$. Under this hypothesis we have the following identity

$$|u(x)|^p=-\int_1^\infty\frac{d}{d\lambda}\,|u(\lambda x)|^p\,d\lambda=-p\int_1^\infty u^{p-1}(\lambda x)\langle x,\nabla u(\lambda x)\rangle\,d\lambda.$$

By using Hölder inequality, it follows that

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx = -p \int_1^\infty \int_{\mathbb{R}^N} \frac{u^{p-1}(\lambda x)}{|x|^{p-1}} \left\langle \frac{x}{|x|}, \nabla u(\lambda x) \right\rangle dx\, d\lambda$$

$$= -p \int_1^\infty \frac{d\lambda}{\lambda^{N+1-p}} \int_{\mathbb{R}^N} \frac{u(y)^{p-1}}{|y|^{p-1}} \frac{\partial u(y)}{\partial r}\, dy$$

$$= -\frac{p}{N-p} \int_{\mathbb{R}^N} \frac{u(y)^{p-1}}{|y|^{p-1}} \frac{\partial u(y)}{\partial r}\, dy$$

$$\leqslant \frac{p}{N-p} \left( \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|y|^p}\, dy \right)^{(p-1)/p} \left( \int_{\mathbb{R}^N} \left| \frac{\partial u(y)}{\partial r} \right|^p dy \right)^{1/p},$$

And then we conclude that

$$\int_{\mathbb{R}^N} \frac{u^p(x)}{|x|^p}\, dx \leqslant \left( \frac{p}{N-p} \right)^p \int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx.$$

*Step* 2. Optimality of the constant. Following the idea of Hardy for the one dimensional case, we show that the best constant is $C_{N,p} = (p/N-p)^p$.

Given $\varepsilon > 0$, take the radial function

$$U(r) = \begin{cases} A_{N,p,\varepsilon} & \text{if } r \in [0,1], \\ A_{N,p,\varepsilon} r^{(p-N)/p-\varepsilon} & \text{if } r > 1, \end{cases} \tag{5}$$

where $A_{N,p,\varepsilon} = p/(N-p+p\varepsilon)$, whose derivative is

$$U'(r) = \begin{cases} 0, & \text{if } r \in [0,1], \\ -r^{-(N/p)-\varepsilon} & \text{if } r > 1. \end{cases} \tag{6}$$

By direct computation we get

$$\int_{\mathbb{R}^N} \frac{U^p(x)}{|x|^p}\, dx = \int_B \frac{U^p(x)}{|x|^p}\, dx + \int_{\mathbb{R}^N - B} \frac{U^p(x)}{|x|^p}\, dx$$

$$= A^p_{N,p,\varepsilon} \omega_N \left( \int_0^1 r^{N-1-p}\, dr + \int_1^\infty r^{-(1+p\varepsilon)}\, d \right)$$

$$= A^p_{N,p,\varepsilon} \omega_N \int_0^1 r^{N-1-p} dr + A^p_{N,p,\varepsilon} \int_{\mathbb{R}^N} |\nabla U(x)|^p\, dx,$$

where $\omega_N$ is the measure of the $(N-1)$-dimensional unit sphere. We conclude by letting $\varepsilon \to 0$. ∎

COROLLARY 2.2. *The same result is true in the unit ball $B \subset \mathbb{R}^N$.*

*Proof.* The proof of the first step is the same. The argument of optimality in the case of the unit ball proceeds by approximation as follows. First, we remark that by the invariance under dilations the optimal constant will be the same for any ball. Second, let $B_R$ be a ball with large radius. We take as test function $v(x) = \psi(x) U(x)$ where $U$ is one of the approximate optimizers explicitly given above and $\psi \in C_0^\infty(B_R)$ is a cutoff function which is identically 1 on $B_{R-1}$ with $|\nabla \psi| \leqslant m$. It is easily seen that for $R \gg 1$ the influence of $\psi$ in the calculation of Step 2 is negligible. ∎

*Remark* 2.3. Sometimes the Hardy inequality in the case $p=2$ is known as *uncertainty principle*, see [15]. We can read the Hardy inequality saying that the embedding of $W_0^{1,p}(\Omega)$ in $L^p$ with respect to the weight $|x|^{-p}$ is continuous.

By normalizing the minimizers that we use in the proof, it is easy to see that the inclusion is *non compact*. This will be the cause of many of our difficulties.

In the sequel, we will denote $\lambda_{N,p} = C_{N,p}^{-1}$.

It will be useful to compare the best constant in the Hardy inequality with the following approximating eigenvalue problems.

THEOREM 2.4. *Consider $\lambda_1(n)$ the first eigenvalue to the problem*

$$\begin{cases} -\Delta_p \psi_1 = \lambda W_n(x)\, |\psi_1|^{p-2}\, \psi_1, & x \in \Omega \subset \mathbb{R}^N, \\ \psi_1(x) = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

*where $W_n(x) = \min\{|x|^{-p}, n\}$. Then $\lambda_1(n) \geqslant \lambda_{N,p}$, and moreover $\lim_{n\to\infty} \lambda_1(n) = \lambda_{N,p}$.*

*Proof.* The first inequality follows immediatly from the definition of the first eigenvalue by the Rayleigh quotient. Also, it is easy to see that $\{\lambda_1(n)\}$ is a nonincreasing sequence; then we have to prove that the limit cannot be bigger than $\lambda_{N,p}$. Assume by contradiction that $\lim_{n\to\infty} \lambda_1(n) = \lambda_{N,p} + \rho$.

Then, we can choose $\phi \in W_0^{1,p}(\Omega)$ such that $(\int_\Omega |\nabla\phi|^p\, dx)/(\int_\Omega \phi^p\, |x|^{-p}\, dx) < \lambda_{N,p} + \rho/2$. But then $\lambda_1(n) \leqslant (\int_\Omega |\nabla\phi|^p\, dx)/(\int_\Omega \phi^p W_n(x)\, dx)$, and this is a contradiction because the last expression has to be smaller than $\lambda_{N,p} + \rho$ for $n$ large. ∎

We will study in the next section the behaviour of elliptic equations related to the Hardy inequality.

## 3. ELLIPTIC EQUATIONS WITH CRITICAL POTENTIAL

The first result in this section is an easy consequence of the Hardy inequality

LEMMA 3.1. *Consider the nonlinear operator*

$$\mathcal{L}_\lambda u \equiv -\Delta_p u - \frac{\lambda}{|x|^p} |u|^{p-2} u \tag{8}$$

*in* $W_0^{1,p}(\Omega)$. *Then*

1. *If* $\lambda \leqslant \lambda_{N,p}$, $\mathcal{L}_\lambda$ *is a positive operator.*
2. *If* $\lambda > \lambda_{N,p}$, $\mathcal{L}_\lambda$ *is unbounded from below.*

*Proof.* (1) It is obvious from the Hardy inequality. (2) An easy consequence of the optimality of the constant and a density argument is the existence of $\phi \in \mathcal{C}_0^\infty(\Omega)$ such that $\langle \mathcal{L}_\lambda \phi, \phi \rangle < 0$. We can assume that $\|\phi\|_p = 1$ and then by defining $u_\mu(x) = \mu^{N/p} \phi(\mu x)$ we have $\|u_\mu\|_p = 1$ and the homogeneity of the operator allows us to conclude that $\langle \mathcal{L}_\lambda u_\mu, u_\mu \rangle = \mu^p \langle \mathcal{L}_\lambda \phi, \phi \rangle < 0$. ∎

Taking into account the previous result, in this section we will study the following problem

$$\begin{cases} \mathcal{L}_\lambda u = f(x) \in W^{-1,p'}(\Omega), & x \in \Omega, \quad \lambda < \lambda_{N,p}, \quad \dfrac{1}{p} + \dfrac{1}{p'} = 1 \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. If $0 \notin \Omega$ then we have a classical problem with a bounded potential. So we will assume hereafter that $0 \in \Omega$.

Consider the energy functional,

$$J(u) = \int_\Omega F(x, u, \nabla u)\, dx$$

where $F(x, u, \xi) = 1/p\, |\xi|^p - \lambda/p\, (u^p/|x|^p) - f(x) u$.

The classical results in the Calculus of Variations characterize the weak lower semicontinuity of $J$ if $F(x, u, \bullet)$ is convex and $F$ verifies a lower bound: positivity, or a lower estimate by a linear combination of $\xi$, etc. (See Tonelli [29], Serrin [28], De Giorgi [11], the book by Dacorogna [10] and the references therein). However, in our case these usual hypotheses are not fulfilled.

*Variational Approach.* The energy functional,

$$J(u)=\frac{1}{p}\int_\Omega |\nabla u|^p\,dx-\frac{\lambda}{p}\int_\Omega \frac{u^p}{|x|^p}\,dx-\int_\Omega fu\,dx,$$

by the Hardy inequality, is continuous, Gateaux differentiable and coercive, namely, there exist constants $\gamma>0$ and $c\in\mathbb{R}$ such that

$$J(u)\geqslant\gamma\int_\Omega |\nabla u|^p\,dx-c.$$

Hence by the *Variational Principle of Ekeland*, see [14], we can find a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$J(u_n)\to\inf J,\qquad \text{and}\qquad J'(u_n)\to 0,\qquad \text{as}\quad n\to\infty.$$

As usually we say that $\{u_n\}_{n\in\mathbb{N}}$ is a *Palais–Smale sequence*. The coercivity of $J$ implies the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$, so we have that for some subsequence: (i) $\nabla u_n\rightharpoonup\nabla u$ in $L^p$; (ii) $u_n$ converges in $L^p$ and a.e.; (iii) $\lambda(u_n^{p-1})/|x|^p$ are bounded as Radon measures. Under these hypotheses we can apply the following convergence theorem by Boccardo and Murat, see [8].

Lemma 3.2. *Let* $\{u_n\}_{n\in\mathbb{N}}$ *verifying* (i), (ii) *and* (iii) *above. Then for some subsequence,*

$$u_{n_j}\to u,\qquad \textit{in}\quad W_0^{1,q}(\Omega),\quad q<p$$

*and*

$$\mathcal{T}_k(u_n)\to\mathcal{T}_k(u)\qquad \textit{in}\quad W_0^{1,p}(\Omega)$$

*for all* $k>0$, *where* $\mathcal{T}_k(s)=s$ *if* $|s|\leqslant k$ *and* $\mathcal{T}_k(s)=ks/|s|$ *if* $|s|\geqslant k$.

According with the previous Lemma, and by a density argument, we can prove the required compactness property. We will call such a compactness property *singular Palais–Smale condition* in the sense of the following Lemma:

Lemma 3.3. *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be the Palais–Smale sequence obtained above. Then* $\{u_n\}_{n\in\mathbb{N}}$ *satisfies the singular Palais–Smale condition, namely there exists a subsequence* $\{u_{n_j}\}_{j\in\mathbb{N}}$ *such that*

$$u_{n_j}\to u,\qquad \textit{in}\quad W_0^{1,q}(\Omega),\quad q<p.$$

An inmediate consequence of Lemma (3.3) is that $u$ is a solution of our problem in the sense of distributions. Moreover by density and taking into account that $u \in W_0^{1,p}(\Omega)$, we conclude that $u$ is solution in the sense of $W_0^{1,p}(\Omega)$.

Finally the homogeneity of the problem implies that $u$ is a minimum for $J$. Consider

$$J(u_k) - \frac{1}{p}\langle J'(u_k), u_k\rangle = \left(\frac{1}{p} - 1\right)\int_\Omega f u_k$$

where in the last term we can pass to the limit by weak $W_0^{1,p}(\Omega)$-convergence. Therefore

$$\begin{aligned}\inf J = \lim_{k\to\infty} J(u_k) &= \lim_{k\to\infty}\left(J(u_k) - \frac{1}{p}\langle J'(u_k), u_k\rangle\right)\\ &= \left(\frac{1}{p} - 1\right)\int_\Omega fu = J(u) - \frac{1}{p}\langle J'(u), u\rangle = J(u)\end{aligned}$$

We would like to point out that this approach solves the minimization problem, namely, the solution is obtained as a minimum of $J$. Also it is interesting to emphasize that this approach will be used to study problems with unbounded energy functionals in the end of this section.

*Truncature Approach.* We will use as an alternative to the variational approach a truncation argument. More precisely, we consider for $n \in \mathbb{N}$ the problems

$$\begin{cases} -\Delta_p u - \lambda W_n(x) u^{p-1} = f, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{10}$$

where $W_n(x) = \min\{1/|x|^p, n\}$ and $f \in W^{-1,p'}(\Omega)$. Problem (10) can be solved by standard minimization arguments, getting a subsequence $\{u_n\}_{n\in\mathbb{N}}$ which verifies:

1. $\nabla u_n \rightharpoonup \nabla u$ weakly in $W_0^{1,p}(\Omega)$, by Hardy inequality.
2. $\lambda W_n(x) u_n^{p-1} \to \lambda u^{p-1}/|x|^p$ in $L^1$.

At this point, we can use again the compactness result by Boccardo–Murat (see [8]), and by density we get:

THEOREM 3.4. *Problem* (9) *has a weak solution* $u \in W_0^{1,p}(\Omega)$.

*Remark* 3.5. (I) The uniqueness in the case $p = 2$ is obvious.

(II) If $p > 2$ the uniqueness is in general not true as the following argument shows (see [13]). Assume $B_{2R} \subset \Omega$ a ball of radius $2R$ such that

$0 \notin B_{2R}$ and consider $u_0 \in \mathscr{C}_0^2(\Omega)$ such that $u=k>0$ in $B_R$ and $u_0=0$ in $\Omega - B_{2R}$. Consider $f(x)=\mathscr{L}_\lambda u$ with $\lambda<\lambda_{N,p}$. In this way $f \in W^{-1,p'}(\Omega)$ and then we can find a solution $v$ by minimization of $J$ as in the Theorem (3.4). Now it is clear that such $v \neq u_0$, because $u_0$ cannot be a minimum for $J$: in fact, taking into account that $p>2$ and the choice of $u_0$, $J$ is twice differentiable and we see that

$$\langle J''(u_0)z, z\rangle = (p-1)\left(\int_\Omega |\nabla u_0|^{p-2}\, |\nabla z|^2\, dx - \lambda \int_\Omega \frac{|u_0|^{p-2}}{|x|^p}\, z^2\, dx\right).$$

Hence, in particular if $z \in \mathscr{C}_0^\infty(\Omega)$ is such that $supp(z) \subset B_R$,

$$\langle J''(u_0)z, z\rangle = -\lambda(p-1)\int_\Omega \frac{|u_0|^{p-2}}{|x|^p}\, z^2\, dx < 0,$$

and then $u_0$ is not a minimum for $J$.

(III) The uniqueness in the case $1<p<2$ seems to be an open problem. ∎

After the last remark we see that no comparison theorem is possible in general. But the counterexample is for data which changes sign; however if $f \geqslant 0$ in $\Omega$ then we can show that the sequence $u_n$ of minimal solutions is nondecreasing, i.e., $u_n \leqslant u_{n+1}$ almost everywhere, and this implies a stronger result, namely the strong convergence in $W_0^{1,p}(\Omega)$. In fact, if $f \geqslant 0$, we have,

$$\begin{aligned}
&\int_\Omega \langle |\nabla u|^{p-2}\, \nabla u, \nabla \mathscr{T}_k(u)\rangle\, dx \\
&\quad = \lim_{n\to\infty} \int_\Omega \langle |\nabla u_n|^{p-2}\, \nabla u_n, \nabla \mathscr{T}_k u\rangle\, dx \\
&\quad = \lim_{n\to\infty} \left(\int_\Omega \lambda W_n(x)\, u_n^{p-1}\, \mathscr{T}_k(u)\right) dx + \int_\Omega f\mathscr{T}_k(u) \\
&\quad = \left(\int_\Omega \lambda \frac{u^{p-1}\mathscr{T}_k(u)}{|x|^p}\, dx + \int_\Omega f\mathscr{T}_k(u)\, dx\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\nabla u|^p\, dx &= \lim_{k\to\infty} \int_\Omega \langle |\nabla u|^{p-2}\, \nabla u, \nabla \mathscr{T}_k u\rangle\, dx \\
&= \lim_{k\to\infty} \left(\lambda \int_\Omega \frac{u^{p-1}\mathscr{T}_k(u)}{|x|^p}\, dx + \int_\Omega f\mathscr{T}_k(u)\right)
\end{aligned}$$

$$= \lambda \int_\Omega \frac{u^p}{|x|^p}\, dx + \int_\Omega fu\, dx$$

$$= \lim_{n\to\infty} \left( \lambda \int_\Omega W_n(x)\, |u_n|^p\, dx + \int_\Omega fu_n\, dx \right)$$

$$= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^p\, dx.$$

The last convergence of the norms and the a.e convergence implies that $u_n$ converges strongly to $u$ in $W_0^{1,p}(\Omega)$.

The strong convergence in the case $p = 2$ is straightforward.

We will use again the variational approach to study the case of unbounded funcionals, more precisely the existence of solution via the *Mountain Pass Lemma.* (See [2]). For instance the following result holds.

THEOREM 3.6. *Consider the Dirichlet problem*

$$-\Delta_p u = \lambda\, |u|^{p-2}\, u\, |x|^{-p} + |u|^{\alpha-2}\, u,$$

$$\lambda < \lambda_{N,p}, \; p < \alpha < Np/(N-p), \qquad u\,|_{\partial\Omega} = 0.$$

*There exists at least a positive solution* $u \in W_0^{1,p}(\Omega)$.

*Proof.* The proof of this theorem follows closely the previous variational approach; instead of minimizing and using the variational principle of Ekeland, the geometry of the energy functional allows us to use the Mountain Pass Lemma of Ambrosetti–Rabinowitz. In fact since $\lambda < \lambda_{N,p}$ then

$$J(u) \equiv \frac{1}{p} \int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{p} \int_\Omega \frac{|u|^p}{|x|^p}\, dx - \frac{1}{\alpha} \int_\Omega |u|^\alpha\, dx$$
$$\geqslant \gamma \int_\Omega |\nabla u|^p\, dx - C(p,\alpha) \left( \int_\Omega |\nabla u|^p\, dx \right)^{\alpha/p}.$$

Then we found the required Palais Smale sequence, which is easy to show has to be bounded in $W_0^{1,p}(\Omega)$. Using again [8], we get the compactness result, in the sense of the *singular Palais Smale condition* defined above. Then we can found a function $u \in W_0^{1,p}(\Omega)$ which is the strong limit in $W_0^{1,q}(\Omega)$, $q < p$, and therefore is a solution in the sense of distributions. Since $u \in W_0^{1,p}(\Omega)$, by density we get that $u$ is a weak solution. Finally, by homogeneity, and taking into account that we have strong convergence in $L^\alpha$, we can see that $u \not\equiv 0$:

$$c = \lim_{n \to \infty} J(u_n) = \lim_{n \to \infty} \left(J(u_n) - \frac{1}{p} \langle J'(u_n), u_n \rangle\right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{p} - \frac{1}{\alpha}\right) \int_\Omega |u_n|^\alpha \, dx = \left(\frac{1}{p} - \frac{1}{\alpha}\right) \int_\Omega |u|^\alpha \, dx$$

and we conclude. ∎

The following nonexistence result shows from a different point of view the critical character of the problem.

LEMMA 3.7. *Consider the problem*

$$\begin{cases} -\Delta_p u = \lambda \dfrac{u^{p-1}}{|x|^p} + \gamma f(u), & x \in \Omega, \quad \lambda > 0, \\ u(x) = 0, & \text{on } \partial\Omega \end{cases} \tag{11}$$

*where $\Omega \subset \mathbb{R}^N$ is bounded, starshaped with respect to the origin, $f$ is a continuous function and*

$$\gamma \left( NF(u) - \frac{N-p}{p} uf(u) \right) \leqslant 0, \qquad F(u) = \int_0^u f(s) \, ds.$$

*Then* (11) *has no positive solution $u \in W_0^{1,p}(\Omega)$.*

*Proof.* We will use a Pohozaev type identity. The idea consists on multiply the equation by $\langle x, \nabla u \rangle$ and integrate by parts. (Observe that the regularity of $u$ does not suffices to justify this calculus directly but as Pohozaev points out in [25] the results are valid also for weak solutions; an argument of aproximation which justifies the previous observation can be seen for instance in [20]. See also [26].)

$$\left(\frac{p-1}{p}\right) \int_{\partial\Omega} |\nabla u|^p \langle x, \nu \rangle \, d\sigma + \left(\frac{N-p}{p}\right) \int_\Omega |\nabla u|^p \, dx$$

$$= \lambda \left(\frac{N-p}{p}\right) \int_\Omega \frac{u^p}{|x|^p} \, dx + \gamma N \int_\Omega F(u) \, dx \tag{12}$$

where $\nu$ is the outwards normal to $\partial\Omega$. On the other hand, multipliying the equation by $u$ and integrating

$$\int_\Omega |\nabla u|^p \, dx = \lambda \int_\Omega \frac{u^p}{|x|^p} \, dx + \gamma \int_\Omega uf(u) \, dx \tag{13}$$

Both identities give

$$\left(\frac{p-1}{p}\right)\int_{\partial\Omega}|\nabla u|^p\langle x,\nu\rangle\,d\sigma=\gamma\int_{\Omega}\left(NF(u)-\frac{N-p}{p}\,uf(u)\right)dx$$

The conclusion is now obvious. ▮

Notice the contrast of Lemma 12 with the results in the papers [9] and [23] where the case $p=2$ with constant potential is considered. (See also [18]).

*Remark* 3.8. In the case $\Omega=\mathbb{R}^N$, $p=2$ and $f(u)=u^{(N+2)/(N-2)}$ an existence result can be seen in [22], Th. I.3, pg. 179. However the previous Lemma proves that this *doubly critical* problem has no positive solution in bounded starshaped domains. This means that the term with the potential cannot be seen as a lower order perturbation of the term with critical Sobolev exponent, although this is the case in terms of the growth in $u$. We can explain this result in an easy way: this behavior is a consequence of the noncompactness of the term $\lambda(u^p/|x|^p)$ in $W_0^{1,p}(\Omega)$, as we point out in Remark (2.3).

We will prove that in fact the potential $\lambda\,|x|^{-p}$ is critical in the sense of the remark above by considering $\lambda\,|x|^{-q}$ with $0<q<p$.

It is known that for the potential $\lambda\,|x|^{-q}$, which belongs to $L^r$ for some $r>N/p+\sigma$, there exists a first isolated and simple eigenvalue $\lambda_1$, for the corresponding Dirichlet problem. We will show that this subcritical potential produces a similar effect to the case $q=0$, in the sense that the lack of compactness arises only from the highest power term. Moreover there are also *bad dimensions*, $p<N<p^2-(p-1)q$. This interval coincides with the known result in the case $q=0$, and decreases when $q\to p$, dissapearing the solution in the limit case $q=p$, according to Lemma 3.7.

THEOREM 3.9. *Consider the Dirichlet problem*

$$\begin{cases}-\Delta_p u=\lambda\,|u|^{p-2}\,u\,|x|^{-q}+|u|^{p^*-2}\,u, & \lambda<\lambda_1\\ u\,|_{\partial\Omega}=0, \qquad 0<q<p, \quad 1<p<N, & p^*=Np/(N-p).\end{cases}\tag{14}$$

*If* $N\geqslant p^2-(p-1)q$, *then there exists at least a positive solution* $u\in W_0^{1,p}(\Omega)$.

*Proof.* The arguments are similar to those in [18] and then we will be sketchy. The geometry of the energy functional, $J$, satisfies the requirements of the *Mountain Pass Theorem* as can be checked following the same calculations as in Theorem 12. Then by using the *concentration-compactness method* by P.L. Lions, see [22], we get a local Palais–Smale condition.

More precisely, let $S$ the optimal constant in the Sobolev embedding and given a Palais–Smale sequence such that

$$J(u_k) \to c < \frac{S^{N/p}}{N}, \; J'(u_k) \to 0, \qquad \text{in} \quad W^{-1,p'}(\Omega)$$

there exists a convergent subsequence. The only thing to check (and this is the main point), is the existence of a Palais–Smale sequence at this subcritical energy level. To get this particular sequence it is well known that it is sufficient to find a direction $v_\varepsilon \in W_0^{1,p}(\Omega)$ for which

$$\sup_{t>0} J(tv_\varepsilon) < c_0 < \frac{S^{N/p}}{N}, \tag{15}$$

because in this way the minimax critical value in the Mountain Pass Theorem verifies $c < c_0$. The natural election is

$$v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{p^*}}, \qquad \text{where} \quad u_\varepsilon = \frac{\phi(x)}{(\varepsilon + |x|^{p/(p-1)})^{(N-p)/p}}$$

are the minimizers of the Sobolev inclusion with a convenient truncation $\phi \in \mathscr{C}_0^\infty$ to adapt their support to $\Omega$. As usual we compute:

1. $\|\nabla v_\varepsilon\|_p^p \approx S + c_1 \varepsilon^{(N-p)/p}$
2. $\int_\Omega (v_\varepsilon^p/|x|^q)\, dx \approx c_2 \varepsilon^{(p^2-(p-1)q-p)/p}$ if $N > p^2 - (p-1)q$.
3. $\int_\Omega (v_\varepsilon^p/|x|^q)\, dx \approx c_2 \varepsilon^{(N-p)/p} |\log \varepsilon|$ if $N = p^2 - (p-1)q$.
4. $\int_\Omega (v_\varepsilon^p/|x|^q)\, dx \approx c_2 \varepsilon^{(N-p)/p}$ if $N < p^2 - (p-1)q$.

Following the proof of Theorem 3.3 in [18] the decay as $\varepsilon \to 0$ of $\|\nabla v_\varepsilon\|_p^p$ must be faster than the decay of $\int_\Omega (v_\varepsilon^p/|x|^q)\, dx$, to get the inequality (15) and this is true if $N \geqslant p^2 - (p-1)q$. So we conclude. ∎

*Remark* 3.10. Something more can be said in the case $N < p^2 - (p-1)q$. Following the same argument as in the proof of Theorem 3 in [4] we find that there exists a positive solution to problem (14) if $\lambda \in (\lambda_1 - A, \lambda_1)$, where $A = S(\int_\Omega |x|^{-Nq/p}\, dx)^{-p/N}$.

## 4. THE PARABOLIC PROBLEM: THE CASE $\lambda < \lambda_{N,p}$

In this section we prove the following result

THEOREM 4.1. *Consider the initial value problem with zero Dirichlet boundary data,*

$$\begin{cases} u_t - \Delta_p u = \dfrac{\lambda}{|x|^p}\,|u|^{p-2}\,u, & x \in \Omega \subset \mathbb{R}^N, \quad t>0, \quad p<N, \\ u(x,0)=f(x), & x\in\Omega \\ u(x,t)=0, & x\in\partial\Omega, \quad t>0. \end{cases} \tag{16}$$

*where* $\lambda < \lambda_{N,p}$ *and* $f \in L^2(\Omega)$. *Then problem* (16) *has a global solution*

$$u \in L^\infty([0,\infty), L^2(\Omega)) \cap L^p((0,T), W^{1,p}(\Omega)), \qquad \textit{for all} \quad T>0$$

*and*

$$u_t \in L^2((\varepsilon,\infty)\times\Omega), \qquad \textit{for all} \quad \varepsilon>0.$$

*Proof.* Consider $W_n(x)=\min\{n, 1/|x|^p\}$. Then the solution for the truncated problem, which we denote $u_n$, verifies

$$\begin{aligned} &\int_\Omega |u_n(x,T)|^2\,dx + \int_0^T \int_\Omega |\nabla u_n(x,t)|^p\,dx\,dt \\ &\quad = \int_\Omega |f(x)|^2\,dx + \lambda \int_0^T \int_\Omega W_n(x)\,|u_n(x,t)|^p\,dx\,dt \\ &\quad \leqslant \int_\Omega |f(x)|^2\,dx + \lambda \int_0^T \int_\Omega \frac{|u_n(x,t)|^p}{|x|^p}\,dx\,dt \\ &\quad \leqslant \int_\Omega |f(x)|^2\,dx + \lambda C_{N,p} \int_0^T \int_\Omega |\nabla u(x,t)|^p\,dx\,dt, \end{aligned}$$

by Hardy inequality. As a conclusion, for some $\alpha>0$ we have

$$\int_\Omega |u_n(x,T)|^2\,dx + \alpha \int_0^T \int_\Omega |\nabla u_n(x,t)|^p\,dx\,dt \leqslant \int_\Omega |f(x)|^2\,dx$$

So we get a global solution defined as the limit of solutions of the problems with truncated potential. In fact, we can pass to the limit by using Theorem 4.1 in [8] and taking into account that $u_n$ is nondecreasing. More precisely we have that the limit $u$ verifies $u \in L^\infty([0,\infty), L^2(\Omega)) \cap L^p((0,T), W_0^{1,p}(\Omega))$. The estimate for $u_t$ can be obtained by multiplying the corresponding equations by $u_{nt}$ and taking limits. ∎

*Remark* 4.2. It is interesting to point out that in the case of a bounded domain, $\lambda \leqslant \lambda_{N,p}$, and $1<p<2$ the solutions have finite time of extinction. See [1] for more details and related results for the Cauchy problem.

## 5. THE CASE $\lambda > \lambda_{N,p}$ AND $p > 2$: BLOW-UP

We will concentrate our attention in the case $p > 2$. We will show the *complete* and *instantaneous* blow up in the case of initial data positive close to the origin. (We will precise below the meaning of *complete blow up* in this case).

The idea is looking for subsolutions of the truncated problems by using the method of separation of variables, i.e., considering solutions to the equation of the type $u(x, t) = T(t)\, X(x)$. By substitution into the problem we arrive to the coupled system,

$$\begin{cases} -\Delta_p X - \lambda W_n(x) X^{p-1} = -\mu X, & x \in \Omega \\ X(x) = 0, & x \in \partial\Omega \\ T'(t) = \mu T^{p-1}(t), \quad T(0) = T_0, & \end{cases} \tag{17}$$

where $W_n(x) = \min\{n, 1/|x|^p\}$ and $\mu > 0$. By integration we obtain

$$T(t) = \frac{T_0}{(1 - (p-2)\, \mu T_0^{p-2} t)^{1/(p-2)}}$$

which blows up for $t = 1/\mu(p-2)\, T_0^{p-2}$. Hence, we have to study the elliptic problems,

$$\begin{cases} -\Delta_p X - \lambda W_n(x) X^{p-1} = -\mu X, & x \in \Omega \\ X(x) = 0, & x \in \partial\Omega. \end{cases} \tag{18}$$

Denote $\alpha X = Y$ with $\mu \alpha^{p-2} = \lambda$. Then we get the equivalent problem

$$\begin{cases} -\Delta_p Y = \lambda( W_n(x)\, Y^{p-1} - Y), & x \in \Omega \\ Y(x) = 0, & x \in \partial\Omega. \end{cases} \tag{19}$$

For this problem, we have the following result

LEMMA 5.1. *Consider the problem* (19). *Then*

(1) *There exists a constant $R_n > 0$ such that if* (19) *has a positive solution u, then $\|u\|_\infty > R_n$.*

(2) *Consider $\lambda_1(n)$ the first eigenvalue of $-\Delta_p$ with weight $W_n$. Then $\lambda_1(n)$ is the unique bifurcation point from infinity for the problem* (19).

*Proof.* The first assertion is an easy calculation: if there is a positive solution $Y$ such that $\|Y\|_\infty < \varepsilon$, then we have $-\Delta_p Y \leqslant \lambda Y(n\varepsilon^{p-2} - 1) < 0$ for $\varepsilon$ small. And this is a contradiction, because the maximum principle. The second statement is an application of Theorem 4.1 in [3]. In this way the unbounded *bifurcation branch* (to be precise, the *bifurcation continuum*) given by Theorem 4.1 in [3] cannot cross the level $\|u\|_\infty = R_n$ by the first assertion, neither, obviously, the hyperplane $\lambda = 0$. Hence we get the existence of at least one positive solution to problem (19) for $\lambda > \lambda_1(n)$. ∎

We have the following corollary:

COROLLARY 5.2. *There exists a $n_0$ such that for $n > n_0$, problem* (18) *has a positive solution.*

*Proof.* It suffices to consider a solution $Y$ of (19) and define $X = \alpha^{-1} Y$ with $\mu = \lambda \alpha^{2-p}$. In fact, if $\lambda > \lambda_{N,p}$ then for $n$ large enough we have $\lambda_1(n) < \lambda$, because, according Theorem (2.4), $\lambda_1(n) \to \lambda_{N,p}$ as $n \to \infty$. ∎

In the sequel we will use often the following comparison result.

LEMMA 5.3. *Consider the problem*

$$\begin{cases} u_{i,t} - \Delta_p u_i = W_n(x)\,|u_i|^{p-2}\,u_i, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u_i(x, 0) = f_i(x), & x \in \Omega \\ u_i(x, t) = g_i(t), & x \in \partial\Omega, \quad t > 0, \quad i = 1, 2. \end{cases} \tag{20}$$

*If $f_1(x) \leqslant f_2(x)$, $g_1(t) \leqslant g_2(t)$ and they are bounded functions, then $u_1 \leqslant u_2$*

*Proof.* By regularity, $|u_i| < M$, $i = 1, 2$ and by the lipschitz condition and the ellipticity condition for the $p$-Laplacian, we obtain

$$\begin{aligned} &\int_{\{u_1 \geqslant u_2\}} |u_1(x,t) - u_2(x,t)|^2\,dx \\ &\qquad + \alpha(p) \int_0^t \int_{\{u_1 \geqslant u_2\}} |\nabla u_1(x,s) - \nabla u_2(x,s)|^p\,dx\,ds \\ &\leqslant C(n, M) \int_0^t \int_{\{u_1 \geqslant u_2\}} |u_1(x,s) - u_2(x,s)|^2\,dx\,ds, \end{aligned}$$

where $\alpha(p)$ is a constant depending only of $p$, then, since the test function $(u_1 - u_2)_+$ is zero in the parabolic boundary, Gronwall's Lemma gives the result. ∎

With these tools we have the following finite time complete blow-up result for the truncated problems.

THEOREM 5.4. *Consider the problem*

$$\begin{cases} u_t - \Delta_p u = \lambda W_n(x)\, |u|^{p-2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad N > p > 2, \quad t > 0, \\ u(x, 0) = f(x) > 0, & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{21}$$

*where* $p > 2$, $f \in L^\infty$ *and* $\lambda > \lambda_{N,p}$. *Assume that* $n$ *is large enough, such that* $\lambda_1(n) < \lambda$. *Then, there exists a* $T = T(f, \Omega, n, p, \lambda) > 0$ *such that the solution to* (21) *blows up in the following sense:* $u(x, t) \geqslant \phi(x, t)$ *with* $\lim_{t \to T} \phi(x, t) = \infty$ *for any* $x \in \Omega$.

*Proof.* Bearing in mind the comparison lemma, and the method of separation of variables, the idea is looking for a subsolution $\phi(x, t) = X(x)\, T(t)$ with $\phi(x, 0) = X(x)\, T(0) \leqslant u(x, \tau)$, with $\tau > 0$. Then, we take $X(x)$ the solution found in Corollary 5.2 for $\mu = 1$, and a constant $\varepsilon > 0$ such that $\varepsilon X(x) \leqslant u(x, \tau)/2$. Next, we consider

$$T(t, \varepsilon) = \varepsilon(1 - (p-2)\varepsilon^{p-2} t)^{-1/(p-2)}$$

Therefore, by comparison, the blow up holds before the time $T = \varepsilon^{2-p}/(p-2)$. ∎

Now we are able to state the main result in this section. We will show that there are instantaneous blow up, in all $L^r$ norms, $1 \leqslant r \leqslant \infty$. Even more, we can prove that the blow up occurs in a stronger sense, that we precise below.

THEOREM 5.5. *Consider the problem*

$$\begin{cases} u_t - \Delta_p u = \dfrac{\lambda}{|x|^p}\, |u|^{p-2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad N > p > 2, \quad t > 0, \quad \lambda \in \mathbb{R} \\ u(x, 0) = f(x), & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{22}$$

*where* $p > 2$, $\lambda > \lambda_{N,p}$, $f \in L^\infty$, $f \geqslant 0$ *and* $f > \delta > 0$ *in a neighbourhood of the origin. Then* (22) *has no local solution, in the sense that for any* $\varepsilon > 0$, *there exists* $r(\varepsilon) > 0$ *such that* $\lim_{n \to \infty} u_n(x, t) = \infty$ *if* $|x| \leqslant r(\varepsilon)$ *and* $t \geqslant \varepsilon$, *where* $u_n$ *are the solutions to the problems with the truncated potentials* $W_n(x)$.

*Proof.* The proof lies in three steps: the bifurcation lemma, a rescaling, and the comparison argument.

Given $\varepsilon > 0$ the bound for the blow up time, take $\mu > 1/(p-2)\varepsilon$. Then, the function $T(t) \equiv T(t, 1) = (1/(1-(p-2)\mu t))^{1/(p-2)}$ blows up before $t = \varepsilon$. Assume that $B = \{|x| < 1\} \subset \Omega$, and take $n_0$ such that the problem

$$\begin{cases} -\Delta_p Y - \lambda W_{n_0}(x) Y^{p-1} = -\mu Y, & x \in B \\ Y(x) = 0, & x \in \partial B. \end{cases} \tag{23}$$

has a positive solution (in fact, it suffices to take $n_0$ large enough in order to have $\lambda_1(n_0, B) < \lambda$; see Theorem 2.4).

In this way, $T(t)\, Y(x)$ is a solution of the equation, which blows up before $t = \varepsilon$. But we cannot take this function as a subsolution to our problem, because of the lack of control on the size of the initial data. The next step is, by using a suitable rescaling, to construct a *local* subsolution.

In fact, taking into account that $(n/n_0)\, W_{n_0}((n/n_0)^{1/p}\, x) = W_n(x)$, if we define $Z_n(x) = (n_0/n)^{1/(p-2)}\, Y((n/n_0)^{1/p}\, x)$ then $Z_n$ is a solution to the problem

$$\begin{cases} -\Delta_p Z_n - \lambda W_n(x) Z_n^{p-1} = -\mu Z_n, & |x| < (n_0/n)^{1/p} \\ Z_n(x) = 0, & |x| = (n_0/n)^{1/p}. \end{cases} \tag{24}$$

Taking $n$ large enough, since $\|Y\|_\infty < C$, we get $\|Z_n\|_\infty < \delta < f(x)$ in the ball $|x| < (n_0/n)^{1/p}$. (Notice that $n$ and $n_0$ (i. e., the radius of the ball) depends on $\lambda$ and the $L^\infty$ norm of $Y$, which depends on $\mu$, which is fixed depending on the blow up time $\varepsilon$).

Then the function $\phi_n(x, t) = T(t)\, Z_n(x)$ is the desired *local* subsolution. Finally, the last step is to use the comparison lemma in the ball $|x| < (n_0/n)^{1/p}$. ∎

*Remark* 5.6. It is interesting to emphasize that this instantaneous *regional* blow up can be seen as the $p > 2$ version of the estimates in Theorem 2.2 of Baras–Goldstein for $p = 2$, see [5]. We have a stronger result due to the time dependence in the separation of variables method. Nevertheless, in the linear case the representation of the solutions by the Green's function implies *infinite speed of propagation* and this is the point that makes easy to prove that the blow up is complete. In our case, if $p > 2$, the speed of propagation is finite if $\lambda = 0$, and the argument to prove that $\lim_{n \to \infty} u_n(x, t) = \infty$, $\forall (x, t) \in \Omega \times (0, \infty)$ must be different.

## 6. THE CASE $\lambda > \lambda_{N,p}$ AND $1 < p < 2$

In this section we will show that the blow-up result by Baras and Goldstein for $p = 2$ does not admits an extension for $1 < p < 2 \leqslant N$. We distinguish several cases because the regularity of the solutions becames worst as $p$ approaches 2.

6.1. *The Case* $\lambda > \lambda_{N,\,p}$ *and* $1 < p < 2N/(N+2)$

First we restrict ourselves to the case $1 < p < 2N/(N+2)$.

THEOREM 6.1. *Problem*

$$\begin{cases} u_t - \Delta_p u = \dfrac{\lambda}{|x|^p}\, u^{p-1}, & x \in \Omega, \quad t > 0, \quad \lambda \in \mathbb{R} \\ u(x, 0) = f(x), & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{25}$$

*where* $f(x) \in L^2(\Omega)$, $f(x) > 0$ *and* $1 < p < 2N/(N+2)$, *has global solutions.*

*Proof.* We can construct local solutions by using a classical device by Fujita. (See [16].)

*Step* 1. Consider for $M > 0$ arbitrary, $L = M + n$ and $f_n(x) = \min\{n, f(x)\}$. Let $w^0$ be the solution to the problem

$$\begin{cases} w^0_t - \Delta_p w^0 = \lambda W_n(x) L^{p-1}, & x \in \Omega, \quad t > 0, \quad \lambda \in \mathbb{R} \\ w^0(x, 0) = f_n(x), & x \in \Omega \\ w^0(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases}$$

and consider $T > 0$ for wich $w^0(x, t) \leqslant L$ if $0 < t < T$. We perform the following iteration defining $w^k$ as the solution to the problem

$$\begin{cases} w^k_t - \Delta_p w^k = \lambda W_n(x)(w^{k-1})^{p-1}, & x \in \Omega, \quad t > 0, \quad \lambda \in \mathbb{R} \\ w^k(x, 0) = f_n(x), & x \in \Omega \\ w^k(x, t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases}$$

Then by comparison in $\Omega \times [0, T]$ the sequence verifies

$$w^0 \geqslant w^1 \geqslant \cdots \geqslant w^k \geqslant \cdots > 0$$

In the limit we get a maximum positive solution of the truncated problem $w_n$, defined in $[0, T]$.

*Step* 2. Next we construct the minimal solution by iteration: We solve the problem

$$\begin{cases} v^0_t - \Delta_p v^0 = 0, & x \in \Omega, \quad t > 0, \quad \lambda \in \mathbb{R} \\ v^0(x, 0) = f_n(x), & x \in \Omega \\ v^0(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases}$$

by an standard comparison argument we obtain $0 < v^0 \leqslant w^0$ in $\Omega \times [0, T]$ and then by iteration the solutions of the problems

$$\begin{cases} v^k_t - \Delta_p v^k = \lambda W_n(x)(v^{k-1})^{p-1}, & x \in \Omega, \quad t > 0, \quad \lambda \in \mathbb{R} \\ v^k(x, 0) = f_n(x), & x \in \Omega \\ v^k(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases}$$

verify

$$v^0 \leqslant v^1 \leqslant \cdots \leqslant v^k \leqslant \cdots w^0, \qquad \text{in} \quad \Omega \times [0, T]$$

hence there exists $v_n = \lim_{k \to \infty} v^k$, which can be shown easily to be the minimal solution.

*Step* 3. We have the following *a priori* estimate. Let $u$ be a solution to the problem

$$\begin{cases} u_t - \Delta_p u = \lambda W_n(x) u^{p-1}, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u(x, 0) = f_n(x), & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{26}$$

then by Hölder and Young inequalities,

$$\begin{aligned} &\int_\Omega |u(x, T)|^2 \, dx + \int_0^T \int_\Omega |\nabla u(x, t)|^p \, dx \, dt \\ &\quad = \int_\Omega |f_n(x)|^2 \, dx + \lambda \int_0^T \int_\Omega W_n(x) \, |u(x, t)|^p \, dx \, dt \\ &\quad \leqslant \int_\Omega |f_n(x)|^2 \, dx + \lambda \int_0^T \left( \int_\Omega |W_n(x)|^{2/(2-p)} \, dx \right)^{(2-p)/2} \\ &\qquad \times \left( \int_\Omega |u(x, t)|^2 \, dx \right)^{p/2} dt \\ &\quad \leqslant \int_\Omega |f_n(x)|^2 \, dx + \lambda \left( \frac{2-p}{2} \int_0^T \int_\Omega |W_n(x)|^{2/(2-p)} \, dx \, dt \right. \\ &\qquad \left. + \frac{p}{2} \int_0^T \int_\Omega |u(x, t)|^2 \, dx \, dt \right). \end{aligned}$$

Call

$$y(T) \equiv \int_\Omega |u(x, T)|^2 \, dx$$

and

$$\beta_n(t)=\int_\Omega |f_n(x)|^2\,dx+\lambda\,\frac{2-p}{2}\int_0^T\int_\Omega |W_n(x)|^{2/(2-p)}\,dx\,dt.$$

Hence

$$y(T)\leqslant \beta_n(T)+\lambda\,\frac{p}{2}\int_0^T y(s)\,ds,$$

and, as a consequence of Gronwall inequality, we have the estimate,

$$\int_\Omega |u(x,T)|^2\,dx\leqslant \beta_n(T)+\int_0^T \beta_n(s)\,e^{\alpha s}\,ds \tag{27}$$

where $\alpha=\alpha(\lambda,p)>0$ is a constant. The first consequence is that $u$ is defined for all $t>0$, namely it is a global solution.

*Final Step.* By definition, the sequence $\beta_n$ is increasing. Moreover

$$\int_\Omega |W_n(x)|^{2/(2-p)}\,dx\leqslant \int_\Omega \left(\frac{1}{|x|^p}\right)^{2/(2-p)} dx$$

and because $1<p<2N/(N+2)$, the last integral is convergent. In particular this implies that any solution to problem (25) is globally defined. The existence of solution follows by passing to the limit in the following way: if we take any sequence of solutions of the truncated problems, $u_n\in L^p([0,T],\ W_0^{1,p}(\Omega))\cap L^\infty([0,T],L^2(\Omega))$, by the last estimate we conclude that the sequence $\{u_n\}$ is uniformly bounded in these spaces, and converges almost everywhere. Hence by using the convergence theorem in [8] and the *a priori estimate* (27), for a suitable subsequence, there exists the limit

$$\lim_{n\to\infty} u_n=u$$

which is a positive global solution of problem (25). ∎

*Remark* 6.2. (I) The above result shows that a *Baras–Goldstein instantaneous complete blow-up* result for small $p$ is not true. We have in general instantaneous $L^\infty$-blow-up, as we will see. But the blow-up is incomplete in the sense of the boundedness of some norms. Again the Sobolev theorem seems to play an important role in this case. In fact, by the parabolicity of the problem we have a direct $L^2$ estimate. On the other hand when $2>p^*$, or equivalently, $1<p<2N/(N+2)$ the potential $1/|x|^p$ is *not too singular*. Both facts imply the result. At the end of the next section we will give a different explanation to this fact.

(II) A result on nonuniqueness of solution to problem (25) will be discussed in the last section.

### 6.2. *The Case $\lambda > \lambda_{N,p}$ and $2 > p \geqslant 2N/(N+2)$*

We begin this section by investigating the existence of a selfsimilar solution for the Cauchy problem in the whole $\mathbb{R}^N$.

#### 6.2.1. *Selfsimilar Solution*

If we look for positive selfsimilar solutions to the Cauchy problem in all $\mathbb{R}^N$, namely, for solutions like $S(r, t) = t^{\alpha} f(t^{\beta} r)$, where $r = |x|$, then:

$$S_t = \alpha t^{\alpha-1} f + \beta t^{\alpha+\beta-1} r f', \qquad S_r = t^{\alpha+\beta} f', \qquad S_{rr} = t^{\alpha+2\beta} f'',$$

and as a consequence:

(1) The similarity exponents satisfy $(\alpha - 1) = \alpha(p-1) + \beta p$

(2) The corresponding ordinary differential equation in the variable $\xi = t^{\beta} r$ is

$$\alpha f + \beta \xi f' = (p-1)\, |f'|^{p-2}\, f'' + \frac{N-1}{\xi}\, |f'|^{p-2} f' + \frac{\lambda}{\xi^p}\, |f|^{p-2}\, f$$

Next we look for solutions of the form $A\, |\xi|^{\gamma}$, for which we found:

(1) $$\gamma = \frac{-p}{2-p},$$

(2) $$|A|^{p-2}\, (\lambda) = \frac{\alpha + \beta\gamma}{(p-1)\, |\gamma|^p + (N-p)\, |\gamma|^{p-2}\, \gamma + \lambda} \equiv B$$

The last identity implies that $B$ must be positive, then there exists solution if and only if $\lambda$ is large enough to fulfill this condition.

In our range of parameters we have $\gamma < 0$ and if $\lambda$ is such that $B > 0$ then the corresponding selfsimilar solution can be considered as a solution in the distributions sense if and only if $p < 2N/(N+1)$. It is clear that $B > 0$ if $\lambda$ is large enough. More precisely, if $1 < p < 2$, we can prove that $B > 0$ for any $\lambda > \lambda_{Np}$, which is the interval under study. This fact will be sufficient if $2N/(N+2) \leqslant p < 2N/(N+1)$ to obtain a global solution in a weaker class for the Cauchy problem, in a bounded domain containing the origin, and with positive and bounded initial data. This result will be discussed in the next subsection.

LEMMA 6.3. *Consider* $1<p<2$ *and* $\lambda>\lambda_{N,p}$ *then* $B(\lambda)>0$.

*Proof.* It suffices to prove that

$$g_p(N)=\left(\frac{N-p}{p}\right)^p-\left(\frac{p}{2-p}\right)^{p-1}\left((N-p)-\frac{p(p-1)}{2-p}\right)\geqslant 0.$$

Fixed $p\in(1,2)$ we have

$$g_p'(N)=\left(\frac{N-p}{p}\right)^{p-1}-\left(\frac{p}{2-p}\right)^{p-1},$$

hence $g_p'(N_0)=0$ if and only if

$$\left(\frac{N_0-p}{p}\right)=\left(\frac{p}{2-p}\right).$$

Moreover, $N_0$ is a point of minimum for $g$ as can be seen directly by calculation of the second derivative. Now

$$g_p(N_0)=0,$$

so we conclude that $B^{-1}=(g_p(N)+(\lambda-\lambda_{N,p}))/(\alpha+\beta\gamma)>0$ if $\lambda>\lambda_{N,p}$. ∎

Hence for all $\lambda>\lambda_{N,p}$ a selfsimilar solution in $\mathbb{R}^N$ is

$$S(r,t)=A(\lambda)\left(\frac{t}{r^p}\right)^{1/(2-p)}.$$

An important consequence of the previous calculations is that the Cauchy problem with trivial initial data has unbounded positive solution if $1<p<2N/(N+1)$. The selfsimilar solution $S(r,t)$ can be seen as the paradigmatic example of how the $L^\infty$-blow up occurs.

### 6.2.2. *Existence of Solution*

When $1<p<2N/(N+2)$ an *a priori* estimate gives the global existence of a weak solution. Such an estimate does not work in the case $p\geqslant 2N/(N+2)$. However, it is easy to see that the selfsimilar solution obtained in the previous subsection can be considered as a supersolution in the distributions sense if and only if $p<2N/(N+1)$. And if $p\geqslant 2N/(N+2)$ and fixed $t>0$, $S(r,t)\notin W_0^{1,p}(\Omega)$.

In this subsection we prove that if $2N/(N+2)\leqslant p<2N/(N+1)$ then it is still possible to get a global solution in a weaker class, for the problem with positive and bounded initial data.

The first step is the construction of a suitable candidate to solve the problem. The second step will be the proof that the function built in the first step is solution at less in the sense of distributions. We will work in the setting of $L^1$ right hand sides. (See for instance [7] and [6] for the elliptic case and [8] for the parabolic case.)

Assume $2N/(N+2) \leqslant p < 2N/(N+1)$ and consider the selfsimilar solution

$$S_\lambda(r, t) = A(\lambda) \left(\frac{t}{r^p}\right)^{1/(2-p)}.$$

With a convenient shift in time $W(x, t) = S_\lambda(|x|, t + t_0)$, we have that $W$ is a supersolution if we have positive bounded initial data, namely, $W(x, 0) \geqslant f(x)$, for $x \in \Omega$ and $W(x, t) > 0$ if $(x, t) \in \partial\Omega \times [0, \infty)$. Consider the sequence of problems

$$\begin{cases} u_t^{(n)} - \Delta_p u^{(n)} = \lambda W_n(x)(u^{(n-1)})^{p-1}, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u^{(n)}(x, 0) = f(x), & x \in \Omega \\ u^{(n)}(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{28}$$

for $n \geqslant 1$ where $W_n(x) = \min\{n, \lambda\, |x|^{-p}\}$ and where $u^0$ is the solution to the problem

$$\begin{aligned} &u_t^{(0)} - \Delta_p u^{(0)} = 0, && x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ &u^{(0)}(x, 0) = f(x), && x \in \Omega \\ &u^{(0)}(x, t) = 0, && x \in \partial\Omega, \quad t > 0. \end{aligned}$$

We have that $(u^{(0)} - W)_+ = 0$ in the parabolic boundary $\Gamma_p = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, \infty))$ and $(u^{(0)} - W)_t - (\Delta_p u^{(0)} - \Delta_p W) \leqslant 0$. Notice that $(u^{(0)} - W)_+$ can be used as a test function to integrate in this last expression; hence $u^{(0)} \leqslant W$. In the same way we prove that

$$u^{(0)} \leqslant u^{(1)} \leqslant \cdots \leqslant u^{(n)} \leqslant \cdots W.$$

Now it is clear that $W \in L^q_{loc}$ if $1 \leqslant q < N((2-p)/p)$, and this is a nonempty interval because the hypothesis on $p$. We define $u = \lim_{n\to\infty} u^{(n)} \leqslant W$ pointwise. By the Lebesgue Theorem we have also

$$\|u\|_q^q = \lim_{n\to\infty} \|u^{(n)}\|_q^q \leqslant \|W\|_q^q,$$

and then the convergence holds in $L^q$. To prove that $u$ is a solution in the sense of distributions we need the following lemmas that we prove by using the ideas in [7] and [6].

LEMMA 6.4. *Let* $\{u^{(n)}\}_{n\in\mathbb{N}}$ *be defined above. Consider* $Q_T=\Omega\times[0,T]$, *then we have the following estimate in the Marcinkiewitz space* $\mathcal{M}^{p_2}$,

$$|\{(x,t)\in Q_T \mid |\nabla u^{(n)}(x,t)|>h\}|\leqslant C(p,N,T)h^{-p_2},$$

*where* $1\leqslant q<N((2-p)/p)$ *and* $p_2=pq/(q+1)$.

*Proof.* Fix $n$ we have $u^{(n)}\leqslant W$ and then

$$|\{(x,t)\in Q_T: |u^{(n)}(x,t)|>k\}|\leqslant\frac{1}{k^q}\int_0^T\int_\Omega W^q\,dx\,dt$$

independently of $n$. Consider for $k,h>0$

$$A(k,h)=|\{(x,t)\in Q_T: |\nabla u^{(n)}|^p>h,\ |u^{(n)}|>k\}|.$$

It is clear that $l\to A(k,l)$ is nonincreasing, and since from the initial remark

$$A(k,0)\leqslant C(N,p,T)k^{-q}$$

then we have

$$A(0,l)\leqslant\frac{1}{l}\int_0^l A(0,s)\,ds\leqslant A(k,0)+\int_0^l (A(0,s)-A(k,s))\,ds,$$

but

$$A(0,s)-A(k,s)=|\{(x,t)\in Q_T: |\nabla u^{(n)}|^p>s,\ |u^{(n)}|\leqslant k\}|,$$

and multiplying in the corresponding equation by the $k$-truncature of $u^{(n)}$ we get

$$\frac{1}{k}\int_0^T\int_{\{|u^{(n)}|<k\}}|\nabla u^{(n)}|^p\,dx\,dt\leqslant M$$

whence

$$\int_0^l (A(0,s)-A(k,s))\,ds\leqslant Mk.$$

The final conclusion is

$$A(0,l)\leqslant Mkl^{-1}+C(N,p,T)k^{-q}.$$

Minimizing in $k$ we have that the minimum is attained in $k = Dl^{1/(q+1)}$. Taking $l = h^p$, our inequality for $A(0, h)$ becames

$$|\{(x, t) \in Q_T \mid |\nabla u^{(n)}(x, t)| > h\}| \leqslant C(p, N, T)\, h^{-(pq/(q+1))}$$

and then we conclude because $p_2 = pq/(q+1)$. ∎

Lemma. 6.5. *In the hypotheses of Lemma* 6.4 *we get*

1. $\nabla u^{(n)} \to \nabla u$, *a.e. and in measure.*
2. $|\nabla u^{(n)}|^{p-2}\, \nabla u^{(n)} \to |\nabla u|^{p-2}\, \nabla u$ *in* $L^1$

*Proof.* (1) Because $\{u^{(n)}\}$ converges in $L^q(Q_T)$, *a fortiori* $u^{(n)} \to u$ in measure as $n \to \infty$. We try to prove that $\nabla u^{(n)}$ converges in measure. It suffices to show that $\{\nabla u^{(n)}\}$ is a Cauchy sequence in measure. For $h > 0$ the set

$$\{(x, t) \in Q_T : |\nabla u^{(n)} - \nabla u^{(m)}| > h\}$$

is a subset of the union of

$$\begin{aligned}
\Gamma(n, A) &= \{(x, t) \in Q_T : |\nabla u^{(n)}| > A\}\\
\Gamma(m, A) &= \{(x, t) \in Q_T : |\nabla u^{(m)}| > A\}\\
\Lambda(k) &= \{(x, t) \in Q_T : |u^{(n)} - u^{(m)}| > k\}\\
D_{A, k, h} &= \{(x, t) : |\nabla u^{(n)}| < A,\ |\nabla u^{(m)}| < A,\ |\nabla u^{(n)} - \nabla u^{(m)}| > h,\\
&\qquad |u^{(n)} - u^{(m)}| < k\}
\end{aligned}$$

for any $A, k > 0$. From Lemma 6.4 we can choose $A$ in such a way that $|\Gamma(n, A)| \leqslant \varepsilon$ for all $n \in \mathbb{N}$. Moreover if $|\xi| < A$, $|\eta| < A$ and $|\xi - \eta| > h$, by monotonicity, we have $\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\rangle \geqslant \mu > 0$ for some $\mu > 0$. Hence we get

$$\begin{aligned}
&\int_0^T \int_{\{|u^{(n)} - u^{(m)}| < k\}} \langle |\nabla u^{(n)}|^{p-2}\, \nabla u^{(n)} - |\nabla u^{(m)}|^{p-2}\, \nabla u^{(m)}, \nabla u^{(n)} - \nabla u^{(m)}\rangle\, dx\, dt\\
&\quad \leqslant \int_0^T \int_\Omega \lambda \left(\frac{|u^{(n)}|^{p-1} - |u^{(m)}|^{p-1}}{|x|^p}\right) \mathcal{T}_k(u^{(n)} - u^{(m)})\, dx\, dt\\
&\quad \leqslant 2k\lambda \int_0^T \int_\Omega \frac{W^{p-1}}{|x|^p}\, dx\, dt = C(p, N, T, \lambda)\, k.
\end{aligned}$$

Then

$$\begin{aligned}|D_{A,k,h}| &\leqslant |\{|\nabla u^{(n)}| < A,\ |\nabla u^{(m)}| > A,\ |u^{(n)} - u^{(m)}| < k,\\ &\langle |\nabla u^{(n)}|^{p-2}\nabla u^{(n)} - |\nabla u^{(m)}|^{p-2}\nabla u^{(m)}, \nabla u^{(n)} - \nabla u^{(m)}\rangle > \mu\}|\\ &\leqslant \frac{1}{\mu}\int_0^T \int_{\{|u^{(n)}-u^{(m)}|<k\}} \langle |\nabla u^{(n)}|^{p-2}\nabla u^{(n)}\\ &\quad - |\nabla u^{(m)}|^{p-2}\nabla u^{(m)}, \nabla u^{(n)} - \nabla u^{(m)}\rangle\, dx\, dt\\ &\leqslant C(p, N, T, \lambda)\,\frac{k}{\mu} \leqslant \varepsilon,\end{aligned}$$

taking $k$ small enough; now for $n, m \geqslant n_0$, $|A_k| \leqslant \varepsilon$, so $\nabla u^{(n)} \to v$ in measure for some measurable function $v$.

(2) From Lemma 6.4, it is easy to check that $|\nabla u^{(n)}|^{p-2}\nabla u^{(n)}$ is bounded in $L^r$ for some $r > 1$; by 1) and the Nemystskii lemma we have $|\nabla u^{(n)}|^{p-2}\nabla u^{(n)} \to |\nabla u|^{p-2}\nabla u$ a.e. and in measure and, as is well known by the Vitali lemma, $|\nabla u^{(n)}|^{p-2}\nabla u^{(n)} \to |\nabla u|^{p-2}\nabla u$ in $L^1$ (See [27]). ▮

As a consequence we obtain the following result.

THEOREM 6.6. *Assume* $2N/(N+2) < p < 2N/(N+1)$, *then the problem*

$$\begin{cases} u_t - \Delta_p u = \lambda \dfrac{u^{p-1}}{|x|^p}, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0,\\ u(x, 0) = f(x), & x \in \Omega, \quad f \in L^\infty, \quad f \geqslant 0,\\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{29}$$

*has a global solution u in the sense of distributions.*

*Moreover* $u \in L^\infty_{loc}((0, \infty), L^q(\Omega))$ *and* $|\nabla u| \in \mathcal{M}^{p_2}$.

*Remark* 6.7. (i) Depending on $p$ and $N$, when $p_2 = pq/(q+1) > 1$, we obtain a better regularity for $u$.

(ii) We can repeat the same proof for unbounded positive data, provided $u_0 \leqslant W$ for some $t_0$.

(iii) In the case $2 > p > 2N/(N+1)$ we can use the same kind of arguments as in the theorem, to find a solution away from the origin.

We would like to point out a last comment about the values of $p$ for which the behaviour of the equation (4) changes.

(1) If $1 < p < 2N/(N+1)$ there exist solution in the distributions sense at least for bounded data, independently of the value of $\lambda > 0$. It is nice to

point out that this is the same range of $p$ for wich the problem with $\lambda = 0$ has extinction in finite time, or the same range where the Harnack inequality is false. In some sense, this means that under the influence of the reaction term, a solution avoids the extinction but cannot reach the *complete blow-up*.

In this case, if moreover $1 < p < 2N/(N+2)$, then the $L^2$-estimate and the Sobolev inclusion gives a strong solution, for data in $L^2$.

(2) If $2 > p \geqslant 2N/(N+1)$ then there exist solution away of the origin. It is an open problem to precise the behaviour of the solutions in this case.

(3) If $p = 2$ we have the Baras–Goldstein result. (See [5]).

(4) If $p > 2$ we have instantaneous blow-up in the sense stated before. In some way the extension to a complete blow up result should imply that the reaction term produces infinite speed of propagation.

## 7. EXISTENCE IN THE CASE $\lambda = \lambda_{N,\,p}$

When $\lambda = \lambda_{N,\,p}$, the proof of the existence result in Section 4 does not work. In fact, the main point in the proof is an energy estimate, which requires that $\lambda < \lambda_{N,\,p}$. However, following the same kind of ideas as in subsection (6.2), we will show the existence of solution (in the sense of distributions) to the problem with positive bounded initial data $f$.

First, if $1 < p < 2N/(N+1)$ we can use as a supersolution a convenient shift in time of the selfsimilar solution $S(x, t)$; hence, the same iterative method of subsection (6.2) allows us to conclude the existence of solution in the sense of distributions.

In the range $p \geqslant 2N/(N+1)$, the argument is similar: we replace the selfsimilar solution $S(x, t)$ by an stationary supersolution $\phi(x) = c|x|^{-(N-p)/p}$. In fact, for any $c \in \mathbb{R}$, this function is a solution (for $x \neq 0$) to the problem

$$-\Delta_p \phi = \lambda_{N,\,p} \phi^{p-1} |x|^{-p}.$$

Then, taking $c$ large enough we have $\phi \geqslant f$. Notice that $\phi \in W^{1,\,q}_{loc}$ for any $q < p$, but $\phi \notin W^{1,\,p}_{loc}$. In other words, we cannot pass to the limit in $W^{1,\,p}_0(\Omega)$; instead of this, we use the same convergence results as in section (6.2) (see Lemma 6.4 and Lemma 6.5), passing to the limit in $W^{1,\,p-1}_0$ in a sequence of solutions of the truncated problems, getting finally a solution in the sense of distributions.

Therefore, we have proved the following theorem:

THEOREM 7.1. *Assume* $1 < p < N$, *then the problem*

$$\begin{cases} u_t - \Delta_p u = \lambda_{N,p} \dfrac{u^{p-1}}{|x|^p}, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u(x,0) = f(x), & x \in \Omega, \quad f \in L^\infty, \quad f \geqslant 0 \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{30}$$

*has a global solution u in the sense of distributions.*

*Remark* 7.2. Observe that the hypothesis of boundedness in the data $f$ is needed only to have that $f \leqslant \phi$ for a suitable constant $c$. This means that the theorem of existence holds for any initial data $f \in L^2$ satisfying this condition.

## 8. SOME REMARKS ON UNIQUENESS

The uniqueness is in general an interesting open problem. However some partial results will be explained in this section. First of all we will explain some facts for $\lambda < \lambda_{N,p}$ in the next Lemma. The second result is on nonuniqueness for $\lambda > \lambda_{N,p}$ and $1 < p < 2N/(N+1)$. In this sense the best constant in Hardy's inequality is the critical value of the parameter $\lambda$ for uniqueness in the range $1 < p < 2N/(N+1)$. We recall that $W_n(x) = \min\{n, |x|^{-p}\}$ is the truncated potential.

LEMMA 8.1. *Assume* $\lambda < \lambda_{N,p}$ *and consider the problem*

$$\begin{cases} u_t - \Delta_p u = \lambda W_n(x)\, |u|^{p-2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{31}$$

*Then,*

(1) *If* $p \geqslant 2$ *and* $f \in L^2(\Omega)$, *there exists an unique solution to* (31).

(2) *For any* $p < 2$, *if* $f \equiv 0$ *the problem has only the trivial solution. Moreover, the same result is true for the unbounded potential* $\lambda/|x|^p$.

*Proof.* (1) If $u, v$ are solutions, by regularity we get $|u| < M$, $|v| < M$; and by the lipschitz condition and the ellipticity condition for the $p$-Laplacian, we obtain

$$\int_{\Omega} |u(x,t)-v(x,t)|^2\,dx+\alpha(p)\int_0^t\int_{\Omega} |\nabla u(x,s)-\nabla v(x,s)|^p\,dx\,ds$$

$$\leqslant C(n,M)\int_0^t\int_{\Omega} |u(x,s)-v(x,s)|^2\,dx\,ds,$$

where $\alpha(p)$ is a constant depending only of $p$, then, Gronwall's Lemma gives the result.

(2) By Hardy inequality we have the estimate

$$\int_{\Omega} |u(x,T)|^2\,dx+\alpha\int_0^T\int_{\Omega} |\nabla u(x,t)|^p\,dx\,dt\leqslant\int_{\Omega} |f(x)|^2\,dx\equiv 0$$

therefore $u=0$. ∎

In the case $p>2$ and the hypothesis of the previous lemma the unique solution to the aproximate problems converge to the minimal solution to the problem with potential $\lambda|x|^{-p}$, however the uniqueness for the problem (4) is an open question. This problem seems to be not easy because the potential $|x|^{-p}\in L^q$ only if $1<q<N/p$.

In the case $1<p<2$, the uniqueness is in general not satisfied as we will see with the following comparison argument.

(I) In order to get a subsolution, we study the eigenvalue problem

$$\begin{cases} -\Delta_p\psi_1=\lambda_1(n)\,W_n(x)\,|\psi_1|^{p-2}\,\psi_1, & x\in\Omega\subset\mathbb{R}^N,\\ \psi_1(x)=0, & x\in\partial\Omega. \end{cases} \tag{32}$$

The results in Theorem (2.4) allows us to conclude that for $\lambda>\lambda_{N,p}$ fixed, we can choose $n$ large enough such that $\lambda_1(n)<\lambda$. Next, we define $w(x,t)=ct^{\alpha}\psi_1(x)$, with $\alpha=1/(2-p)$. Then, if $c$ is small enough,

$$0<\mu(x,t)\equiv\frac{w_t-\Delta_p w}{\lambda\,|x|^{-p}\,w^{p-1}}\leqslant c_1(N,p,\Omega)\,c^{2-p}\psi_1^{2-p}+\frac{\lambda_1(n)}{\lambda}<1$$

where $c_1(N,p,\Omega)=C_{N,p}\alpha\delta_{\Omega}^p$, $C_{N,p}$ being the best constant in Hardy inequality and $\delta_{\Omega}$ the diameter of $\Omega$. Therefore, we have a subsolution to the problem

$$\begin{cases} u_t-\Delta_p u=\lambda\,|x|^{-p}\,|u|^{p-2}\,u, & x\in\Omega\subset\mathbb{R}^N,\quad t>0,\\ u(x,0)=0, & x\in\Omega\\ u(x,t)=0, & x\in\partial\Omega,\quad t>0. \end{cases}$$

(II) Now, we can take as a supersolution the selfsimilar solution in the whole space-time obtained in the previous section, $S(x, t) = A(\lambda)(t/|x|^p)^{1/(2-p)}$. Then we can perform the following iteration:

$$\begin{cases} u_{(k+1)t} - \Delta_p u_{k+1} = \lambda\, |x|^{-p}\, |u_k|^{p-2}\, u_k, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u_{k+1}(x, 0) = 0, & x \in \Omega \\ u_{k+1}(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases}$$

starting with $u_0(x, t) = w(x, t)$. If we choose $c$ small enough we can get $w(x, t) \leqslant S(x, t)$. Then the iterative scheme gives:

THEOREM 8.2. *Given the problem*

$$\begin{cases} u_t - \Delta_p u = \lambda\, |x|^{-p}\, |u|^{p-2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u(x, 0) = 0, & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases}$$

*with* $1 < p < 2N/(N+1)$, *and* $\lambda > \lambda_{N,\,p}$, *then there exists a positive solution in the sense of distributions. Moreover, if* $1 < p < 2N/(N+2)$, *the equation is satisfied in the weak* $W_0^{1,\,p}(\Omega)$ *sense.*

*Proof.* The argument has been already used in the previous sections, and then we will be sketchy. The only point to justify is the comparison theorem. In the range $1 < p < 2N/(N+2)$ there is no problem, because $S(\cdot, t) \in W_0^{1,\,p}(\Omega)$. In the range $2N/(N+2) \leqslant p < 2N/(N+1)$, this is false, but $(u_k - S)_+ (\cdot, t) \in W_0^{1,\,p}(\Omega)$, and can be taken as test function in order to prove the comparison. ∎

Now we explain some elementary nonuniqueness results related to the classical ideas by Fujita. On the other hand we prove a new result for the p-laplacian.

LEMMA 8.3. *Consider the problem,*

$$\begin{cases} u_t - \Delta_p u = \lambda\, |u|^{q-2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \quad \lambda > 0 \\ u(x, 0) = 0, & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{33}$$

*where,* $1 < q < p \leqslant 2$. *Then problem* (33) *has a positive solution.*

*Proof.* We follow the ideas in [16]. There are two steps in the proof. First of all, we construct a maximum solution. Second, we show that such a maximum solution is positive. Let $M$ be a positive number and consider $u_0$ the solution of

$$\begin{cases} u_t - \Delta_p u = \lambda M^{q-1}, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \\ u(x, 0) = 0, & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{34}$$

Let $T > 0$ such that $u_0(x, t) < M$ in $\Omega \times [0, T]$. By iteration we construct $u_k$ solution of

$$\begin{cases} u_{kt} - \Delta_p u_k = \lambda u_{k-1}^{q-1}, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \quad p \leqslant N, \\ u(x, 0) = 0, & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{35}$$

The weak comparison theorem gives us that $u_0 \geqslant u_1 \geqslant \cdots \geqslant u_k \geqslant \cdots \geqslant 0$, hence $\bar{u} = \lim_{k \to \infty} u_k$ is a solution of problem (33) with trivial initial data.
Now we will use the following result.

Lemma 8.4. *Let $\bar{u}$, $M$ be defined above. If $v$ is a subsolution to problem* (35), *such that $v(x, t) < M$ in $(x, t) \in \Omega \times [0, T]$ then $v \leqslant \bar{u}$.*

*Proof.* Since $v \leqslant M$ we have that

$$v_t - \Delta_p v \leqslant \lambda M^{q-1} = u_{0t} - \Delta_p u_0,$$

therefore by comparison $v \leqslant u_0$. By iteration of this argument, $v \leqslant u_k$ for all $k \in \mathbb{N}$. Hence $v \leqslant \bar{u}$. ∎

To finish, define $w(x, t) = \mu(\varepsilon t) \phi_1(x)$ where $\mu' = \mu^{q-1}$, $\mu(0) = 0$ and $\phi_1 > 0$ is the positive solution of

$$\begin{cases} -\Delta_p \phi_1 = \lambda \, |\phi_1|^{q-2} \, \phi_1, & x \in \Omega \subset \mathbb{R}^N, \\ \phi_1(x) = 0, & x \in \partial\Omega. \end{cases} \tag{36}$$

Direct computations give

$$\alpha(x, t) = \frac{w_t - \Delta_p w}{\lambda w^{q-1}} < 1, \qquad t \in [0, T]$$

because $0 < \alpha(x, t) \leqslant \varepsilon \phi_1^{2-q} / \lambda + \mu(\varepsilon t)^{p-q} < 1$, for $\varepsilon > 0$ small enough. From the result in Lemma 8.4 we conclude $\bar{u} \geqslant w(x, t) > 0$. ∎

By the same argument as above we obtain:

COROLLARY 8.5. *Problem*

$$\begin{cases} u_t - \Delta_p u = \lambda\, |u|^{p-2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \quad \lambda > \lambda_1 \\ u(x, 0) = 0, & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{37}$$

*where, $p < 2$ and $\lambda_1$ is the first eigenvalue of the p-Laplacian, has a positive solution.*

However, if we admit in the reaction term an exponent larger than $p$, then in some cases we can prove the following uniqueness result, which seems to be new. We emphasize that the second member is not *lipschitz continous*. Also we would like to remark that in the semilinear case, when $q > 2$ the nonlinearity is *lipschitz*, hence this difficulty does not appear in the semilinear case.

LEMMA 8.6. *Consider the problem*

$$\begin{cases} u_t - \Delta_p u = \lambda\, |u|^{q-2}\, u, & x \in \Omega \subset \mathbb{R}^N, \quad t > 0, \quad \lambda > 0 \\ u(x, 0) = 0, & x \in \Omega \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{38}$$

*where, $2N/(N+2) < p < q \leqslant 2$.*

*Then the unique solution $u \in L^\infty([0, T];\ L^2(\Omega)) \cap L^p([0, T],\ W_0^{1,p}(\Omega))$ to the problem (38) is $u \equiv 0$.*

*Proof.* The idea is very elementary. The usual energy estimate gives us

$$\int_\Omega |u(x, T)|^2\, dx + \int_0^T \int_\Omega |\nabla u(x, t)|^p\, dx\, dt = \int_0^T \int_\Omega |u(x, t)|^q\, dx\, dt$$

and the hypotheses on $p$ give that $p^* > 2$. By Sobolev and Hölder inequalities we obtain

$$\int_\Omega |u(x, T)|^2\, dx + c_p \int_0^T \left( \int_\Omega |u(x, t)|^2\, dx \right)^{p/2} dt$$

$$\leqslant |\Omega|^{(2-q)/2} \int_0^T \left( \int_\Omega |u(x, t)|^2\, dx \right)^{q/2} dt$$

If we call $y(t) \equiv (\int_\Omega |u(x,t)|^2\,dx)^{1/2}$ the above inequality can be written as

$$0 \leqslant y^2(T) \leqslant \int_0^T (|\Omega|^{(2-q)/2} y^q(t) - c_p y^p(t))\,dt.$$

And this is impossible because $\lim_{t\to 0} y(t)=0$. ∎

*Note added in proof.* With respect to Remark 3.5 (III), we have received a personal communication by Xiao Zhong (Jyvaskyla University, Finland), proving the non-uniqueness in the case $1<p<2$, by using some convenient weighted Hardy inequalities.

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