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Wavelets with composite dilations and their MRA properties

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Abstract

Affine systems are reproducing systems of the form

$$\mathcal{A}_{\mathcal{C}} = \{D_c T_k \psi^\ell \colon 1 \leqslant \ell \leqslant L,\ k \in \mathbb{Z}^n,\ c \in \mathcal{C}\},$$

which arise by applying lattice translation operators T_k to one or more generators ψ^ℓ in $L^2(\mathbb{R}^n)$, followed by the application of dilation operators D_c, associated with a countable set $\mathcal{C}$ of invertible matrices. In the wavelet literature, $\mathcal{C}$ is usually taken to be the group consisting of all integer powers of a fixed expanding matrix. In this paper, we develop the properties of much more general systems, for which $\mathcal{C} = \{c = ab\colon a \in A,\ b \in B\}$ where A and B are not necessarily commuting matrix sets. $\mathcal{C}$ need not contain a single expanding matrix. Nonetheless, for many choices of A and B, there are wavelet systems with multiresolution properties very similar to those of classical dyadic wavelets. Typically, A expands or contracts only in certain directions, while B acts by volume-preserving maps in transverse directions. Then the resulting wavelets exhibit the geometric properties, e.g., directionality, elongated shapes, scales, oscillations, recently advocated by many authors for multidimensional signal and image processing applications. Our method is a systematic approach to the theory of affine-like systems yielding these and more general features.

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1. Introduction

There is considerable interest, both in mathematics and its applications, in the study of efficient representations of multidimensional functions. The motivation comes partly from signal processing, where such representations are useful in image compression and feature extraction, and from the investigation of certain classes of singular integral operators. For example, it was pointed out in several recent research papers that *oriented* oscillatory waveforms play a fundamental role in the construction of representations for multidimensional functions and signals (cf. [3,5,8] and articles in [20]). In particular, it was shown that, in order to be optimally sparse in a certain sense, such representations must contain basis elements with many more locations, scales, shapes and directions than the "classical" wavelets (cf. [4]).

In this paper, we introduce a new class of representation systems which have exactly the features we have described, as well as several other properties which are closely analogous to the properties of systems constructed in [7] and, therefore, for the reasons explained there as well as in [4,5,8], have great potential in applications. We call these systems *affine systems with composite dilations*, and they have the form

$$\mathcal{A}_{AB}(\Psi) = \{D_a D_b T_k \Psi\colon\ k \in \mathbb{Z}^n,\ b \in B,\ a \in A\}, \tag{1.1}$$

where $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^n)$, T_k are the *translations*, defined by $T_k f(x) = f(x-k)$, D_a are the *dilations*, defined by $D_a f(x) = |\det a|^{-1/2} f(a^{-1}x)$, and A, B are countable subsets of $GL_n(\mathbb{R})$. By choosing Ψ, A, and B appropriately, we can make $\mathcal{A}_{AB}(\Psi)$ an orthonormal (ON) basis or, more generally, a Parseval frame (PF) for $L^2(\mathbb{R}^n)$. In this case, we call Ψ an *ON AB-multiwavelet* or a *PF AB-multiwavelet*, respectively. If the system has only one generator, that is, $\Psi = \{\psi\}$, then we use the expression *wavelet* rather than *multiwavelet* in this definition.

As we will show, the mathematical theory of these systems provides a simple and flexible framework for the construction of several classes of orthonormal bases and Parseval frames. For example, in Section 5, we construct PF AB-wavelets with good time-frequency decay properties, whose elements contain "long and narrow" waveforms with many locations, scales, shapes and directions. These examples have similarities to the *curvelets* [4] and *contourlets* [7], which have been recently introduced in order to obtain efficient representations of natural images. Our approach is more general and presents a simple method for obtaining several such orthonormal bases and Parseval frames that exhibit these and other geometric features. In particular, our approach extends naturally to higher dimensions and allows a multiresolution construction which appears to be well suited to a fast numerical implementation. For example, the fan filter approach developed in [7] can be used in some cases.

The paper is organized as follows. In Section 2 we introduce the study of AB-multiwavelets by constructing some examples of such systems in $L^2(\mathbb{R}^2)$. In Section 3 we examine the conditions on $A, B \in GL_n(\mathbb{R})$ that ensure the existence of AB-multiwavelets and present several classes of these systems for $L^2(\mathbb{R}^n)$. In Sections 4 and 5, we describe the AB-multiwavelets generated using a generalization of the classical MRA. Finally, in Section 6, we describe an example of a singly generated orthonormal AB-wavelet.

2. Example

In this paper, we shall present a variety of affine systems with composite dilations. Perhaps, the most efficient way of entering into the study of these systems is to examine in some detail a particular example of such a system.

Throughout this paper, we shall consider the points $x \in \mathbb{R}^n$ to be column vectors, i.e., $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, and the points $\xi \in \hat{\mathbb{R}}^n$ (the frequency domain) to be row vectors, i.e., $\xi = (\xi_1, \dots, \xi_n)$. A vector x multiplying a matrix $a \in GL_n(\mathbb{R})$ on the right is understood to be a column vector, while a vector ξ multiplying a on the left is a row vector. Thus, $ax \in \mathbb{R}^n$ and $\xi a \in \hat{\mathbb{R}}^n$. The Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} \, \mathrm{d}x,$$

where $\xi \in \hat{\mathbb{R}}^n$, and the inverse Fourier transform is

$$\check{f}(x) = \int_{\hat{\mathbb{R}}^n} f(\xi) e^{2\pi i \xi x} \, \mathrm{d}\xi.$$

Let $a = \begin{pmatrix} 2 & 0 \\ 0 & \epsilon \end{pmatrix}$, where $\epsilon \neq 0$, $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $G = \{(b^j, k)\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^2\}$. Then G is a group with group multiplication:

$$(b^\ell, m)(b^j, k) = (b^{\ell+j}, k + b^{-j} m). \tag{2.1}$$

In particular, we have $(b^j, k)^{-1} = (b^{-j}, -b^j k)$. The multiplication (2.1) is consistent with the operation that maps $x \in \mathbb{R}^2$ into $b^j(x+k) \in \mathbb{R}^2$. This is clarified by introducing the unitary representation π of G, acting on $L^2(\mathbb{R}^2)$, defined by

$$\big(\pi(b^j, k) f\big)(x) = f(b^{-j}x - k) = \big(D_b^j T_k f\big)(x) \tag{2.2}$$

for $f \in L^2(\mathbb{R}^2)$. The observation that

$$\big(D_b^\ell T_m\big)\big(D_b^j T_k\big) = \big(D_b^{\ell+j} T_{k+b^{-j}m}\big),$$

where $\ell, j \in \mathbb{Z}$, $k, m \in \mathbb{Z}^2$, shows how the group operation (2.1) is associated with the unitary representation (2.2).

Let $S_0 = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ |\xi_1| \leqslant 1\}$ and define

$$V_0 = L^2(S_0)^\vee = \big\{f \in L^2(\mathbb{R}^n)\colon\ \operatorname{supp} \hat{f} \subset S_0\big\}.$$

Since, for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^2$, we have

$$\big(\pi(b^j, k) f\big)^\wedge(\xi) = \big(D_b^j T_k f\big)^\wedge(\xi) = e^{-2\pi i \xi b^j k} \hat{f}(\xi b^j), \tag{2.3}$$

and $\xi b^j = (\xi_1, \xi_2) b^j = (\xi_1, \xi_2 + j\xi_1)$, then the action of b^j maps the vertical strip domain S_0 into itself and, thus, the space V_0 is invariant under the action of $\pi(b^j, k)$. The same invariance property holds similarly for the vertical strips

$$S_i = S_0 a^i = \big\{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ |\xi_1| \leqslant 2^i\big\},$$

$i \in \mathbb{Z}$, and, as a consequence, the spaces $V_i = L^2(S_i)^\vee$ are also invariant under the action of the operators $\pi(b^j, k)$. The spaces $\{V_i\}_{i \in \mathbb{Z}}$ also satisfy the basic MRA properties:

(1) $V_i \subset V_{i+1}$, $i \in \mathbb{Z}$;
(2) $D_a^{-i} V_0 = V_i$;
(3) $\bigcap V_i = \{0\}$; and
(4) $\overline{\bigcup V_i} = L^2(\mathbb{R}^n)$.

The complete definition of an MRA includes the assumption that V_0 is generated by the integer translates of a $\phi \in V_0$, called the *scaling function*, and that these translates $\{T_k\phi\colon k \in \mathbb{Z}^2\}$ are an orthonormal basis of V_0. In our situation, as we will discuss later on, there is an analogous property that will replace the "scaling" property.

Let $A = \{a^i\colon i \in \mathbb{Z}\}$ and $B = \{b^j\colon j \in \mathbb{Z}\}$, and W_0 be the orthogonal complement of V_0 in V_1, that is, $V_1 = V_0 \oplus W_0$. We shall now show how to construct an ON AB-multiwavelet generated by three mutually orthogonal functions $\psi^1, \psi^2, \psi^3 \in W_0$ of norm 1. It will be convenient to work in the Fourier domain. Thus, $\hat{V}_1 = \hat{V}_0 \oplus \hat{W}_0$ and, consequently, $\hat{W}_0 = L^2(R_0)$, where $R_0 = S_1 \setminus S_0 = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon 1 < |\xi_1| \leqslant 2\}$. We begin by constructing a particular orthonormal basis of W_0 that it is mapped into itself by the representation π. To do this, define the following subsets of $R_0 = S_1 \setminus S_0$:

$$I_1 = I_1^+ \cup I_1^-, \qquad I_2 = I_2^+ \cup I_2^-, \qquad I_3 = I_3^+ \cup I_3^-,$$

where

$$I_1^+ = \bigl\{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon 1 < \xi_1 \leqslant 2,\ 0 \leqslant \xi_2 < 1/2\bigr\},$$
$$I_2^+ = \bigl\{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon 1 < \xi_1 \leqslant 2,\ 1/2 \leqslant \xi_2 < 1\bigr\},$$
$$I_3^+ = \bigl\{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon 1 < \xi_1 \leqslant 2,\ 1 \leqslant \xi_2 < \xi_1\bigr\},$$

and $I_\ell^- = \{\xi \in \hat{\mathbb{R}}^2\colon -\xi \in I_\ell^+\}$, $\ell = 1, 2, 3$. These sets are shown in Fig. 1. We then define ψ^ℓ, $\ell = 1, 2, 3$, by setting $\hat{\psi}^\ell = \chi_{I_\ell}$, $\ell = 1, 2, 3$. Observe that each set I_ℓ is a fundamental domain of $\mathbb{Z}^2$, that is, the functions $\{e^{2\pi i \xi k}\colon k \in \mathbb{Z}^2\}$, restricted to I_ℓ, form an orthonormal basis of $L^2(I_\ell)$. It follows that the collection

$$\bigl\{e^{2\pi i \xi k} \hat{\psi}^\ell(\xi)\colon k \in \mathbb{Z}^2\bigr\}$$

is an orthonormal basis of $L^2(I_\ell)$, $\ell = 1, 2, 3$. A simple direct calculation shows that the sets $\{I_\ell b^j\colon j \in \mathbb{Z},\ \ell = 1, 2, 3\}$ are a partition of R_0, that is,

$$\bigcup_{\ell=1}^{3} \bigcup_{j \in \mathbb{Z}} I_\ell b^j = R_0,$$

where the union is disjoint. It follows that the collection

$$\bigl\{e^{2\pi i \xi k} \hat{\psi}^\ell(\xi b^j)\colon k \in \mathbb{Z}^2,\ j \in \mathbb{Z},\ \ell = 1, 2, 3\bigr\} \tag{2.4}$$

is an orthonormal basis of $L^2(R_0)$ and, thus, by taking the inverse Fourier transform of (2.4), we have that

$$\bigl\{\pi(b^j, k)\psi^\ell\colon k \in \mathbb{Z}^2,\ j \in \mathbb{Z},\ \ell = 1, 2, 3\bigr\} \tag{2.5}$$

is an orthonormal basis of $W_0 = L^2(R_0)^\vee$. Notice that, since, for each $j \in \mathbb{Z}$ fixed, b^j maps $\mathbb{Z}^2$ into itself, the collection $\{e^{2\pi i \xi b^j k}\colon k \in \mathbb{Z}^2\}$ is equal to the collection $\{e^{2\pi i \xi k}\colon k \in \mathbb{Z}^2\}$.

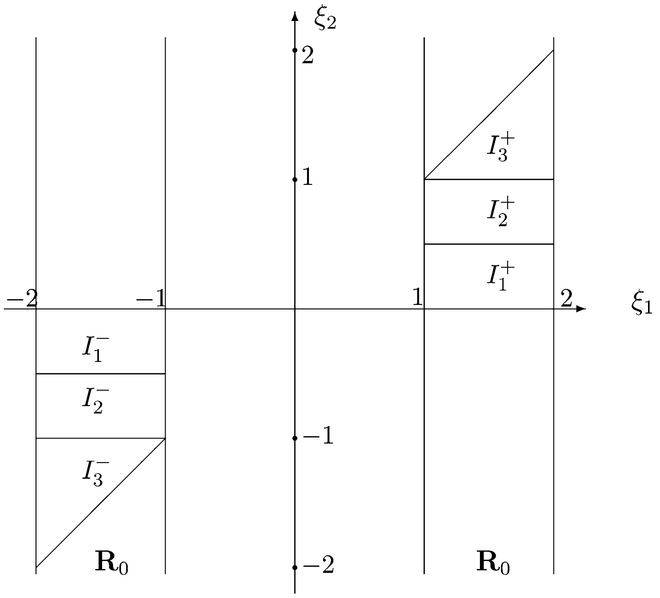

Fig. 1. Example of ON AB-multiwavelet. The sets $\{I_\ell b^j\colon\ j \in \mathbb{Z},\ \ell = 1, 2, 3\}$ are a disjoint partition of R_0.

Observe that the number of generators, three, of the orthonormal basis (2.5) of W_0 is independent of the choice of the functions ψ^ℓ. That is, if

$$\left\{\pi(b^j, k)\phi^\ell\colon\ k \in \mathbb{Z}^2,\ j \in \mathbb{Z},\ \ell = 1, \dots, L\right\},$$

for some functions $\phi^\ell \in L^2(\mathbb{R}^2)$, is an orthonormal basis of $L^2(R_0)$, then ℓ must range through the set $\{1, 2, 3\}$. This is a consequence of the following general result:

Proposition 2.1. *Let G be a countable set and, for each $u \in G$, let T_u be a unitary operator acting on a Hilbert space $\mathcal{H}$. Assume that, for each T_u, there is a unique $u^* \in G$ such that $T_{u^*} = T_u^*$. Suppose $\Phi = \{\phi^1, \dots, \phi^N\}$, $\Psi = \{\psi^1, \dots, \psi^M\} \subset \mathcal{H}$, where $N, M \in \mathbb{N} \cup \{\infty\}$. If $\{T_u\phi^k\colon\ u \in G,\ 1 \leqslant k \leqslant N\}$ and $\{T_u\psi^i\colon\ u \in G,\ 1 \leqslant i \leqslant M\}$ are each orthonormal bases for $\mathcal{H}$, then $N = M$.*

Observe that if G is a group and T_u, $u \in G$, is a unitary representation of G acting on $\mathcal{H}$, then the assumption of this proposition are satisfied. This is the situation we encounter in the case of AB-wavelets.

Proof of Proposition 2.1. It follows from the assumptions that, for each $1 \leqslant k \leqslant N$,

$$\|\phi^k\|^2 = \sum_{u\in G}\sum_{i=1}^{M}\left|\langle\phi^k, T_u\psi^i\rangle\right|^2.$$

Thus, using the unitary property of T_u, we have

$$N = \sum_{k=1}^{N}\|\phi^k\|^2 = \sum_{k=1}^{N}\sum_{u\in G}\sum_{i=1}^{M}\left|\langle\phi^k, T_u\psi^i\rangle\right|^2 = \sum_{i=1}^{M}\sum_{u^*\in G}\sum_{k=1}^{N}\left|\langle T_{u^*}\phi^k, \psi^i\rangle\right|^2 = \sum_{i=1}^{M}\|\psi^i\|^2 = M. \qquad \square$$

In order to obtain the desired ON AB affine system for $L^2(\mathbb{R}^2)$, we apply the dilations D_a^i, $i \in \mathbb{Z}$, to the orthonormal system (2.5). This is easily seen in the Fourier domain, since the action of these dilations on the region R_0 generates the sets

$$R_i = R_0 a^i = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ 2^i < |\xi_1| \leqslant 2^{i+1}\},$$

and we have that $\bigcup_{i\in\mathbb{Z}} R_i = \hat{\mathbb{R}}^2$, where the union is disjoint. Since the dilations D_a^i are unitary operators, they map an orthonormal basis into an orthonormal basis and, thus, for each $i \in \mathbb{Z}$, the set $\{D_a^i \pi(b^j, k)\psi^\ell\colon\ k \in \mathbb{Z}^2,\ j \in \mathbb{Z},\ \ell = 1, 2, 3\}$ is an orthonormal basis of $L^2(R_i)^\vee = W_i$. Since the spaces $L^2(R_i)$ (and thus the spaces W_i) are mutually orthogonal, it follows that the system

$$\begin{aligned}&\{D_a^i \pi(b^j, k)\psi^\ell\colon\ k \in \mathbb{Z}^2,\ i, j \in \mathbb{Z},\ \ell = 1, 2, 3\}\\ &\quad = \{D_a^i D_b^j T_k \psi^\ell\colon\ k \in \mathbb{Z}^2,\ i, j \in \mathbb{Z},\ \ell = 1, 2, 3\}\end{aligned} \tag{2.6}$$

is an orthonormal basis of $L^2(\mathbb{R}^2) = \bigoplus_{i\in\mathbb{Z}} W_i$, that is, $\Psi = \{\psi^1, \psi^2, \psi^3\}$ is an ON AB-multiwavelet.

The number of generators of this ON AB-multiwavelet is fixed. Indeed, by Proposition 2.1, if we could replace Ψ in (2.5) by a $\Phi = \{\phi^1, \dots, \phi^L\}$, then $L = 3$, and this applies to (2.6) as well. As we will show in Section 5.2, the Fourier transform of the multiwavelets $\hat{\phi}^\ell$ need not be characteristic functions.

Recall that a countable family $\{e_j\colon\ j \in \mathcal{J}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a *frame* if there exist constants $0 < A \leqslant B < \infty$ satisfying

$$A\|v\|^2 \leqslant \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2 \leqslant B\|v\|^2$$

for all $v \in \mathcal{H}$. A frame is *tight* if A and B can be chosen so that $A = B$, and is a *Parseval frame* (PF) (also called *normalized tight frame*) if $A = B = 1$. Thus, if $\{e_j\colon\ j \in \mathcal{J}\}$ is a Parseval frame in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2$$

for each $v \in \mathcal{H}$. This is equivalent to the reproducing formula

$$v = \sum_{j\in\mathcal{J}} \langle v, e_j\rangle e_j \tag{2.7}$$

for all $v \in \mathcal{H}$, where the series in (2.7) converges in the norm of $\mathcal{H}$. Equations (2.7) shows that a Parseval frame provides a basis-like representation. In general, however, a PF need not be a basis. We refer the reader to [9,14] for more details about frames.

We will now show how we can construct MRA PF AB wavelet systems with a *single* generator. To do this we modify the construction of ON systems that led to (2.6). We begin with $T = T^+ \cup T^-$, where T^+ is the trapezoidal region with vertices $(1/2, 0)$, $(1/2, 1/2)$, $(1, 0)$ and $(1, 1)$, and $T^- = \{\xi \in \hat{\mathbb{R}}^2\colon\ -\xi \in T^+\}$; let $R = S_0 \setminus S_{-1} = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ 1/2 < |\xi_1| \leqslant 1\}$. This is illustrated in Fig. 2. A direct computation shows that $\bigcup_{j\in\mathbb{Z}} T b^j = R$, where the union is disjoint. It follows from the Plancherel theorem (using the fact that T is contained inside a fundamental domain) that the function $\chi_T(\xi)$ satisfies $\sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, e^{2\pi i(\cdot)k}\chi_T\rangle|^2 = \|\hat{f}\|^2$ for all $\hat{f} \in L^2(R)$, and thus the collection

$$\{D_b^j e^{2\pi i\xi k}\chi_T(\xi)\colon\ k \in \mathbb{Z}^2,\ j \in Z\}$$

is a Parseval frame of $L^2(T)$. Similarly to the construction above, we have that $\bigcup_{i\in\mathbb{Z}} R a^i = \hat{\mathbb{R}}^2$, where the union is disjoint, and so it follows that the set

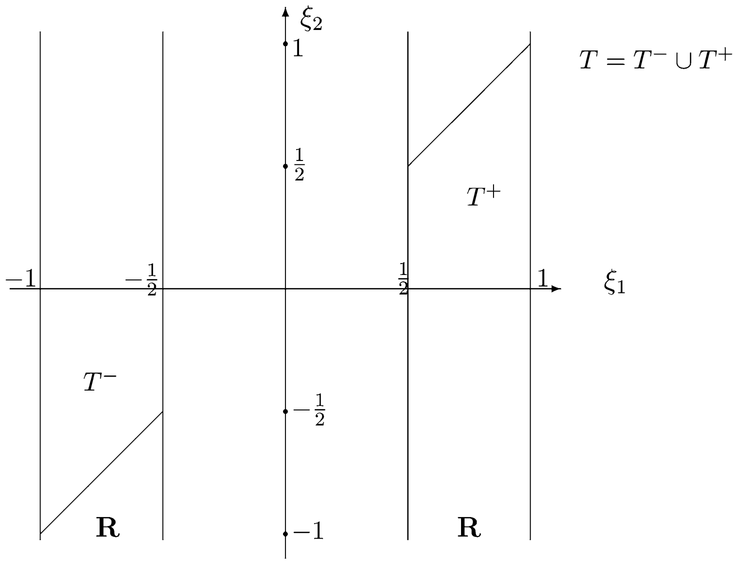

Fig. 2. Example of PF AB-wavelet. The sets $\{Tb^j\colon\ j \in \mathbb{Z}\}$ are a disjoint partition of R.

$$\left\{D_a^i \pi(b^j, k)\psi\colon\ k \in \mathbb{Z}^2,\ i, j \in \mathbb{Z}\right\} = \left\{D_a^i D_b^j T_k \psi\colon\ k \in \mathbb{Z}^2,\ i, j \in \mathbb{Z}\right\}$$

where $\psi = (\chi_T)^\vee$ is a PF for $L^2(\mathbb{R}^2) = \bigoplus_{i \in \mathbb{Z}} L^2(Ra^i)^\vee$, that is, ψ is a Parseval frame AB-wavelet.

It is not hard to see that, by modifying the function ψ, one can obtain singly generated ON AB-wavelets (cf. [11]). It is important to point out that, as we will discuss in Sections 5 and 6, those singly generated ON AB-wavelets are not of MRA type. These remarks make clear that the construction of AB-Parseval frames is simpler than the corresponding construction of ON AB-multiwavelets. Because of this fact, and because Parseval frames are as effective as ON bases in many applications, in the following we will concentrate mostly on the construction of Parseval frames AB-wavelets, that are not necessarily orthonormal bases.

We end this section by stating some basic properties of the translation and dilation operators, that will be used throughout the paper.

Proposition 2.2. *Let*

$$G = \left\{U = D_a T_y\colon\ (a, y) \in GL_n(\mathbb{R}) \times \mathbb{R}^n\right\}.$$

G is a subgroup of the group of unitary operators on $L^2(\mathbb{R}^n)$ which is preserved by the action of the operator $U \mapsto \hat{U}$, where $\hat{U}\hat{f} = (Uf)^\wedge$. In particular, we have:

(i) $D_a T_y = T_{ay} D_a$;
(ii) $D_{a_1} D_{a_2} = D_{a_1 a_2}$ *for each* $a_1, a_2 \in GL_n(\mathbb{R})$;
(iii) *for* $U = D_a T_y$, $\hat{U} = \hat{D}_a M_{-y}$, *where* $\hat{D}_a \hat{f}(\xi) = |\det a|^{1/2} \hat{f}(\xi a)$;
(iv) *for a measurable set* $S \subset \hat{\mathbb{R}}^n$ *and* $L^2(S) = \{\hat{f} \in L^2(\hat{\mathbb{R}}^n)\colon\ \operatorname{supp} \hat{f} \subseteq S\}$, *we have* $\hat{D}_a L^2(S) = L^2(Sa^{-1})$.

3. The admissibility condition

In Section 2, we have examined some special cases of *affine systems* associated with the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, a countable collection $\mathcal{C} \subset GL_n(\mathbb{R})$ containing the $n \times n$ identity matrix I_n, and a set $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$, having the form

$$\mathcal{A}_{\mathcal{C}}(\Psi) = \{D_c T_k \Psi\colon\ c \in \mathcal{C},\ k \in \mathbb{Z}^n\}. \tag{3.1}$$

Our main concern here is to establish conditions on $\mathcal{C}$ that guarantee the existence of a finite set of functions Ψ such that $\mathcal{A}_{\mathcal{C}}(\Psi)$, given by (3.1), is either an orthonormal basis or a Parseval frame for $L^2(\mathbb{R}^n)$. When this is the case, we say that Ψ is an *orthonormal (ON) $\mathcal{C}$-multiwavelet* or a *Parseval frame $\mathcal{C}$-multiwavelet*, respectively, for $L^2(\mathbb{R}^n)$. More generally, when $S \subset \hat{\mathbb{R}}^n$ has positive Lebesgue measure and $Sc = S$ for each $c \in \mathcal{C}$, we say that Ψ is an ON or a Parseval frame $\mathcal{C}$-multiwavelet for $L^2(S)^\vee$, if $\mathcal{A}_{\mathcal{C}}(\Psi)$ is an ON basis or a Parseval frame, respectively, for $L^2(S)^\vee$. For example, in the construction of Section 2, we consider affine systems on $L^2(S_i)^\vee$, $i \in \mathbb{Z}$, where the strip domains $S_i \subset \hat{\mathbb{R}}^2$ are invariant with respect to the matrices $b \in B$.

It is an open problem to give necessary and sufficient conditions on $\mathcal{C}$ for which $\mathcal{C}$ multiwavelets for $L^2(S)^\vee$ exist. In all known cases where they exist, $\mathcal{C}$ satisfies a geometric condition that we call the *tiling* property. Namely, if there exist measurable subsets $R_1, \dots, R_L$ of S such that a.e. $\xi \in S \subset \hat{\mathbb{R}}^n$ uniquely determines an index $1 \leqslant i \leqslant L$, $\eta \in R_i$, and a $c \in \mathcal{C}$, for which $\xi = \eta c^{-1}$, we say that the sets $\{R_\ell\colon\ \ell = 1, \dots, L\}$ are *S-tiling sets for the dilation set $\mathcal{C}^{-1}$*. Equivalently, we have that

$$S = \bigcup_{c \in \mathcal{C}} \bigcup_{1 \leqslant \ell \leqslant L} R_\ell c^{-1}, \tag{3.2}$$

where the union is disjoint in measure. If $S = \mathbb{R}^n$, we simply say that the sets $\{R_\ell\colon\ \ell = 1, \dots, L\}$ are tiling sets for $\mathcal{C}^{-1}$. The property (3.2) ensures that $L^2(S)^\vee$ is the orthogonal direct sum

$$L^2(S)^\vee = \bigoplus_{c \in \mathcal{C},\ 1 \leqslant \ell \leqslant L} L^2(R_\ell c^{-1})^\vee.$$

Therefore, for $\Psi = \{\psi^1, \dots, \psi^L\}$, where $\psi^\ell = (\chi_{R_\ell})^\vee$, the system $\mathcal{A}_{\mathcal{C}}(\Psi)$ given by (3.1) is a Parseval frame for $L^2(S)^\vee$ if and only if, for each $1 \leqslant i \leqslant L$, the collection

$$\left\{(T_k \psi^i)^\wedge = e^{2\pi i k \xi} \chi_{R_i}\colon\ k \in \mathbb{Z}^n\right\}$$

is a Parseval frame for $L^2(R_i)^\vee$. By an elementary Fourier series argument, this occurs precisely if the sets $R_1, \dots, R_L$ satisfy

$$(R_\ell + k) \cap R_\ell = 0 \quad \text{for } k \in \hat{\mathbb{Z}}^n \setminus \{0\},\ 1 \leqslant \ell \leqslant L,\ \text{up to sets of measure zero}, \tag{3.3}$$

in which case we say that the sets $\{R_\ell\colon\ \ell = 1, \dots, L\}$ are *packing sets for $\mathbb{Z}^n$ translations*.[3] Observe that this condition implies that the measure of each set R_ℓ cannot be larger than one. Therefore we have the following:

Proposition 3.1. *Let $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(S)^\vee$, where $\psi^\ell = (\chi_{R_\ell})^\vee$ for $1 \leqslant \ell \leqslant L$. Ψ is a Parseval frame $\mathcal{C}$-multiwavelet for $L^2(S)^\vee$ if and only if* (3.2) *and* (3.3) *hold.*

[3] Recall that in Section 2 we introduced the notion of "fundamental domain." Observe that a packing set for $\mathbb{Z}^n$ translations is a subset of a fundamental domain for $\mathbb{Z}^n$.

Whenever Ψ is of the form given by Proposition 3.1, we say that Ψ is a *tiling* (or *MSF*) $\mathcal{C}$-multiwavelet of $L^2(S)^\vee$. In Section 5, we show how tiling $\mathcal{C}$-multiwavelets can be smoothed off to obtain more general $\mathcal{C}$-multiwavelets. However, it is not known whether the existence of a $\mathcal{C}$-multiwavelet implies the existence of a tiling $\mathcal{C}$-multiwavelet.

Note that in the example of Parseval frame AB-wavelet from Section 2, we construct a set $T \subset \hat{\mathbb{R}}^2$ having the properties

(i) $\bigcup_{i,j} T(a^i b^j)^{-1} = \hat{\mathbb{R}}^2 \setminus \{(0,\xi_2)\colon \xi_2 \in \mathbb{R}\}$;
(ii) $(T+k) \cap T = \varnothing$ for all $k \in \mathbb{Z}^2 \setminus \{0\}$.

This shows that Eqs. (3.2) and (3.3) are satisfied, and so it follows that $\psi = (\chi_T)^\vee$ is a PF $\mathcal{C}$-wavelet for $L^2(\mathbb{R}^2)$, where $\mathcal{C} = \{a^i b^j\colon i, j \in \mathbb{Z}\}$.

The set $\mathcal{C}$ is called *S-admissible* if tiling $\mathcal{C}$-multiwavelets for $L^2(S)^\vee$ exist. In case $S = \hat{\mathbb{R}}^n$, we will simply say *admissible* (rather than $\hat{\mathbb{R}}^n$-admissible). In the following, we will briefly examine the relationship between the notion of admissibility that we have just introduced, and the theory of continuous wavelets (Section 3.1). Next, in Section 3.2 we will show that the admissibility condition is closely related to a condition that we call local admissibility. In Section 3.3 we examine the admissibility for dilation sets of the form $\mathcal{C} = AB$, and look at two types of examples unlike those in Section 2. In Section 3.4 we give a complete discussion of the theory that generalizes the examples in Section 2.

3.1. Connection to the theory of continuous multiwavelets

For $\mathcal{C}$ and S defined as in the previous section, we say that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(S)^\vee$ is a *continuous* $\mathcal{C}$-multiwavelet if

$$\|f\|^2 = \sum_{\ell=1}^{L} \sum_{c \in \mathcal{C}} \int_{\mathbb{R}^n} \left|\langle f, D_c T_y \psi^\ell \rangle\right|^2 \mathrm{d}y \tag{3.4}$$

for all $f \in L^2(S)^\vee$. By a trivial extension of an argument in [16], one shows that Ψ satisfies (3.4) if and only if it satisfies the *Calderòn equation*

$$\sum_{\ell=1}^{L} \sum_{c \in \mathcal{C}} \left|\hat{\psi}^\ell(\xi c)\right|^2 = 1 \quad \text{for a.e. } \xi \in S. \tag{3.5}$$

It is easy to see that every tiling $\mathcal{C}$-multiwavelet is also a continuous $\mathcal{C}$-multiwavelet. In fact, if $\Psi = \{\psi^1, \dots, \psi^L\}$ where $\psi^\ell = (\chi_{R_\ell})^\vee$ and the sets $\{R_\ell\colon 1 \leqslant \ell \leqslant L\}$ satisfy Eqs. (3.2) and (3.3), then (3.5) is immediately satisfied. More generally, it is shown in [13] that, when $\mathcal{C}$ satisfies a technical property called the *local integrability condition* (LIC), then the Calderòn equation is one of a family of equations characterizing $\mathcal{C}$-multiwavelets for $L^2(\mathbb{R}^n)$. The LIC is satisfied, for example, when $\mathcal{C}$ is of the form $\mathcal{C} = \{a^i\colon i \in \mathbb{Z}\}$ where $a \in GL_n(\mathbb{R})$ is an *expanding* matrix (that is, all the eigenvalues λ of a satisfy $|\lambda| > 1$). Finally, observe that there are no known examples of $\mathcal{C}$-multiwavelets which are not *continuous* $\mathcal{C}$-multiwavelets, i.e., do not satisfy the Calderòn equation.

3.2. The local admissibility condition

As above, let $\mathcal{C} \subset GL_n(\mathbb{R})$ be a countable set containing the identity matrix I. We say that $\mathcal{C}$ is *locally admissible* if, for a.e. $\xi \in \hat{\mathbb{R}}^n$, there is an open neighborhood U of ξ such that, for $c_1, c_2 \in \mathcal{C}$ with $c_1 \neq c_2$, we have that $Uc_1^{-1}c_2 \cap U = \emptyset$. In particular, this means that the set of points $\{\xi c^{-1}\colon c \in \mathcal{C}\}$ is discrete in the topology of $\hat{\mathbb{R}}^n$. We can assume that U is contained in a cube of side 1 centered at ξ. Then, for $S = \bigcup_{c\in\mathcal{C}} Uc^{-1}$, the set $\mathcal{C}$ is S-admissible and $\chi_U^\vee$ is a tiling wavelet for $L^2(S)^\vee$.

Under certain assumptions on $\mathcal{C}$, one can take $S = \hat{\mathbb{R}}^n$. Consider for example the situation where $\mathcal{C} = \{a^i\colon i \in \mathbb{Z}\}$ and $a \in GL_n(\mathbb{R})$ is an expanding matrix. $\mathcal{C}$ is clearly locally admissible. Let $U \subset [-1/2, 1/2]^n \subset \hat{\mathbb{R}}^n$ be an open neighborhood of the origin (this implies that U is a packing set for $\mathbb{Z}^n$ translations). Since a is expanding, we can pick such a U so that $Ua^{-1} \subset U$. Therefore, if we let $T = U \setminus (Ua^{-1})$, then T is a tiling set for the set of dilations $\mathcal{C}$ (observe that $\mathcal{C}$ is a group and so $\mathcal{C} = \mathcal{C}^{-1}$). This shows that a tiling $\mathcal{C}$-wavelet for $L^2(\mathbb{R}^n)$ exists. The following section elaborates this situation further, by showing an example of a dilation set that is not locally admissible.

Example of a nonadmissible dilation set

Consider the set $\mathcal{C} = \{2^i 3^j\colon i, j \in \mathbb{Z}\}$. This set is not locally admissible in view of the fact that $\ln 3/\ln 2$ is irrational and so $\{\ln(2^i 3^j)\colon i, j \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Using this fact, the following argument will show that no $\mathcal{C}$-wavelets for $L^2(\mathbb{R})$ exist.

In fact, if such a wavelet ψ exists, then it satisfies the Calderón condition

$$\sum_{i,j\in\mathbb{Z}} \left|\hat{\psi}(2^i 3^j \xi)\right|^2 = 1 \quad \text{a.e. } \xi \in \hat{\mathbb{R}}.$$

We claim that no such ψ exists. If it did, then we could find an $n \in \mathbb{Z}$ and a measurable set $R \subset [n, n+1]$ of positive measure such that $|\hat{\psi}(\xi)| \geqslant \delta$ for some $\delta > 0$ and for all $\xi \in R$. Fix such n and δ. Since $\|\psi\| \leqslant 1$, it follows that $\int_n^{n+1} |\hat{\psi}(\xi)|^2\, \mathrm{d}\xi \geqslant \delta^2 |R|$ and so

$$\delta^2 \leqslant \frac{1}{|R|} \leqslant \frac{|n|+1}{|R|}. \tag{3.6}$$

It is easy to see that there is a countably infinite set $\mathcal{P}$ of elements p of the form $p = 2^j 3^i$, $i, j \in \mathbb{Z}$, such that

$$1 < p < 1 + \frac{\delta^2 |R|}{2(|n|+1)}. \tag{3.7}$$

From (3.6) and (3.7) we have that $p < 1 + |R|/(2|R|) = 3/2$ and, thus, $1/p > 1/2$. Using this observation, (3.7) and the fact that $\|\psi\| \leqslant 1$, we have

$$\begin{aligned}\int_n^{n+1} \left|\hat{\psi}(p\xi)\right|^2 \mathrm{d}\xi &= \frac{1}{p} \int_{pn}^{p(n+1)} \left|\hat{\psi}(\xi)\right|^2 \mathrm{d}\xi \geqslant \frac{1}{2}\left(\int_n^{n+1} \left|\hat{\psi}(\xi)\right|^2 \mathrm{d}\xi - \int_n^{pn} \left|\hat{\psi}(\xi)\right|^2 \mathrm{d}\xi \right) \\ &\geqslant \frac{1}{2}\big(\delta^2|R| - n(p-1)\big) \geqslant \frac{1}{2}\big(\delta^2|R| - n\delta^2|R|/(2|n|+1)\big) \geqslant \frac{\delta^2|R|}{4}. \end{aligned} \tag{3.8}$$

Thus, using (3.8) and the Calderón condition we have

$$1 = \int_n^{n+1} \sum_{i,j\in\mathbb{Z}} \left|\hat{\psi}(2^i 3^j \xi)\right|^2 \mathrm{d}\xi \geqslant \int_n^{n+1} \sum_{p\in\mathcal{P}} \left|\hat{\psi}(p\xi)\right|^2 \mathrm{d}\xi \geqslant \frac{\delta^2|R|}{4} \sum_{p\in\mathcal{P}} 1 = \infty.$$

This contradicts the Calderón condition. It is easy to see that the same argument applies to any $A = \{a^j b^i \colon i, j \in \mathbb{Z}\}$, with $a, b \in \mathbb{Z} \setminus \{0, 1\}$ relatively prime. The same argument also applies if one replaces $L^2(\mathbb{R})$ by $L^2(S)^\vee$, where $S \subset \hat{R}$ is a set of positive measure.

Consider, on the other hand,

$$\mathcal{C}' = \left\{ \begin{pmatrix} 2^i & 0 \\ 0 & 3^j \end{pmatrix} \colon i, j \in \mathbb{Z} \right\}.$$

This set is locally admissible, and an argument similar to the one described in the second paragraph of Section 3.2, where $\mathcal{C} = \{a^i \colon i \in \mathbb{Z}\}$ and a is an expanding matrix, shows that tiling $\mathcal{C}'$-wavelets for $L^2(\mathbb{R}^2)$ exist.

3.3. Admissibility condition. The AB case

If $B \subset GL_n(\mathbb{R})$ is S-admissible and $c \in GL_n(\mathbb{R})$, then cB is Sc^{-1}-admissible since the unitary operator D_c maps the PF $\mathcal{A}_B(\psi)$ for $L^2(S)^\vee$ onto the PF $\mathcal{A}_{cB}(\psi)$ for $L^2(Sc^{-1})^\vee$, where ψ is a PF cB-wavelet for $L^2(Sc^{-1})^\vee$. In particular, this holds for $c = b^{-1}$, where $b \in B$. In this case, $b^{-1}B$ is still S-admissible since $Sb = S$, and thus there is no loss of generality in assuming $I_n \in B$. We will be especially interested in the situation where B is S-admissible and there is a countable set $A \subset GL_n(\mathbb{R})$ for which S is a tiling set for A. Then

$$L^2(\mathbb{R}^n) = \bigoplus_{a\in A} L^2(Sa^{-1})^\vee = \bigoplus_{a\in A} D_a\bigl(L^2(S)\bigr)^\vee, \tag{3.9}$$

and it follows that the set $\mathcal{C} = AB = \{ab \colon a \in A,\ b \in B\}$ is admissible, and ψ is a PF AB-wavelet whenever ψ is a B-wavelet for $L^2(S)^\vee$. It is clear that a similar approach holds for multiwavelets $\Psi \subset L^2(\mathbb{R}^n)$. A particular instance of this phenomenon was illustrated in Section 2, where $A = \{a^i \colon i \in \mathbb{Z}\}$ with $a = \left(\begin{smallmatrix} 2 & 0 \\ 0 & \epsilon \end{smallmatrix}\right)$ and $B = \{b^j \colon j \in \mathbb{Z}\}$ with $b = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Observe that in these examples the right action of A doubles the first coordinate. On the other hand, the action of A on the second coordinate is irrelevant, since the right action of B leaves the first coordinate fixed and uses the first coordinate to control the second one.

In the following sections, will show that there are many possible choices for A and B, and that they do not have to be subgroups of $GL_n(\mathbb{R})$. Unifying all these examples of admissible AB-multiwavelets that we are going to construct is a not necessarily linear change of coordinates map $\phi(t, s)$ from $\hat{\mathbb{R}}^k \times \hat{\mathbb{R}}^{n-k}$ onto a set of full measure in $\hat{\mathbb{R}}^n$. Like in the two-dimensional example above, the action of A will be "upper triangular," in the sense that, for $a \in A$, $\phi(t, s)a = \phi(ta', s')$, where $a \in A'$ and A' is a set (or a group) of operators on $\hat{\mathbb{R}}^k$ that admits tiling sets for the A' dilations. The action of A on the coordinate $s \in \hat{\mathbb{R}}^{n-k}$ is irrelevant. On the other hand, the action of B will leave the coordinate $t \in \hat{\mathbb{R}}^k$ invariant: for $b \in B$ we have that $\phi(t, s)b = \phi(t, \sigma_t(s, b))$ for some transformation $\sigma_t(\cdot, b)$ on $\hat{\mathbb{R}}^{n-k}$. As t varies over a compact set $K \in \hat{\mathbb{R}}^k$, we will be able to construct a set R which is an S-tiling set for the B dilations, where S is the strip domain $K \times \hat{\mathbb{R}}^{n-k}$. This general procedure will be illustrated in Sections 3.3.1 and 3.3.2

for the case of spherical and hyperbolic coordinates, respectively. Next, in Section 3.4, we consider the linear coordinate systems, by generalizing the examples in Section 2.

3.3.1. Orthogonal AB-multiwavelets

Perhaps the simplest class of admissible AB-multiwavelets is obtained when B is a finite group. Such a B is conjugate to a subgroup of the orthogonal group $O_n(\mathbb{R})$; i.e., given any finite group B, there is a $P \in GL_n(\mathbb{R})$ and a subgroup $\tilde{B} \subseteq O_n(\mathbb{R})$ such that $PBP^{-1} = \tilde{B}$. Thus, without loss of generality, by conjugating both A and B by P, we may assume that $B \subset O_n(\mathbb{R})$. Let $S_0 \subset \hat{\mathbb{R}}^n$ be a compact region, starlike with respect to the origin, with the property that B maps S_0 into itself. In many situations, one can find a lattice $L \subset \mathbb{R}^n$ and a region $U_0 \subseteq S_0$ such that U_0 is both a S_0-tiling set for the B dilations and a packing set for the Λ translations (i.e., $(U_0 + \lambda) \cap U_0 = 0$ for $\lambda \in \Lambda \setminus \{0\}$), where $\Lambda = \{\lambda \in \hat{\mathbb{R}}^n\colon \lambda l \in \mathbb{Z}, \forall l \in L\}$ is the lattice dual to L. Then

$$\Phi_B = \{D_b T_l (\chi_{U_0})^\vee\colon b \in B,\ l \in L\}$$

is a PF for $L^2(S_0)^\vee$. Next suppose that $A = \{a^i\colon i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$ is expanding, $aBa^{-1} = B$ and $S_0 \subseteq S_0 a = S_1$. These assumptions imply that each region $S_i = S_0 a^i$, $i \in \mathbb{Z}$, is B-invariant and the family of disjoint regions $S_{i+1} \setminus S_i$, $i \in \mathbb{Z}$, tiles $\hat{\mathbb{R}}^n$. Thus, one can decompose $L^2(\mathbb{R}^n)$ as in (3.9). Since B is finite, there exist many choices of a measurable set $R \subset S_1 \setminus S_0$ for which R is a $(S_1 \setminus S_0)$-tiling set for the B dilations. Since a is expanding, we can always take S_0 to be contained in a small neighborhood of the origin, and thereby ensuring that R is a packing set for the Λ translations. Then

$$\Psi_{AB} = \{D_a^i D_b T_l (\chi_R)^\vee\colon b \in B,\ i \in \mathbb{Z},\ l \in L\}$$

is a PF for $L^2(\mathbb{R}^n)$. On the other hand, if U_0 is a tiling region for the Λ translations, that is, $\bigcup_{\lambda\in\Lambda}(U_0 + \lambda) = \hat{\mathbb{R}}^n$ where the union is disjoint, every such tiling set has the same measure as U_0. If $|\det a| \in \mathbb{N}$, then $|S_1| = |\det a||S_0| = |\det a| \operatorname{card}(B)|U_0|$ and it follows that no single subset R of $S_1 \setminus S_0$ can be both a $(S_1 \setminus S_0)$-tiling set for the B dilations and a tiling set for the Λ translations. Instead, if R is a $(S_1 \setminus S_0)$-tiling set for the B dilations, then one can decompose R into a disjoint union of subregions $R_1, \ldots, R_N$ (where $N = |\det a| - 1$) each of which is a tiling set for the Λ translations. It follows that

$$\widetilde{\Psi}_{AB} = \{D_a^i D_b T_l (\chi_{R_\ell})^\vee\colon i \in \mathbb{Z},\ b \in B,\ l \in L,\ \ell = 1, \ldots, N\}$$

is a an ON AB-multiwavelet for $L^2(\mathbb{R}^n)$. Moreover, in this case, the set Φ_B is a ON basis for $L^2(S_0)^\vee$. Some special examples of this construction can be found in [10] and [11, Section 2.2].

3.3.2. Hyperbolic AB-wavelets

By using a nonlinear system of coordinates, we can construct a variant of the system described in Section 2, where B does not consist of shear matrices.

Fix $\lambda > 1$ and let

$$B = \left\{ b_j = \begin{pmatrix} \lambda^j & o \\ 0 & \lambda^{-j} \end{pmatrix}\colon j \in \mathbb{Z} \right\}.$$

For $k > 0$, the set $H_k = \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon \xi_1\xi_2 = k\}$ consists of four hyperbolas. Observe that, for any $\xi = (\xi_1, \xi_2) \in H_k$, every other point ξ' on the same hyperbola has the unique representation $\xi' = (\xi_1\lambda^t, \xi_2\lambda^{-t})$ for some $t \in \mathbb{R}$. We can parametrize any $\xi = (\xi_1, \xi_2)$ in the first quadrant by $\xi(r, t) = (\sqrt{r}\lambda^t, \sqrt{r}\lambda^{-t})$, where $r \geqslant 0$, $t \in \mathbb{R}$. Then, for any $k_1 < k_2$, the set $T^1(k_1, k_2) = \{\xi(r, t)\colon k_1 \leqslant r < k_2,\ 0 \leqslant t < 1\}$ is a hyperbolic trapezoidal region. Also observe that, for any $k \neq 0$, the right action of B preserves the set H_k since

$$\xi b_j = (\xi_1, \xi_2)\begin{pmatrix} \lambda^j & o \\ 0 & \lambda^{-j} \end{pmatrix} = (\xi_1 \lambda^j, \xi_2 \lambda^{-j}) = (\eta_1, \eta_2)$$

and $\eta_1\eta_2 = \xi_1\xi_2$. Therefore, the set $T^1(k_1, k_2)$ is an $S^1(k_1, k_2)$-tiling set for the B dilations, where $S^1(k_1, k_2)$ is the hyperbolic strip $\{\xi(r, t)\colon\ k_1 \leqslant r < k_2\}$. Proceeding similarly in the other quadrants, we obtain that the similarly defined trapezoidal regions $T^\ell(k_1, k_2)$, $\ell = 2, 3, 4$, are $S^\ell(k_1, k_2)$-tiling sets for the B dilations. By taking unions, we have that $T(k_1, k_2) = \bigcup_{\ell=1^4} T^\ell(k_1, k_2)$ is a $S(k_1, k_2)$-tiling set for the B dilations, where $S(k_1, k_2) = \bigcup_{\ell=1^4} S^\ell(k_1, k_2)$.

Now let $A = \{a^i\colon\ i \in \mathbb{Z}\} \subset GL_2(\mathbb{R})$, where a is diagonal with $m = |\det a| > 1$. Then, for each $k > 0$, $H_k a = H_{mk}$. Thus, for any $k_0 > 0$, $S(k_0/m, k_0)$ is a tiling set for the A dilations. By choosing k_0 small enough, the set $T = T(k_0/m, k_0)$ is contained in the fundamental domain $[1/2, 1/2)^2$ and, thus, $\psi = (\chi_T)^\vee$ is a PF AB-wavelet, where $AB = \{a^i b\colon\ i \in \mathbb{Z},\ b \in B\}$.

3.4. The shear group

We would like to find a general setting in which the systems $\{D_a^i D_b^j T_k \psi^\ell\colon\ i, j \in \mathbb{Z},\ k \in \mathbb{Z}^2,\ \ell = 1, 2, 3\}$ described in Section 2 are included. Observe that the matrix b satisfies $(b - I_2)^2 = 0$. Let us first characterize all such matrices in the n-dimensional case. We say that a matrix $b \in \mathbb{R}^{n\times n}$ is a *shear matrix* if

$$(b - I_n)^2 = 0.$$

Each such b has a Jordan form that consists of k blocks of the form $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, with $k \leqslant n/2$, followed by an $(n - 2k) \times (n - 2k)$ identity matrix. That is, $b = pJp^{-1}$, where $p \in GL_n(\mathbb{R})$, $J = I_n + \sum_{j=1}^k e_{2j-1}\hat{e}_{2j}$, and $\{e_1, \ldots, e_k\}$, $\{\hat{e}_1, \ldots, \hat{e}_k\}$ are the canonical bases vectors of $\mathbb{R}^n$ and $\hat{\mathbb{R}}^n$, respectively. This implies that a general shear matrix has the form

$$b = I_n + \sum_{j=1}^{k} y^{(j)}\eta^{(j)}, \tag{3.10}$$

where $\eta^{(j)}y^{(i)} = 0$ for each $1 \leqslant i, j \leqslant k$ and $y^{(j)} = pe_{2j-1}$, $\eta^{(j)} = \hat{e}_{2j}p^{-1}$ (observe that, for $y \in \mathbb{R}^n$ and $\eta \in \hat{\mathbb{R}}^n$, $y\eta$ is the $n \times n$ matrix with entries $(y_i\eta_j)$, $1 \leqslant i, j \leqslant n$, and ηy is the scalar $\sum_{i=1}^n \eta_i y_i$).

Let $y \in \mathbb{R}^n$ and $\eta \in \hat{\mathbb{R}}^n$. If $b = I_n + y\eta$, where $\eta y = 0$, then $(b - I_n)^2 = y(\eta y)\eta = 0$ and, thus, b is a shear matrix. We will call an *elementary shear matrix* any matrix of this form. Observe that, if $y\eta = 0$ and b is an elementary shear matrix, then the mapping $\xi \mapsto \xi b = \xi + (\xi y)\eta$ has the property that $\xi \in \hat{\mathbb{R}}^n$ is fixed by b if and only if ξ lies in the hyperplane $y^\perp = \{z \in \mathbb{R}^n\colon\ zy = 0\}$, otherwise ξ is translated in the direction $\eta \in y^\perp$ (see Fig. 3). In the examples from Section 2, $y = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\eta = (0, 1)$ and so, for each $j \in \mathbb{Z}$,

$$b^j = I_n + jy\eta = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}. \tag{3.11}$$

As we observed there, in this situation, $\xi \in \hat{\mathbb{R}}^2$ is fixed under the right action of b^j if and only if $\xi = (0, \xi_2)$, otherwise ξ is translated in the vertical direction.

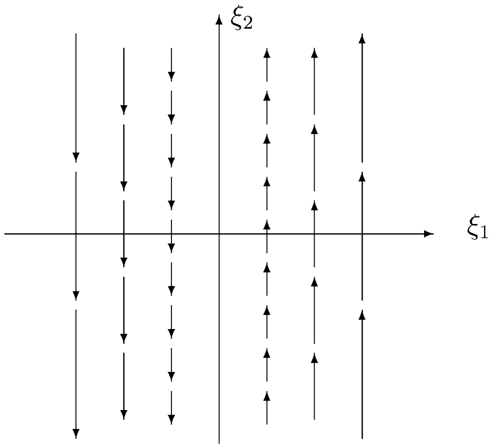

Fig. 3. Shearing transformation. Vector field induced by the right action of the shear group $B = \{b^j\colon\ j \in \mathbb{Z}\}$, where b^j is given by (3.11).

A direct computation shows that, when $b_1 = I_n + y^{(1)}\eta^{(1)}$ and $b_2 = I_n + y^{(2)}\eta^{(2)}$ are elementary shear matrices, then $b_1 b_2$ is a shear matrix if and only if $b_1 b_2 = b_2 b_1$. This occurs precisely when $\eta^{(1)} y^{(2)} = \eta^{(2)} y^{(1)} = 0$, with $b_1 b_2 = I_n + \sum_{i=1}^{2} y^{(i)}\eta^{(i)}$. Similarly, it follows that a general shear matrix b given by (3.10) is a shear matrix, where $b = b_1 b_2 \dots b_k$, and the matrices b_i, $1 \leqslant i \leqslant k$, are commuting elementary shear matrices.

We will say that a subgroup B of $GL_n(\mathbb{R})$ is an *admissible shear group* if B is locally admissible and is generated by finitely many commuting elementary shear matrices. In this case, B is *maximal* if B is not a proper subgroup of any other shear group in $GL_n(\mathbb{R})$.

Characterization of the maximal locally admissible shear groups

As we will show below in Theorem 3.3, after a change of coordinates, the general maximal locally admissible shear group $B \subset GL_n(\mathbb{R})$ has the form

$$\left\{ b_{(j_1,\dots,j_{n-k})} = \begin{pmatrix} I_k & j_1 e_1 & \dots & j_k e_k & j_{k+1} c_{k+1} & \dots & j_{n-k} c_{n-k} \\ 0 & & & & I_{n-k} & & \end{pmatrix}\colon\ j_1,\dots,j_{n-k} \in \mathbb{Z} \right\}, \tag{3.12}$$

where $k \leqslant n/2$, $\{e_1,\dots,e_k\}$ is the canonical basis of $\mathbb{R}^k$ and $\{c_{k+1},\dots,c_{k-n}\}$ are general nonzero column vectors in $\mathbb{R}^k$.

In the following we will illustrate some special cases of such B. Let $\{\hat{e}_1,\dots,\hat{e}_n\}$ be the dual basis of $\hat{\mathbb{R}}^n$ and, for $i \neq j$, let $b^{i,j} = I_n + e_i \hat{e}_j$.

(i) For $k = 1$, the simplest B of form (3.12) arises by choosing $c_i = e_1$ for $2 \leqslant i \leqslant n-1$. This yields the maximal admissible shear group

$$B = \left\{ b_{(j_1,\dots,j_{n-1})} = \begin{pmatrix} 1 & j_1 \cdots j_{n-1} \\ 0 & I_{n-1} \end{pmatrix}\colon\ j_1,\dots,j_{n-1} \in \mathbb{Z} \right\},$$

generated by $\{b^{1,j}\colon\ 2 \leqslant j \leqslant n\}$.

(ii) For $k = n/2$, we have $n - k = k$ and the expression of B given by (3.12) simplifies since there are no vectors $\{c_i\}$ to be chosen. Then

$$B = \left\{ b_{(j_1,\dots,j_k)} = \begin{pmatrix} I_k & \begin{pmatrix} j_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & j_k \end{pmatrix} \\ 0 & I_k \end{pmatrix} \colon\ j_1,\dots,j_k \in \mathbb{Z} \right\}$$

is the admissible shear group generated by $\{b^{j,k+j}\colon\ 1 \leqslant j \leqslant k\}$.

(iii) Suppose $k \geqslant 2$, $\ell = \ell_1 + \dots + \ell_k$, where $\ell_j \in \mathbb{N}$, and $n = k + \ell_1 + \dots + \ell_k$. For $1 \leqslant i \leqslant k$, let B_i be the subgroup of $GL_{(\ell_i+1)}(\mathbb{R})$ of the form (i). In $GL_n(\mathbb{R})$ we can form the group

$$B = \left\{ b = \begin{pmatrix} \beta_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \beta_k \end{pmatrix} \colon\ \beta_i \in B_i,\ 1 \leqslant i \leqslant k \right\},$$

and regard B as the outer direct product of the groups $B_1,\dots,B_k$. By rearranging the order of the columns, we can recast B as the set of all matrices of the form (3.12) where $\ell_i - 1$ of the column vectors $\{c_{k+1},\dots,c_{n-k}\}$ are chosen to be equal to e_i for $1 \leqslant i \leqslant k$.

In the following, we describe some examples of groups of shear matrices that are not locally admissible, but contain locally admissible subgroups or subsets.

(iv) For $n = 2$, the noncommuting elementary shear matrices $b^{1,2} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $b^{2,1} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generate $SL_2(\mathbb{Z})$. It is easy to verify that $SL_2(\mathbb{Z})$ is not locally admissible, although subgroups of $SL_2(\mathbb{Z})$ not generated by elementary shear matrices my be locally admissible. Consider, for example, the hyperbolic shear group in Section 3.3.2 or the finite group of the isometries of the square $[-1,1]^2$ (a special case of the finite groups in Section 3.3.1).

(v) For $n = 3$, the noncommuting elementary shear matrices $b^{1,2}$ and $b^{2,3}$ generate the integral Heisenberg group

$$H_3 = \left\{ b_{(i,j,k)} = \begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} \colon\ i,j,k \in \mathbb{Z} \right\}.$$

For $\xi = (\xi_1,\xi_2,\xi_3) \in \hat{\mathbb{R}}^3$, we have $\xi b_{(i,j,k)} = (\xi_1, \xi_2 + i\xi_1, \xi_3 + j\xi_2 + k\xi_1)$. If $\xi_1/\xi_2 \notin \mathbb{Q}$, then $\{j\xi_2 + k\xi_1\colon\ j,k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ and thus the H_3 orbit is not discrete in $\hat{\mathbb{R}}^3$. Observe that H_3 is not a shear group. However, the subgroup $\{b_{(i,0,k)}\colon\ i,k \in \mathbb{Z}\}$ of H_3 is a maximal admissible shear group of the form given by the example (i), and the subset $\{b_{(i,j,0)}\colon\ i,j \in \mathbb{Z}\}$ is locally admissible. More generally, for $n \geqslant 3$, let B_i be the shear group generated by $b^{i,i+1}$ for $1 \leqslant i \leqslant n$. Then the set product

$$B_{n-1}B_{n-2}\dots B_1 = \left\{ b_{(j_1,\dots,j_{n-1})} = \begin{pmatrix} 1 & j_1 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \dots & 1 & j_{n-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \colon\ j_1,\dots,j_{n-1} \in \mathbb{Z} \right\}$$

is locally admissible. Indeed, the set product is locally admissible for any ordering of the noncommuting groups $B_1,\dots B_{n-1}$.

The following proposition elaborates the above observations further.

Proposition 3.2. *Let $\widetilde{B}$ be a subset of $GL_n(\mathbb{R})$ containing the group B generated by two noncommuting elementary shear matrices $b_1 = I_n + y^{(1)}\eta^{(1)}$ and $b_2 = I_n + y^{(2)}\eta^{(2)}$. Then $\widetilde{B}$ is not admissible.*

Proof. Since $b_1 b_2 \neq b_2 b_1$, then either $\eta^{(1)} y^{(2)}$ or $\eta^{(2)} y^{(1)}$ is nonzero. In the case when $\eta^{(1)} y^{(2)} = 0$ and $\eta^{(2)} y^{(1)} \neq 0$, B is isomorphic to the integral Heisenberg group H_3 and is not locally admissible for the same reason discussed in Example (iv). When both $\eta^{(1)} y^{(2)}$ and $\eta^{(2)} y^{(1)}$ are nonzero, we can assume that their product is positive by replacing b_1 with b_1^{-1} if needed. Using the rescaling $y\eta = (ky)(\eta/k)$ for $k > 0$, we may assume that $\eta^{(1)} y^{(2)} = c^{-1}$ and $\eta^{(2)} y^{(1)} = c$, for some $c > 0$. Then B is isomorphic to $SL_2(\mathbb{Z})$ if $c = 1$ and, in general, B is conjugate to the subgroup B_c of $GL_n(\mathbb{R})$ generated by

$$\begin{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & I_{n-2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} & 0 \\ 0 & I_{n-2} \end{pmatrix}.$$

It is easy to see that B_c is not locally admissible for any c. Thus, in all cases, the group B generated by b_1 and b_2 is not locally admissible and, so, any subset $\widetilde{B}$ of $GL_n(\mathbb{R})$ containing B is not locally admissible. □

Observe that Proposition 3.2 does not apply to the locally admissible subgroups mentioned in Example (iii) (that are not generated by elementary shear matrices), and does not apply to the locally admissible sets of Example (v), obtained as products of noncommuting elementary shear matrices.

We can now state the main result of this section.

Theorem 3.3. *Let $B \subset GL_n(\mathbb{R})$ be a maximal locally admissible shear group.*

(a) *There is a unique index $k \leqslant n/2$ and a change of basis matrix P such that $\widetilde{B} = P^{-1} B P$ is of the form given by Example* (ii).
(b) *If $a \in GL_n(\mathbb{R})$ is such that $P^{-1} a P = \left(\begin{smallmatrix} c & * \\ 0 & d \end{smallmatrix}\right)$, where $c \in GL_k(\mathbb{R})$ is expanding and $d \in GL_{n-k}(\mathbb{R})$, then $AB = \{a^i b\colon\ i \in \mathbb{Z},\ b \in B\}$ is admissible.*

Proof. (a) Let ℓ be the minimal numbers of elementary shear matrix generators for B and $\{b_i = I_n + y^{(i)}\eta^{(i)}\colon\ 1 \leqslant i \leqslant \ell\}$ a particular set of such generators. For $V = \operatorname{span}\{y^{(i)}\colon\ 1 \leqslant i \leqslant \ell\}$ and $k = \dim V$, we will show that $\ell = n - k$ and $\{\eta^{(1)}, \ldots, \eta^{(\ell)}\}$ is a basis for the $V^{\perp}$, the annihilator of V, given by $\{\nu \in \hat{\mathbb{R}}^n\colon\ \nu v = 0,\ \forall v \in V\}$. Let $W = \operatorname{span}\{\eta^{(i)}\colon\ 1 \leqslant i \leqslant \ell\}$. Since $\eta^{(i)} y^{(j)} = 0$ for $1 \leqslant i, j \leqslant \ell$, then $W \subseteq V^{\perp}$. For any $v \in V$, $\eta \in V^{\perp} \setminus W$, the elementary shear matrix $\tilde{B} = I_n + v\eta$ commutes with every member of B. Let $\widetilde{B}$ be the shear group generated by B and $\tilde{b}$. Since B is locally admissible and $\eta \notin W$, then $\widetilde{B}$ is locally admissible and this contradicts the maximality of B. Hence $V^{\perp} = W$ and $\ell \geqslant n - k$.

In order to prove that $\ell = n - k$, we argue by contradiction and assume that $\eta^{(1)}, \ldots, \eta^{(\ell)}$ are linearly dependent. Let $m < \ell$ be the largest index for which $\eta^{(1)}, \ldots, \eta^{(m)}$ are linearly independent, B_{m+1} be the subgroup of B generated by $\{b_i\colon\ 1 \leqslant i \leqslant m+1\}$ and W_m be the m-dimensional subspace of $V^{\perp}$ spanned by $\{\eta^{(i)}\colon\ 1 \leqslant i \leqslant m+1\}$. By assumption, $\eta^{(m+1)} = \sum_{i=1}^{m} c_i \eta^{(i)}$ for some scalars $c_1, \ldots, c_m$. Since B is locally admissible, so is B_{m+1}, that is, the orbit $\Gamma_\xi = \{\xi - \xi b\colon\ b \in B_{m+1}\}$ is discrete in $\hat{\mathbb{R}}^n$ for a.e. $\xi \in \hat{\mathbb{R}}^n$. Since $B_{m+1} = \{I_n + \sum_{i=1}^{m+1} j_i v^{(i)} \eta^{(i)}\colon\ (j_1, \ldots, j_{m+1}) \in \mathbb{Z}^{m+1}\}$, then Γ_ξ is the additive subgroup of W_m generated by the linearly dependent vectors $(\xi y^{(i)})\eta^{(i)}$, $1 \leqslant i \leqslant m+1$, and Γ_ξ is discrete in W_m if and only if these vectors are linearly dependent over the rational numbers $\mathbb{Q}$. It follows that, for a.e. $\xi \in \hat{\mathbb{R}}^n$,

$\xi y^{(i)} \neq 0$ for $1 \leqslant i \leqslant m+1$, and $c_i \xi y^{(m=1)}$ is a rational multiple of $\xi y^{(i)}$ for $1 \leqslant i \leqslant m$. By suppressing all indices for which $c_i = 0$ and renaming the remaining indices, we can assume that $c_i \neq 0$ for each i. Since the quotient $q(\xi)$ of two linear functions over $\hat{\mathbb{R}}^n$ can take values in $\mathbb{Q}$ for a.e. $\xi \in \hat{\mathbb{R}}^n$ if and only if q is constant, it follows that $y^{(m+1)}$ and $y^{(i)}$ are linearly dependent for each $1 \leqslant i \leqslant m$. By rescaling $y\eta = (ky)(\eta/y)$, we may then assume that $y^{(1)} = y^{(2)} = \cdots = y^{(m+1)} = y$ for some $y \in V$. Then, for all $\xi \in \hat{\mathbb{R}}^n$, we have $\Gamma_\xi = (\xi y)\Gamma$, where $\Gamma = \mathbb{Z}\eta^{(1)} + \cdots + \mathbb{Z}\eta^{(m+1)}$. Since Γ is a lattice in W, we can replace $\eta^{(1)}, \ldots, \eta^{(m+1)}$ by a lattice basis $\nu^{(1)}, \ldots, \nu^{(m+1)}$. This means that the elementary shear matrices $b'_i = I_n + y\nu^{(i)}$, $1 \leqslant i \leqslant m$, are an alternative set of generators for B_{m+1} and $b'_1, \ldots, b'_m, b_{m+2}, \ldots, b_\ell$ is a generating set for B with $\ell - 1$ members. This contradicts the assumption that ℓ is the minimal number of elementary shear matrix generators for B. Thus we conclude that $\{\eta^{(i)}\colon 1 \leqslant i \leqslant \ell\}$ is a linearly independent set, hence $\ell = n-k$ and $\{\eta^{(i)}\colon 1 \leqslant i \leqslant n-k\}$ is a basis for $W = V^\perp$.

By reordering the $\{b_i\}$, we may assume that $\{y^{(i)}\colon 1 \leqslant i \leqslant k\}$ is a basis for V and choose a set of vectors $v^{(k+1)}, \ldots, v^{(n)}$ in $\hat{\mathbb{R}}^n$ for which $\mathcal{B} = \{y^{(1)}, \ldots, y^{(k)}, v^{(k+1)}, \ldots, v^{(n)}\}$ is a basis for $\hat{\mathbb{R}}^n$ with $\eta^{(i)} v^{(k+j)} = \delta_{i,j}$ for $1 \leqslant i, j \leqslant n-k$. Let P be the change of basis matrix mapping $\mathcal{B}$ to the standard basis $\{e_1, \ldots, e_n\}$ for $\hat{\mathbb{R}}^n$. Then $\widetilde{B} = P^{-1}BP$ is of the form given by (3.12).

(b) By our proof of (a), there is no loss of generality in assuming B to be of the form (3.12), and (using the hypotheses) in letting $a = \left(\begin{smallmatrix} c & * \\ 0 & d \end{smallmatrix}\right)$, where $c \in GL_k(\mathbb{R})$ is expanding and $d \in GL_{n-k}(\mathbb{R})$. For notational convenience, let $c_i = e_i$ for $1 \leqslant i \leqslant k$. We can regard $\hat{\mathbb{R}}^n$ as $\hat{\mathbb{R}}^k \times \hat{\mathbb{R}}^{n-k}$ and select a small annular set K, about the origin in $\hat{\mathbb{R}}^k$, so that $\hat{\mathbb{R}}^k = \bigcup_{i \in \mathbb{Z}} Kc^{-i}$ is a disjoint union. For $\xi = (\nu, \eta) \in \hat{\mathbb{R}}^k \times \hat{\mathbb{R}}^{n-k}$, with $\nu \in \hat{\mathbb{R}}^k \setminus \{0\}$, there is a unique index $i \in \mathbb{Z}$ for which $\xi' = \xi a^i = (\nu', \eta')$, with $\nu' = \nu c^i \in K$ and $\eta' \in \hat{\mathbb{R}}^{n-k}$. For $b = b_{(j_1, \ldots, j_{n-k})}$ as in (3.12), we have that $\xi'' = \xi' b = (\nu', \eta'')$, where, for each $1 \leqslant i \leqslant n-k$, $\eta''_i = j_i(\nu' c_i) + \eta'_i$ is the i-component of η'' and η'_i is the i-component of η'. Observe that, for each i, we have $\nu' c_i \neq 0$ on a set of full measure in K. Therefore there is a unique choice of $j_1, \ldots, j_{n-k}$ for which $0 \leqslant \eta''/(\nu' c_i) < 1$ for each i. Finally, let $T_{\nu'}$ be the set of all elements $\eta'' \in \hat{\mathbb{R}}^{n-k}$ satisfying these inequalities. It follows that the set $R = \bigcup_{\nu' \in K} \{\nu'\} \times T_{\nu'}$ is a tiling set for the $(AB)^{-1}$-dilations. By taking K small enough, we can ensure that R is also a packing set for the $\hat{\mathbb{Z}}^n$ translations. Thus $\{D_{a^i b} T_k (\chi_R)^\vee\colon i \in \mathbb{Z},\ b \in B,\ k \in \mathbb{Z}^n\}$ is a PF AB-wavelet and the dilation set AB is admissible. □

3.5. The contourlets

In this section, we describe a variation of the example given in Section 2, that is similar to the *contourlets* recently introduced by Do and Vetterli [7]. The contourlet construction, that is inspired by the curvelets, uses a multiresolution analysis framework with the decomposition

$$L^2(\mathbb{R}^2) = V_{i_0} \oplus \bigoplus_{i < i_0} W_j,$$

where $V_i = L^2(S_i)^\vee$, $S_i = \{\xi \in \hat{\mathbb{R}}^2\colon \|\xi\|_{\ell^1} \leqslant 2^{-i}\}$, $W_i = V_{i-1} \cap V_i^\perp$. In addition, for each $i < i_0$, each subspace W_i is subdivided into the "directional" components:

$$W_i = \bigoplus_{j=0}^{2^{l_i}} W_{i,j}^{(l_i)}.$$

We will obtain a very similar construction using the general setting of the AB-wavelets.

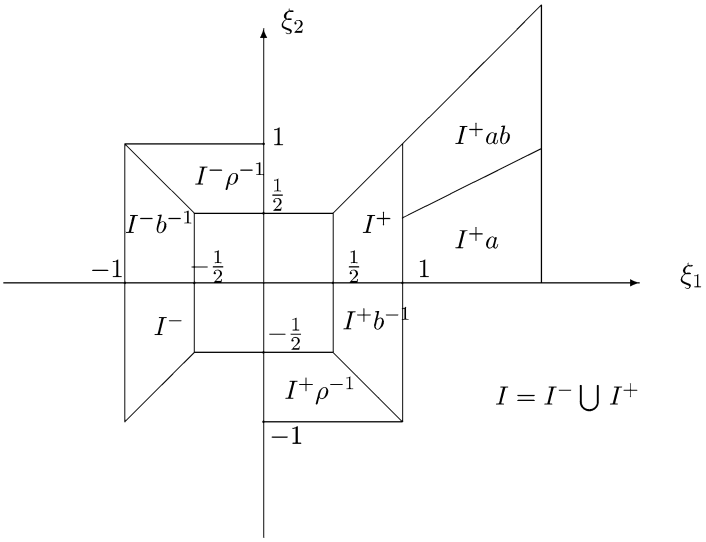

Fig. 4. I is an AB tiling set of the cone H and $\psi = (\chi_I)^\vee$ is a PF $\widetilde{AB}$-wavelet for $L^2(\hat{\mathbb{R}}^2 \setminus [-1/2, 1/2]^2)^\vee$ (see Section 3.5).

Let $a = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $T(\alpha, \beta) = T^+(\alpha, \beta) \cup T^-(\alpha, \beta)$, where $T^+(\alpha, \beta)$ is the trapezoidal region with vertices $(\alpha, 0)$, (α, α), $(\beta, 0)$ and (β, β), and $T^-(\alpha, \beta) = \{\xi \in \hat{\mathbb{R}}^2\colon -\xi \in T^+(\alpha, \beta)\}$. We denote by H the truncated cone

$$H = \left\{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon |\xi_1| \geqslant 1/2,\ 0 \leqslant |\xi_2/\xi_1| \leqslant 1\right\},$$

and let $I = T(1/2, 1)$. These sets are illustrated in Fig. 4. Then a simple computation shows that the sets $\{Ia^i b^j\colon i \geqslant 0, -2^i \leqslant j \leqslant 2^i - 1\}$ form a tiling for H. Thus, for

$$AB = \left\{a^{-i} b^{-j} = (b^j a^i)^{-1}\colon i \geqslant 0,\ -2^i \leqslant j < 2^i - 1\right\}, \tag{3.13}$$

the function $\psi = (\chi_I)^\vee$ is a PF AB-wavelet for $L^2(H)^\vee$, and the set AB is H-admissible.

Next, let $\rho = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Since this matrix produces a rotation by $\pi/2$, then $V = H\rho^{-1} = \rho H$ is the truncated cone:

$$V = \left\{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon |\xi_2| \geqslant 1/2,\ 0 \leqslant |\xi_1/\xi_2| \leqslant 1\right\}.$$

Observe that $(D_\rho \psi)^\wedge(\xi) = \hat{\psi}(\xi\rho) = \chi_{I\rho^{-1}}(\xi)$ (the set $I\rho^{-1}$ is illustrated in Fig. 4) and, thus, by the properties of T, the sets

$$Ib^j a^i \rho^{-1} = (I\rho^{-1})\rho b^j a^i \rho^{-1} \quad \text{for } i \geqslant 0,\ -2^i \leqslant j \leqslant 2^i$$

form a tiling for V. This shows that $D_\rho \psi$ is a PF $(AB\rho^{-1})$-wavelet for $L^2(V)^\vee$. Moreover, since $H \cup V = \hat{\mathbb{R}}^2 \setminus [-1/2, 1/2]^2$, it follows that ψ is a PF $\widetilde{AB}$-wavelet for $L^2(\hat{\mathbb{R}}^2 \setminus [-1/2, 1/2]^2)^\vee$, where $\widetilde{AB} = AB \cup \rho AB\rho^{-1}$.

Expression (3.13) shows that when the scale index i is increased by 1, the number of directions j is doubled. Observe that, in the contourlet construction of Do and Vetterli, as well as in the case of curvelets, the number of directions doubles every time i is increased by 2, and this ensures that the elements of the systems satisfy a *parabolic scaling*, that is, the essential support of these systems obeys approximately the relationship

$$\text{length} \approx 2^{-i} \ \text{width} \approx 2^{-2i}.$$

As shown in [4,7], this property is needed to obtain representations that are optimally sparse for functions in a certain class. In the construction above, we made a different choice of the width-to-length ratio, in order use to same matrices as in the example from Section 2. However, we can easily modify this construction by choosing $a = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and letting $I = T(1/4, 1)$. By doing so, we obtain a Parseval frame of elements satisfying a parabolic scaling relation.

Finally, let us observe that the system we have obtained disregards the low-frequency region $[-1/2, 1/2]^2 \subset \hat{\mathbb{R}}^2$, where standard (nondirectional) wavelets are used (this is similar to the curvelets and contourlets construction).

4. *AB*-multiresolution analysis. Part I

As we already observed in Section 2, there are examples of AB-multiwavelets that can be constructed within a framework very similar to the classical multiresolution analysis (MRA). In this section and in the following one we are going to develop a generalization of this theory that will be useful to construct more examples of AB-multiwavelets, as well as examples with properties that are of great potential in applications.

Let B be a countable subset of $\widetilde{SL}_n(\mathbb{Z}) = \{b \in GL_n(\mathbb{Z}): |\det b| = 1\}$ and $A = \{a^i: i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{Z})$ (notice that a is an *integral* matrix). Also assume that *a normalizes B*, that is, $aba^{-1} \in B$ for every $b \in B$. We say that a sequence $\{V_i\}_{i\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is an *AB-multiresolution analysis (AB-MRA)* if the following holds:

(i) $D_b T_k V_0 = V_0$ *for any* $b \in B$, $k \in \mathbb{Z}^n$;
(ii) *for each* $i \in \mathbb{Z}$, $V_i \subset V_{i+1}$, *where* $V_i = D_a^{-i} V_0$;
(iii) $\bigcap V_i = \{0\}$ *and* $\overline{\bigcup V_i} = L^2(\mathbb{R}^n)$;
(iv) *there exists* $\phi \in L^2(\mathbb{R}^n)$ *such that* $\Phi_B = \{D_b T_k \phi: b \in B, k \in \mathbb{Z}^n\}$ *is a semi-orthogonal Parseval frame for* V_0; *that is,* Φ_B *is a Parseval frame for* V_0 *and, in addition,* $D_b T_k \phi \perp D_{b'} T_{k'} \phi$ *for any* $b \neq b'$, $b, b' \in B$, $k, k' \in \mathbb{Z}^n$.

The space V_0 is called an *AB scaling space* and the function ϕ is an *AB scaling function* for V_0. If, in addition, Φ_B is an orthonormal basis, then we say that ϕ is an *ON AB scaling function*.

Observe that one could consider a more general definition, where A is not necessarily a group, but simply a countable collection, that is, $A = \{a_i: i \in \mathbb{Z}\}$. Furthermore, one could consider the situation where the set Φ_B is simply a Parseval frame for V_0 (not necessarily semi-orthogonal). The assumptions that we made in the above definition are the "simplest," and they ensure that the properties of the AB-MRA are very similar to those of the classical MRA. Also observe that there is a basic difference in the definition of AB-MRA that we just gave, from the definition of the classical MRA. In fact, in our definition, the space V_0 is invariant with respect to the integer translations *and* with respect to the B-dilations. On the other hand, in the classical MRA, the space V_0 is only invariant with respect to the integer translation.

Therefore, in order to examine in detail the main features of the AB-MRA, it will be useful to study the properties of the subspaces of $L^2(\mathbb{R}^n)$ that are invariant with respect to the integer translations and with respect to the B-dilations. This will be done in Section 4.2. Before doing this, in Section 4.1, we will briefly recall some basic results from the theory of shift-invariant spaces.

4.1. Shift-invariant spaces

A $\mathbb{Z}^n$*-invariant space* (or a *shift-invariant space*) of $L^2(\mathbb{R}^n)$ is a closed subspace $V \subset L^2(\mathbb{R}^n)$ for which $T_k V = V$ for each $k \in \mathbb{Z}^n$. For $\phi \in L^2(\mathbb{R}^n) \setminus \{0\}$, we denote by $\langle\phi\rangle$ the shift-invariant space generated by ϕ, that is,

$$\langle\phi\rangle = \overline{\operatorname{span}}\{T_k\phi\colon\ k \in \mathbb{Z}^n\}.$$

Given $\phi_1, \phi_2 \in L^2(\mathbb{R}^n)$, their *bracket product* is defined as

$$[\phi_1, \phi_2](x) = \sum_{k\in\mathbb{Z}^n} \phi_1(x-k)\overline{\phi_2(x-k)}. \tag{4.1}$$

Let $\mathbb{T}^n$ be the n-torus $\mathbb{R}^n/\mathbb{Z}^n \simeq [0,1]^n$ and regard $L^2(\mathbb{T}^n)$ as the space of the measurable $\mathbb{Z}^n$-periodic functions t for which $\|t\|_{L^2(\mathbb{T}^n)} = \int_{[0,1]^n} |t(x)|^2\,\mathrm{d}x < \infty$. As usual, $\hat{\mathbb{T}}^n$ denotes the corresponding space of row vectors. The following properties of the bracket product are easy to verify, and they can be found, for example, in [21, Section 3].

Proposition 4.1. *Let $\phi, \phi_1, \phi_2 \in L^2(\mathbb{R}^n)$.*

(i) *The series* (4.1) *converges absolutely a.e. to a function in $L^1(\mathbb{T}^n)$.*
(ii) *The spaces $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$ are orthogonal if and only if $[\hat{\phi}_1, \hat{\phi}_2](\xi) = 0$ a.e.*
(iii) *Let $V(\phi) = \{T_k\phi\colon\ k \in \mathbb{Z}^n\}$. Then $V(\phi)$ is a orthonormal basis for $\langle\phi\rangle$ if and only if $[\hat{\phi}, \hat{\phi}](\xi) = 1$ a.e., and $V(\phi)$ is a Parseval frame for $\langle\phi\rangle$ if and only if $[\hat{\phi}, \hat{\phi}](\xi) = \chi_{\Omega_\phi}(\xi)$ a.e., where $\Omega_\phi = \{\xi \in \hat{\mathbb{T}}^n\colon\ \hat{\phi}(\xi + \hat{k}) \neq 0$ for some $\hat{k} \in \hat{\mathbb{Z}}^n\}$.*
(iv) *Let $[\hat{\phi}, \hat{\phi}](\xi) = \chi_{\Omega_\phi}(\xi)$. Then $f \in \langle\phi\rangle$ if and only if $\hat{f} = m\hat{\phi}$ for some $m \in L^2(\Omega_\phi)$ satisfying $\|f\| = \|m\|_{L^2(\Omega_\phi)}$.*

Let $U \subseteq \hat{\mathbb{R}}^n$ be measurable and $\Omega_U = \Omega_{\chi_U} = \bigcup_{\hat{k}\in\hat{\mathbb{Z}}^n}(U + \hat{k})$. If this is a disjoint union (modulo a null set), then we say that U is a *Ω_U-tiling set for $\hat{\mathbb{Z}}^n$ translations*. It is clear that this is the case if and only if $[\chi_U, \chi_U](\xi) = \chi_{\Omega_U}(\xi)$ a.e., or, equivalently, if and only if $V((\chi_U)^\vee) = \{T_k(\chi_U)^\vee\colon\ k \in \mathbb{Z}^n\}$ is a Parseval frame for $\langle(\chi_U)^\vee\rangle$. Observe that, for any $\Omega \subseteq \hat{\mathbb{R}}^n$, every Ω-tiling set for $\hat{\mathbb{Z}}^n$ translations is contained in a $\hat{\mathbb{R}}^n$-tiling set for $\hat{\mathbb{Z}}^n$ translations, and all such tiling sets have measure one. Thus, when $\phi = (\chi_U)^\vee$ and $V(\phi)$ is a Parseval frame for $\langle\phi\rangle$, then $|U| \leqslant 1$, with equality if and only if $V(\phi)$ is an orthonormal basis for $\langle\phi\rangle$. Also observe that when U is contained in a tiling set for $\hat{\mathbb{Z}}^n$ translations, then $\langle(\chi_U)^\vee\rangle = L^2(U)^\vee \subseteq L^2(\mathbb{R}^n)$ since any $\hat{f} \in L^2(U)$ extends uniquely to $m \in L^2(\Omega_U)$ with $\hat{f} = m\chi_U$.

Let V be a shift-invariant space of $L^2(\mathbb{R}^n)$. $\Phi = \{\phi^1, \dots, \phi^N\}$, with $N \in \mathbb{N} \cup \{\infty\}$, is a $\mathbb{Z}^n$*-orthonormal set of generators* for V if, for each $1 \leqslant j \leqslant N$, the set $\{T_k\phi^j\colon\ k \in \mathbb{Z}^n\}$ is an orthonormal basis for $\langle\phi^j\rangle$.

Equivalently, we have that $[\hat\phi^i,\hat\phi^j]=\delta_{i,j}$ a.e. In addition, if this is the case, we have that $V=\bigoplus_{j=1}^N\langle\phi^j\rangle$ and we can show that, for each $f\in V$,

$$\hat f=\sum_{j=1}^N[\hat f,\hat\phi^j]\hat\phi^j, \tag{4.2}$$

with pointwise a.e. convergence if $N<\infty$ and L^2-convergence if $N=\infty$. In fact, if $f\in V$, then

$$f=\sum_{j=1}^N\sum_{k\in\mathbb{Z}^n}c_{j,k}T_k\phi^j,$$

where $c_{j,k}=\langle f,T_k\phi^j\rangle$ with pointwise a.e. convergence if $N<\infty$ and L^2-convergence if $N=\infty$. Next, by taking the Fourier transform on both sides and using a periodization argument, we obtain that

$$\begin{aligned}\hat f&=\sum_{j=1}^N\sum_{k\in\mathbb{Z}^n}\langle\hat f,M_{-k}\hat\phi^j\rangle M_{-k}\hat\phi^j=\sum_{j=1}^N\sum_{k\in\mathbb{Z}^n}\left(\int_{\mathbb{T}^n}[\hat f,\hat\phi^j](\xi)e^{2\pi ik\xi}\,\mathrm{d}\xi\right)M_{-k}\hat\phi^j\\&=\sum_{j=1}^N\sum_{k\in\mathbb{Z}^n}[\hat f,\hat\phi^j]\phi^j.\end{aligned}$$

Observe that, by an application of Proposition 2.1 with $G=\mathbb{Z}^n$, any two $\mathbb{Z}^n$-orthonormal sets of generators for the same shift-invariant spaces V must have the same number of generators.

Also observe that, while not every shift-invariant space V admits a set of generators that is $\mathbb{Z}^n$-orthonormal, one can always find a *semi-orthogonal set of generators* $\Phi=\{\phi^1,\dots,\phi^N\}$ for V, in the sense that

$$V=\bigoplus_{i=1}^N\langle\phi^i\rangle,$$

with $[\hat\phi^i,\hat\phi^i]=\chi_{\Omega_i}$, $1\leqslant i\leqslant N$, where $\Omega_i=\Omega_{\phi^i}$. In this situation, N is not uniquely determined by V. However, an extension of the argument in Proposition 2.1 shows that the multiplicity function

$$m_V=\sum_{i=1}^N\chi_{\Omega_i}:\mathbb{T}^n\mapsto\mathbb{N}\cup\{0,\infty\}$$

is independent (a.e.) of the choice of Φ.

4.2. $B\ltimes\mathbb{Z}^n$-invariant spaces

Let $\widetilde{SL}_n(\mathbb{Z})=\{b\in GL_n(\mathbb{Z})\colon |\det b|=1\}$. If B is a subgroup of $\widetilde{SL}_n(\mathbb{Z})$, then $B\ltimes\mathbb{Z}^n$ is a subgroup of the integral affine group $\widetilde{SL}_n(\mathbb{Z})\ltimes\mathbb{Z}^n$ (= the semidirect product of $\widetilde{SL}_n(\mathbb{Z})$ and $\mathbb{Z}^n$). We define the $B\ltimes\mathbb{Z}^n$ *invariant spaces* as those closed subspaces $V\subseteq L^2(\mathbb{R}^n)$ for which $D_bT_kV=V$ for each $(b,k)\in B\ltimes\mathbb{Z}^n$. We will show that these spaces share many properties with the classical shift-invariant spaces.

For $\phi\in L^2(\mathbb{R}^n)$, denote by $\langle\langle\phi\rangle\rangle$ the $B\ltimes\mathbb{Z}^n$ invariant spaces generated by ϕ, that is

$$\langle\langle\phi\rangle\rangle=\overline{\operatorname{span}}\{D_bT_k\phi\colon\ b\in B,\ k\in\mathbb{Z}^n\}.$$

For $b \in \widetilde{SL}_n(\mathbb{Z})$, we have

$$\{D_b T_k\colon\ k \in \mathbb{Z}^n\} = \{T_{k'} D_b\colon\ k' \in \mathbb{Z}^n\}$$

and, as a consequence, $D_b\langle\phi\rangle = \langle D_b\phi\rangle$ for each $\phi \in L^2(\mathbb{R}^n)$. We also have that $\mathbb{Z}^n b = \mathbb{Z}^n$ and, thus,

$$\big[\hat{D}_b\hat{\phi}_1, \hat{\phi}_2\big](\xi) = \big[\hat{\phi}_1, \hat{D}_{b^{-1}}\hat{\phi}_2\big](\xi b) \tag{4.3}$$

for each $\phi_1, \phi_2 \in L^2(\mathbb{R}^n)$, $\xi \in \hat{\mathbb{R}}^n$.

The following simple observations follow easily from Proposition 4.1.

Proposition 4.2. *Let $\phi \in L^2(\mathbb{R}^n)$.*

(i) *The spaces $\langle D_b\phi\rangle$ and $\langle\phi\rangle$ are orthogonal if and only if $[\hat{D}_b\hat{\phi}, \hat{\phi}](\xi) = 0$ a.e.*
(ii) *Let $V_B(\phi) = \{D_b T_k\phi\colon\ b \in B,\ k \in \mathbb{Z}^n\}$. Then $V_B(\phi)$ is an orthonormal basis for $\langle\langle\phi\rangle\rangle$ if and only if $[\hat{D}_b\hat{\phi}, \hat{\phi}](\xi) = 0$ a.e. for each $b \in B \setminus \{I\}$ and $[\hat{\phi}, \hat{\phi}](\xi) = 1$ a.e.*
(iii) *If $V_B(\phi)$ is an orthonormal basis for $\langle\langle\phi\rangle\rangle$, then the map $f \mapsto ([\hat{f}, \hat{D}_b\hat{\phi}])$, where $b \in B$, is an isometry from $\langle\langle\phi\rangle\rangle$ onto the Hilbert space $\ell^2(B, L^2(\mathbb{T}^n)) = \{m = (m_b)_{b\in B}\colon\ m_b \in L^2(\mathbb{T}^n)$ and $\|m\|^2 = \sum_{b\in B}\|m_b\|^2_{L^2(\mathbb{T}^n)} < \infty\}$. In particular,*

$$\hat{f} = \sum_{b\in B}\big[\hat{f}, \hat{D}_b\hat{\phi}\big]\hat{D}_b\hat{\phi}$$

for each $f \in \langle\langle\phi\rangle\rangle$, with convergence in $L^2(\hat{\mathbb{R}}^n)$.

The set $V_B(\phi) = \{D_b T_k\phi\colon\ b \in B,\ k \in \mathbb{Z}^n\}$ is called a *semi-orthogonal Parseval frame* for the $B \ltimes \mathbb{Z}^n$-invariant space $\langle\langle\phi\rangle\rangle$ if

$$\langle\langle\phi\rangle\rangle = \bigoplus_{b\in B} D_b\langle\phi\rangle$$

and $\{T_k\phi\colon\ k \in \mathbb{Z}^n\}$ is a Parseval frame for $\langle\phi\rangle$. A simple extension of Proposition 4.2(ii) gives that $V_B(\phi)$ is a semi-orthogonal Parseval frame for $\langle\langle\phi\rangle\rangle$ if and only if $[\hat{D}_b\hat{\phi}, \hat{\phi}](\xi) = 0$ for each $b \in B \setminus \{I_n\}$ and $[\hat{\phi}, \hat{\phi}](\xi) = \chi_{\Omega_\phi}$ a.e., where $\Omega_\phi = \{\xi \in \hat{\mathbb{R}}^n\colon\ \hat{\phi}(\xi + k) \neq 0$ for some $k \in \hat{\mathbb{Z}}^n\}$.

As a special case, consider $\phi \in L^2(\mathbb{R}^n)$ defined by $\hat{\phi} = \chi_U$ where $U \subseteq \hat{\mathbb{R}}^n$ is measurable and $0 < |U| < \infty$. In this case, $(D_b\phi)^\wedge = \chi_{Ub^{-1}}$ and we have that $[\hat{\phi}, \hat{D}_b\hat{\phi}] = 0$ a.e. if and only if $|U \cap Ub^{-1}| = 0$. Also, $\Omega_U = \Omega_\phi = \bigcup_{\hat{k}\in\hat{\mathbb{Z}}^n}(U + \hat{k})$ and, therefore, $[\hat{\phi}, \hat{\phi}] = \chi_{\Omega_U}$ a.e. if and only if U is a Ω_U-tiling set for the $\hat{\mathbb{Z}}^n$ translations. It follows that $\{D_b\phi\colon\ b \in B\}$ is a semi-orthogonal Parseval frame generator for $\langle\langle\phi\rangle\rangle$ if and only if U is both an S-tiling set for B dilations, where $S = \bigcup_{b\in B} Ub$, and a Ω_U-tiling set for $\hat{\mathbb{Z}}^n$ translations. In this case, $|U| \leqslant 1$ with equality if and only if $[\hat{\phi}, \hat{\phi}] = 1$ a.e., $\langle\phi\rangle = (L^2(U))^\vee$ and $\langle\langle\phi\rangle\rangle = (L^2(S))^\vee = \bigoplus_{b\in B}(L^2(Ub^{-1}))^\vee$.

Let V be a $B \ltimes \mathbb{Z}^n$-invariant space of $L^2(\mathbb{R}^n)$. The set $\Phi = \{\phi^1, \dots, \phi^N\}$, with $N \in \mathbb{N} \cup \{\infty\}$, is a *$B \ltimes \mathbb{Z}^n$-orthonormal set of generators* for V if the set $\{D_b T_k\phi^i\colon\ (b,k) \in B \ltimes \mathbb{Z}^n,\ 1 \leqslant i \leqslant N\}$ is an orthonormal basis for V. Equivalently, we have that $[\hat{D}_b\hat{\phi}^i, \hat{\phi}^j] = \delta_{i,j}\delta_{b,I_n}$ a.e. We make the following observation.

Proposition 4.3. *Let $\Phi = \{\phi^1, \ldots, \phi^N\}$ and $\Psi = \{\psi^1, \ldots, \psi^M\}$ be two $B \ltimes \mathbb{Z}^n$-orthonormal sets of generators for the $B \ltimes \mathbb{Z}^n$-invariant spaces V and W, respectively. If $W \subseteq V$, then $M \leqslant N$ with $M = N$ if and only if $W = V$.*

Proof. We first observe that

$$M = \sum_{i=1}^{M} \|\psi^i\|^2 = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{b \in B} \big\| \big[\hat{\psi}^i, \hat{D}_b \hat{\phi}^j\big] \big\|^2_{L^2(\mathbb{T}^n)}. \tag{4.4}$$

On the other hand, for each $1 \leqslant j \leqslant N$, the function $(\sum_{i=1}^{M} \sum_{b \in B} [\hat{\phi}^i, \hat{D}_b \hat{\psi}^j] \hat{\psi}^i)^\vee$ is the orthogonal projection of ϕ^j into W. Thus,

$$1 = \|\phi^j\|^2 \geqslant \sum_{i=1}^{M} \sum_{b \in B} \big\| \big[\hat{\phi}^j, \hat{D}_b \hat{\psi}^i\big] \big\|^2_{L^2(\mathbb{T}^n)}. \tag{4.5}$$

By (4.3) and the fact that $b \in \widetilde{SL}_n(\mathbb{Z})$ (this implies that the map $\xi \mapsto \xi b$ on $\hat{\mathbb{R}}^n$ is a measure preserving map from $\mathbb{T}^n$ onto $\mathbb{T}^n$) it follows that

$$\big\| \big[\hat{D}_b \hat{\psi}^i, \hat{\phi}^j\big] \big\|^2_{L^2(\mathbb{T}^n)} = \big\| \big[\hat{\psi}^i, \hat{D}_{b^{-1}} \hat{\phi}^j\big] \big\|^2_{L^2(\mathbb{T}^n)}.$$

Using this observation, from (4.4) and (4.5) we obtain

$$M = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{b \in B} \big\| \big[\hat{\psi}^i, \hat{D}_b \hat{\phi}^j\big] \big\|^2_{L^2(\mathbb{T}^n)} = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{b \in B} \big\| \big[\hat{\phi}^j, \hat{D}_{b^{-1}} \hat{\psi}^i\big] \big\|^2_{L^2(\mathbb{T}^n)} \leqslant \sum_{j=1}^{N} \|\phi^j\|^2 = N,$$

with $N = M$ if and only if $\phi^j \in W$ for each $1 \leqslant j \leqslant N$ (which is equivalent to $W = V$). $\square$

Recall that $a \in GL_n(\mathbb{Z})$ normalizes B if $aba^{-1} \in B$ for every $b \in B$. Since B is a group, then aBa^{-1} is a subgroup of B. We have the following result.

Proposition 4.4. *Suppose that $a \in GL_n(\mathbb{Z})$ normalizes B and that the quotient space $B/(aBa^{-1})$ has finite order N. If $\phi \in L^2(\mathbb{R}^n)$ satisfies the relation $[\hat{D}_b \hat{\phi}, \hat{\phi}] = \delta_{b,I_n}$ a.e. for each $b \in B$, then there exists a $B \ltimes \mathbb{Z}^n$-orthonormal set of generators Φ for the space $D_a^{-1}\langle\langle\phi\rangle\rangle$ with cardinality $N|\det a|$.*

Before proving this proposition, we need to make some observations. Recall that, for $a \in GL_n(\mathbb{Z})$, $a\mathbb{Z}^n$ is a subgroup of $\mathbb{Z}^n$ and the quotient group $\mathbb{Z}^n/(a\mathbb{Z}^n)$ has order $M = |\det a|$. Thus, we can choose a complete set of representatives of $\mathbb{Z}^n/(a\mathbb{Z}^n)$, i.e., a set $\alpha_0, \ldots, \alpha_{M-1} \in \mathbb{Z}^n/(a\mathbb{Z}^n)$ so that each element $k \in \mathbb{Z}^n$ can be uniquely expressed in the form

$$k = ak' + \alpha_i,$$

with $k' \in \mathbb{Z}^n$ and $0 \leqslant i \leqslant M - 1$. This shows that, for each $k \in \mathbb{Z}^n$, we have

$$D_a^{-1} T_k = T_{a^{-1}k} D_a^{-1} = T_{k'} T_{a^{-1}\alpha_i} D_a^{-1} = T_{k'} D_a^{-1} T_{\alpha_i}, \tag{4.6}$$

with $k' \in \mathbb{Z}$ and $0 \leqslant i \leqslant M - 1$. For any $\phi \in L^2(\mathbb{R}^n) \setminus \{0\}$, the space $D_a^{-1}\langle\phi\rangle$ is then the shift-invariant space generated by $\Phi = \{\phi^i = D_a^{-1} T_{\alpha_i} \phi\colon\ 0 \leqslant i \leqslant M - 1\}$. Since D_a^{-1} is unitary, then Φ is a $\mathbb{Z}^n$-orthonormal generating set for $D_a^{-1}\langle\phi\rangle$ if and only if ϕ is a $\mathbb{Z}^n$-orthonormal generating set for $\langle\phi\rangle$ and this holds if and only if $[\hat{\phi}, \hat{\phi}] = 1$ a.e. Thus, if Φ is a $\mathbb{Z}^n$-orthonormal generating set for $D_a^{-1}\langle\phi\rangle$, we have

$$D_a^{-1}\langle\phi\rangle = \bigoplus_{i=0}^{M-1}\langle\phi^i\rangle, \tag{4.7}$$

with $[\hat{\phi}^i, \hat{\phi}^i] = 1$ a.e. for $0 \leqslant i \leqslant M-1$.

We can now prove Proposition 4.4.

Proof of Proposition 4.4. Take a complete collection of distinct representatives $\beta_0, \ldots, \beta_{N-1}$ for $B/(aBa^{-1})$. Thus, each $b \in B$ uniquely determines $b' \in B$ and $j \in \{0, \ldots, N-1\}$ for which $b = (ab'a^{-1})\beta_j$. Then

$$D_a^{-1} D_b\langle\phi\rangle = D_{a^{-1}b}\langle\phi\rangle = D_{b'} D_{a^{-1}} D_{\beta_j}\langle\phi\rangle = D_{b'} D_{a^{-1}}\langle D_{\beta_j}\phi\rangle. \tag{4.8}$$

Take a complete collection of distinct representatives $\alpha_0, \ldots, \alpha_{M-1}$ for the quotient space $\mathbb{Z}^n/(a\mathbb{Z}^n)$, where $M = |\det a|$. By Eq. (4.7), we have

$$D_{a^{-1}}\langle D_{\beta_j}\phi\rangle = \bigoplus_{i=0}^{N-1}\langle\phi_{i,j}\rangle,$$

where $\phi_{i,j} = D_a^{-1} D_{\alpha_i} D_{\beta_j}\phi$ with $0 \leqslant i \leqslant M-1$, $0 \leqslant j \leqslant N-1$. We also have

$$D_a^{-1}\langle\langle\phi\rangle\rangle = D_a^{-1}\left(\bigoplus_{b\in B} D_b\langle\phi\rangle\right) = \bigoplus_{b\in B} D_a^{-1} D_b\langle\phi\rangle.$$

Thus, using (4.8), from the last expression we obtain

$$D_a^{-1}\langle\langle\phi\rangle\rangle = \bigoplus_{b'\in B}\bigoplus_{j} D_{b'} D_{a^{-1}}\langle D_{\beta_j}\phi\rangle = \bigoplus_{b'\in B} D_{b'}\left(\bigoplus_{i,j}\langle\phi_{i,j}\rangle\right) = \bigoplus_{i,j}\langle\langle\phi_{i,j}\rangle\rangle.$$

Since the unitary operator D_a^{-1} maps an orthonormal basis for $\langle\langle\phi\rangle\rangle$ to an orthonormal basis for $D_a^{-1}\langle\langle\phi\rangle\rangle$, it follows that the set $\Phi = \{\phi_{i,j}\colon\ 0 \leqslant i \leqslant M-1,\ 0 \leqslant j \leqslant N-1\}$ is a $B \ltimes \mathbb{Z}^n$-orthonormal set of generators for $D_a^{-1}\langle\langle\phi\rangle\rangle$. $\square$

5. AB-multiresolution analysis. Part II

In this section, we apply the techniques developed in Section 4 to obtain a number of basic results about AB-multiresolution analyses.

5.1. Basic results

Let $\{V_i\}_{i\in\mathbb{Z}}$ be an AB-MRA as defined in Section 4. As in the classical multiresolution analysis, let W_0 be the orthogonal complement of V_0 in V_1, that is, $W_0 = V_1 \cap (V_0)^{\perp}$. Then, $V_1 = V_0 \oplus W_0$. We have the following elementary result:

Proposition 5.1. (i) *Let $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ be such that $\{D_b T_k \psi^\ell\colon b \in B,\ \ell = 1, \dots, L,\ k \in \mathbb{Z}^n\}$ is a PF for W_0. Then Ψ is a PF AB-multiwavelet.*

(ii) *Let $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ be such that $\{D_b T_k \psi^\ell\colon b \in B,\ k \in \mathbb{Z}^n,\ \ell = 1, \dots, L\}$ is an orthonormal basis for W_0. Then Ψ is an ON AB-multiwavelet.*

Proof. Define the spaces W_j as $W_j = V_{j+1} \cap (V_j)^\perp$, $j \in \mathbb{Z}$. It follows from the definition of AB-MRA that $L^2(\mathbb{R}^n) = \bigoplus_{j\in\mathbb{Z}} W_j$. Since $\{D_b T_k \psi^\ell\colon b \in B,\ \ell = 1, \dots, L,\ k \in \mathbb{Z}^n\}$ is a PF for W_0, then $\{D_a^i D_b T_k \psi^\ell\colon b \in B, \ell = 1, \dots, L, k \in \mathbb{Z}^n\}$ is a PF for W_i. Thus $\{D_a D_b T_k \psi^\ell\colon b \in B, a \in A, \ell = 1, \dots, L, k \in \mathbb{Z}^n\}$ is a PF for $L^2(\mathbb{R}^n)$.

The proof for the orthonormal case is similar. □

In the situation described by the hypotheses of Proposition 5.1 (where Ψ is not only a PF for $L^2(\mathbb{R}^n)$, but it is also derived from an AB-MRA), we say that Ψ is a *PF MRA AB-multiwavelet* or an *ON MRA AB-multiwavelet*, respectively.

We say that the PF MRA AB-wavelet ψ is of *finite filter (FF) type* if there exists an AB scaling function ϕ for V_0 and a finite set $\{b_1, \dots, b_k\} \subset B$ such that

$$\hat{\phi}(\xi a) = \sum_{j=1}^{k} m_0^{(j)}(\xi)\hat{\phi}(\xi b_j), \qquad \hat{\psi}(\xi a) = \sum_{j=1}^{k} m_1^{(j)}(\xi)\hat{\phi}(\xi b_j),$$

where $m_0^{(j)}, m_1^{(j)}$, $1 \leqslant j \leqslant k$, are periodic functions. Similarly, the ON MRA AB-multiwavelet Ψ is of finite filter (FF) type if there exists an AB scaling function ϕ for V_0 and a finite set $\{b_1, \dots, b_k\} \subset B$ such that

$$\hat{\phi}(\xi a) = \sum_{j=1}^{k} m_0^{(j)}(\xi)\hat{\phi}(\xi b_j), \qquad \hat{\psi}^\ell(\xi a) = \sum_{j=1}^{k} m_{1,\ell}^{(j)}(\xi)\hat{\phi}(\xi b_j), \quad \ell = 1, \dots, L,$$

where $m_0^{(j)}, m_{1,\ell}^{(j)}$, $1 \leqslant j \leqslant k$, are periodic functions. The reader can easily check that the examples of AB-multiwavelets presented in Section 2 are indeed MRA AB-multiwavelets of finite filter type.

It turns out that, while it is possible to construct a PF AB-wavelet using a single generator, that is, $\Psi = \{\psi\}$, in the case of *orthonormal* MRA AB-multiwavelets, multiple generators are needed, that is, $\Psi = \{\psi^1, \dots, \psi^L\}$, where $L > 1$. This situation is similar to the classical MRA case (cf., for example, [21]). The following result establishes the number of generators needed to obtain an ON MRA AB-wavelet.

Theorem 5.2. *Let $\Psi = \{\psi^1, \dots, \psi^L\}$ be an ON MRA AB-multiwavelet for $L^2(\mathbb{R}^n)$, and let $N = |B/aBa^{-1}|$ (= the order of the quotient group B/aBa^{-1}). Assume that $|\det a| \in \mathbb{N}$. Then $L = N|\det a| - 1$.*

Proof. Let $V_0 = \langle\langle\psi^0\rangle\rangle$ be the AB scaling space for the AB-MRA, and let ψ^0 be the corresponding ON AB scaling function. Then $V_1 = D_a^{-1} V_0 = V_0 \oplus W_0$, where $W_0 = \bigoplus_{\ell=1}^{L} \langle\langle\psi^\ell\rangle\rangle$. Hence $\{\psi^0, \psi^1, \dots, \psi^L\}$ is an ON $B \ltimes \mathbb{Z}^n$ generating set for V_1. By Proposition 4.4, $1 + L = N|\det a|$ and so $L = N|\det a| - 1$. □

In the case of the examples of ON AB-multiwavelets given in Section 2, where $B = \{b^j\colon j \in \mathbb{Z}\}$ with $b = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $A = \{a^i\colon i \in \mathbb{Z}\}$ with $a = \left(\begin{smallmatrix}2 & 0\\ 0 & a_{2,2}\end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ (we need to assume $a_{2,2} = 1$ or 2 to apply Theorem 5.2), we have used three generators. This number is confirmed by the formula given by Theorem 5.2. In fact, a calculation shows that $|B/aBa^{-1}| = 2|a_{2,2}|^{-1}$ and, thus, by Theorem 5.2, the number of generators must be $L = 2|a_{2,2}|^{-1}2|a_{2,2}| - 1 = 3$.

Observe that the condition on the number of generators described by this theorem is not needed if the AB affine system does not come from an ON AB-MRA. In Section 6 we present an example of an ON AB-wavelet ψ (a single generator) where $A = \{a^i\colon i \in \mathbb{Z}\}$, $|\det a| = 2$ and $N = |B/aBa^{-1}| = 2$. It is clear, by Theorem 5.2, that this example of AB-wavelet is *not* of MRA type.

The following theorem describes how to construct tiling ON AB-multiwavelets arising from an AB-MRA.

Theorem 5.3. *Let $B \subset \widetilde{SL}_n(\mathbb{Z})$, $a \in GL_n(\mathbb{Z})$ with $aBa^{-1} \subseteq B$, and $L = NM - 1$, where $N = |B/aBa^{-1}|$ and $M = |\det a| > 1$. Suppose that $U \subset \mathbb{R}^n$ is a measurable set and $\phi = (\chi_U)^\vee \in L^2(\mathbb{R}^n)$ is an ON AB scaling function for $V_0 = \overline{\operatorname{span}}\{D_bT_k\phi\colon k \in \mathbb{Z}^n, b \in B\}$, with $V_0 \subseteq D_a^{-1}V_0$. Then there are sets $T_\ell \subset \mathbb{R}^n$, $\ell = 1, \ldots, L$, for which $\Psi = \{\psi^\ell = (\chi_{T_\ell})^\vee\colon \ell = 1, \ldots, L\}$ is an ON MRA AB-multiwavelet, and Ψ is of FF type.*

Proof. By hypothesis, $\{D_bT_k\phi\colon b \in B, k \in \mathbb{Z}^n\}$ is an ON basis of V_0, and, since $\hat{\phi} = \chi_U$, then $V_0 = L^2(S_0)^\vee$, where $S_0 = \bigcup_{b\in B} Ub$ and the union is disjoint. For $i \in \mathbb{Z}$, let $V_i = D_a^{-i}V_0 = L^2(S_i)^\vee$. Then $V_i \subseteq V_{i+1}$ and $S_i = S_0a^i \subset S_{i+1}$. It follows easily that $\bigcap_{i\leqslant 0} S_i$ and $\hat{\mathbb{R}}^n \setminus \bigcup_{i\geqslant 0} S_i$ are null sets. Next let $W_0 = V_1 \cap (V_0)^\perp = L^2(S_1 \setminus S_0)^\vee$. We will show that there are sets T_ℓ, $1 \leqslant \ell \leqslant L$, such that each T_ℓ is a tiling set for $\hat{\mathbb{Z}}^n$ translations and the disjoint union $\bigcup_{\ell=1}^{L} T_\ell$ is a $(S_1 \setminus S_0)$-tiling set for B dilations. In order to do that, let $\beta_0, \ldots, \beta_{N-1}$ be a complete collection of coset representatives of B/aBa^{-1}, with $\beta_0 = I_n$, and let $U_1 = \bigcup_{j=0}^{N-1} U\beta_j a$. Since each $b \in B$ uniquely determines a $b' \in B$ and a j for which $b = \beta_j(ab'a^{-1})$, we have

$$S_1 = S_0a = \bigcup_{b\in B} Uba = \bigcup_{b'\in B} U_1b'.$$

Thus U_1 is an S_1-tiling set for B dilations and, as a consequence, $\tilde{U} = U_1 \cap S_0$ is an S_0-tiling set for B dilations and $T = U_1 \setminus \tilde{U}$ is an $(S_1 \setminus S_0)$-tiling set for B dilations. Note that $|\tilde{U}| = |U| = 1$ since $|\det b| = 1$ for each $b \in B$. Also, $|U_1| = N|\det a| = NM$ and so $|T| = NM - 1 = L$. By an easy calculation, $[\chi_{U_1}, \chi_{U_1}] = NM$ a.e. Thus, for a.e. $\xi \in \hat{\mathbb{R}}^n$, there are precisely NM points in $(\xi + \hat{\mathbb{Z}}^n) \cap T$ and exactly one of these points lies in $\tilde{U}$. This implies that $[\chi_T, \chi_T] = L$ a.e. Now one can decompose T into disjoint subsets T_ℓ, $1 \leqslant \ell \leqslant L$, with $[\chi_{T_\ell}, \chi_{T_\ell}] = 1$ a.e. for each ℓ. The sets T_ℓ have precisely the properties we were looking for, and, as a consequence, $\Psi = \{\psi^\ell = (\chi_{T_\ell})^\vee\colon \ell = 1, \ldots, L\}$ is an ON MRA AB-multiwavelet.

In order to prove the final statement, observe that $T_\ell a^{-1} \subseteq U_1a^{-1} = \bigcup_{j=0}^{N-1} U\beta_j$ and $Ua^{-1} \subseteq U_1a^{-1}$. This implies that, for all $0 \leqslant \ell \leqslant L$, using the notation $\psi^0 = \phi$ and $T_0 = U$, we have

$$\begin{aligned}\hat{\psi}^\ell(\xi a) = \chi_{T_\ell}(\xi a) = \chi_{T_\ell a^{-1}}(\xi) &= \sum_{j=0}^{N} \chi_{(T_\ell a^{-1}\cap U\beta_j)}(\xi) \\ &= \sum_{j=0}^{N} \chi_{(T_\ell a^{-1}\cap U\beta_j)}(\xi)\chi_{(U\beta_j)}(\xi) = \sum_{j=0}^{N} m_\ell^j(\xi)\hat{\phi}\big(\xi\beta_j^{-1}\big),\end{aligned}$$

where $m_\ell^j(\xi)$ is the $\hat{\mathbb{Z}}^n$ periodic extension of $\chi_{(T_\ell a^{-1}\cap U\beta_j)}(\xi)$. $\square$

5.2. Well-localized AB-wavelets

Up to this point, our construction of AB-multiwavelets has been limited to systems arising from compact tiling sets in the frequency domain $\hat{\mathbb{R}}^n$. Such AB-multiwavelets are smooth in $\mathbb{R}^n$ but have slow decay. In this section, we will give an explicit construction of smooth AB-wavelets with fast decay both in $\mathbb{R}^n$ and $\hat{\mathbb{R}}^n$. Systems with these properties are very important for applications since fast decay is essential for their numerical implementation. In the previous section we have seen how filters arise naturally in AB-MRA systems. As is the case with classical MRA wavelets, the filters' role will be even more prominent in the constructions of this section.

5.2.1. Example 1

Let $\psi_1 \in L^2(\mathbb{R})$ be a (one-dimensional) dyadic band-limited wavelet with $\operatorname{supp} \hat{\psi}_1 \subset [-\Omega, \Omega]$, $\Omega > 0$, and $\psi_2 \in L^2(\mathbb{R})$ be another band-limited function with $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$ and satisfying

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_2(\xi + j)\right|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{5.1}$$

Recall that, since ψ_1 is a dyadic wavelet, it satisfies the Calderòn equation (cf. Section 3.1)

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_1(2^j\xi)\right|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{5.2}$$

As we will show later on, there are several choices of functions ψ_1 and ψ_2 satisfying these properties.

For any $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, $\omega_1 \neq 0$, define $\psi \in L^2(\mathbb{R}^2)$ by

$$\hat{\psi}(\omega) = \hat{\psi}_1(2^s\omega_1)\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right), \tag{5.3}$$

where $s \in \mathbb{Z}$ satisfies $2^s \geqslant 2\Omega$. This assumption ensures that $\operatorname{supp} \hat{\psi} \subset [-1/2, 1/2]^2$. In fact, since $\operatorname{supp} \hat{\psi}_1 \subset [-\Omega, \Omega]$ and $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$, it follows from (5.3) that $\hat{\psi}(\omega_1, \omega_2) = 0$ for $|\omega_1| > 1/2$ and $|\omega_2| > 1/2$. It is now simple to show that ψ is a PF AB-wavelet, where

$$A = \left\{a^k = \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix}\colon k \in \mathbb{Z}\right\} \quad \text{and} \quad B = \left\{b^j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\colon j \in \mathbb{Z}\right\}.$$

Indeed, observing that $\omega a^k b^j = (2^k\omega_1, j2^k\omega_1 + \omega_2)$, and using (5.1), (5.2) and (5.3), we have that

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}(\omega a^k b^j)\right|^2 &= \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}_1(2^{s+k}\omega_1)\right|^2 \left|\hat{\psi}_2\left(2^{-k}\frac{\omega_2}{\omega_1} + j\right)\right|^2 \\ &= \sum_{k\in\mathbb{Z}} \left|\hat{\psi}_1(2^{s+k}\omega_1)\right|^2 \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_2\left(2^{-k}\frac{\omega_2}{\omega_1} + j\right)\right|^2 = 1 \quad \text{a.e.}\end{aligned}$$

The fact that ψ is a PF AB-wavelet now follows from the following general observation.

Proposition 5.4. *Let $\psi \in L^2(\mathbb{R}^n)$ be such that $\operatorname{supp} \hat{\psi} \subset Q = [-1/2, 1/2]^n$, and*

$$\sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}(\omega a^k b^j)\right|^2 = 1 \quad \text{a.e. } \omega \in \hat{\mathbb{R}}^n,$$

where $a, b \in GL_n(\mathbb{R})$. Then ψ is a PF AB-wavelet, where $A = \{a^i\colon i \in \mathbb{Z}\}$ and $B = \{b^j\colon j \in \mathbb{Z}\}$.

Proof. For $i, k \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, let $\psi_{i,j,k} = D_a^i D_b^j T_k \psi$. Using the hypotheses on ψ, the change of variable $\eta = \xi a^i b^j$ and Plancherel theorem, for each $f \in L^2(\mathbb{R}^n)$ we have that

$$\sum_{i,j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \bigl|\langle f, \psi_{i,j,k}\rangle\bigr|^2 = \sum_{i,j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left| \int_{\hat{\mathbb{R}}^n} \hat{f}(\omega)\overline{\hat{\psi}(\omega a^i b^j)} e^{2\pi i \omega a^i b^j k} |\det a|^{i/2} |\det b|^{j/2}\, \mathrm{d}\omega \right|^2$$

$$= \sum_{i,j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left| \int_{Q} \hat{f}(\eta b^{-j} a^{-i})\overline{\hat{\psi}(\eta)} e^{2\pi i \eta k} |\det a|^{-i/2} |\det b|^{-j/2}\, \mathrm{d}\eta \right|^2$$

$$= \sum_{i,j\in\mathbb{Z}} \int_{Q} \bigl|\hat{f}(\eta b^{-j} a^{-i})\bigr|^2 \bigl|\hat{\psi}(\eta)\bigr|^2 |\det a|^{-i} |\det b|^{-j}\, \mathrm{d}\eta$$

$$= \sum_{i,j\in\mathbb{Z}} \int_{\hat{\mathbb{R}}^n} \bigl|\hat{f}(\omega)\bigr|^2 \bigl|\hat{\psi}(\omega a^i b^j)\bigr|^2\, \mathrm{d}\omega = \|f\|^2. \qquad \square$$

As we mentioned before, there are many choices for the functions ψ_1 and ψ_2 that satisfy the assumptions we have described above. For example, we can choose ψ_1 to be the Lemariè–Meyer wavelet (see [14, Section 1.4]) defined by $\hat{\psi}_1(\xi) = e^{i\pi\xi} b(\xi)$, where

$$b(\xi) = \begin{cases} \sin\bigl(\frac{\pi}{2}(3|\xi| - 1)\bigr), & \frac{1}{3} \leqslant |\xi| \leqslant \frac{2}{3}, \\ \sin\bigl(\frac{3\pi}{4}\bigl(\frac{4}{3} - |\xi|\bigr)\bigr), & \frac{2}{3} < |\xi| \leqslant \frac{4}{3}, \\ 0, & \text{otherwise}. \end{cases}$$

In order to construct ψ_2, let ϕ be a compactly supported C^∞ bump function, with $\operatorname{supp}\phi \subset [-1, 1]$ (examples can be found in [19, Section 3.3] or [15, Section 1.4]), and define ψ_2 by

$$\hat{\psi}_2(\xi) = \frac{\phi(\xi)}{\sqrt{\sum_{k\in\mathbb{Z}} |\phi(\xi + k)|^2}}.$$

It is clear that $\psi_2 \in C^\infty(\mathbb{R})$ and satisfies (5.1). It follows that $\hat{\psi}$, given by (5.3), is in $C^\infty(\mathbb{R}^2)$ and this implies that $|\psi(x)| \leqslant K_N(1 + |x|)^{-N}$, $K_N > 0$, for any $N \in \mathbb{N}$.

Finally, let us observe that it is easy to generalize this construction for $n > 2$. For example, let $\psi_1, \psi_2 \in L^2(\mathbb{R})$ be defined as above and, for any $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$, $\omega_1 \neq 0$, define $\psi \in L^2(\mathbb{R}^n)$ by

$$\hat{\psi}(\omega) = \hat{\psi}_1(2^s\omega_1)\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right) \ldots \hat{\psi}_2\left(\frac{\omega_n}{\omega_1}\right), \tag{5.4}$$

where $s \in \mathbb{Z}$ satisfies $2^s \geqslant 2\Omega$. It turns out that ψ is a PF AB-wavelet, where

$$A = \left\{ a^i = \begin{pmatrix} 2^i & 0 \\ 0 & I_{n-1} \end{pmatrix}\colon\ i \in \mathbb{Z} \right\} \quad \text{and} \quad B = \left\{ b_j = \begin{pmatrix} 1 & j \\ 0 & I_{n-1} \end{pmatrix}\colon\ j \in \mathbb{Z}^{n-1} \right\},$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. The proof is exactly as in the case $n = 2$ once is observed that, for $j = (j_1, \ldots, j_{n-1})$ with $j_1, \ldots, j_{n-1} \in \mathbb{Z}$, we have

$$\hat{\psi}(\omega a^k b_j) = \hat{\psi}_1(2^{s+k}\omega_1)\hat{\psi}_2\left(2^{-k}\frac{\omega_2}{\omega_1} + j_1\right) \ldots \hat{\psi}_2\left(2^{-k}\frac{\omega_n}{\omega_1} + j_{n-1}\right).$$

A similar idea can be applied to more general shear groups B.

The next example shows how to construct AB-wavelets for $L^2(\mathbb{R}^2)$ of MRA type that are well localized both in $\mathbb{R}^n$ and $\hat{\mathbb{R}}^n$.

5.2.2. Example 2

Let $\psi_1 \in L^2(\mathbb{R})$ be a (one-dimensional) dyadic band-limited MRA wavelet with $\operatorname{supp} \hat{\psi}_1 \subset [-\Omega, \Omega]$, $\Omega > 0$, and let ϕ_1 be its associated scaling function. Let m_0 and m_1 be the low pass and high pass filters, respectively, associated with ϕ_1 and ψ_1, that is, m_0 and m_1 are the periodic functions satisfying the equations

$$\hat{\phi}_1(\omega_1) = m_0\left(\frac{\omega_1}{2}\right)\hat{\phi}_1\left(\frac{\omega_1}{2}\right) \quad \text{and} \quad \hat{\psi}_1(\omega_1) = m_1\left(\frac{\omega_1}{2}\right)\hat{\phi}_1\left(\frac{\omega_1}{2}\right).$$

Let $\psi_2 \in L^2(\mathbb{R})$ be defined by

$$\psi_2(x) = e^{i(N+1)\pi x}\left(\frac{\sin \pi x}{\pi x}\right)^{N+1},$$

where $N \in \mathbb{N}$. That is, $\hat{\psi}_2$ is a basic spline of order N (cf. [14, Section 4.2]). This implies that $\operatorname{supp} \hat{\psi}_2 \subset [0, N+1]$ and $\hat{\psi}_2$ satisfies the so-called *two scale equation*

$$\hat{\psi}_2(\xi) = \sum_{k=0}^{N+1} d_k^{(N)} \hat{\psi}_2(2\xi - k), \tag{5.5}$$

where $d_k^{(N)} = 2^{-N}\binom{N+1}{k}$.

For $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, $\omega_1 \neq 0$, let $\phi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\phi}(\omega) = \hat{\phi}_1(2^s\omega_1)\frac{\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right)}{\sqrt{\sum_{m\in\mathbb{Z}}\left|\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}+m\right)\right|^2}},$$

where $s \in \mathbb{Z}$ satisfies $2^s \geqslant 4\Omega(\frac{N}{2}+1)$. This assumption on s ensures that

$$\operatorname{supp} \hat{\phi} \subset \left\{(\omega_1, \omega_2) \in \hat{\mathbb{R}}^2\colon\ |\omega_1| < \frac{1}{4}(N/2+1)^{-1},\ |\omega_2| < \frac{1}{4}\right\}. \tag{5.6}$$

Also, let $\psi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\psi}(\omega) = \sum_{k=0}^{N+1} d_k^{(N)} m_1(2^{s-1}\omega_1) M_0(a^{-1}\omega)\hat{\phi}(\omega a^{-1} b^{-k}),$$

where the matrices a and b are as in Section 5.2.1, the coefficients $d_k^{(N)}$ are those in (5.5), and $M_0(\omega)$ is the $\mathbb{Z}^2$-periodic function which, restricted to the fundamental region $[-\frac{1}{2}, \frac{1}{2}]^2$, is given by

$$M_0(\omega) = \left(\frac{\sum_{m\in\mathbb{Z}}\left|\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}+m\right)\right|^2}{\sum_{m\in\mathbb{Z}}\left|\hat{\psi}_2\left(2^{-1}\frac{\omega_2}{\omega_1}+m\right)\right|^2}\right)^{1/2}, \quad \omega \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2.$$

Using (5.5), we have that

$$\sum_{k=0}^{N+1} d_k^{(N)} M_0(\omega a^{-1})\hat{\phi}(\omega a^{-1}b^{-k})$$
$$= \hat{\phi}_1(2^{s-1}\omega_1)\sum_{k=0}^{N+1} d_k^{(N)} \frac{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(2\frac{\omega_2}{\omega_1}+m\bigr)\bigr|^2}}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(\frac{\omega_2}{\omega_1}+m\bigr)\bigr|^2}} \frac{\hat{\psi}_2\bigl(2\frac{\omega_2}{\omega_1}-k\bigr)}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(2\frac{\omega_2}{\omega_1}+m\bigr)\bigr|^2}}$$
$$= \hat{\phi}_1(2^{s-1}\omega_1)\sum_{k=0}^{N+1} d_k^{(N)} \frac{\hat{\psi}_2\bigl(2\frac{\omega_2}{\omega_1}-k\bigr)}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(\frac{\omega_2}{\omega_1}+m\bigr)\bigr|^2}} = \hat{\phi}_1(2^{s-1}\omega_1)\frac{\hat{\psi}_2\bigl(\frac{\omega_2}{\omega_1}\bigr)}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(\frac{\omega_2}{\omega_1}+m\bigr)\bigr|^2}}.$$

Applying this observation, the fact that $\omega a^i b^j = (2^i\omega_1, j2^i\omega_1+\omega_2)$, and the observation that ψ_1 satisfies Calderòn equation (5.2), we obtain that

$$\sum_{i,j\in\mathbb{Z}}\bigl|\hat{\psi}(\omega a^i b^j)\bigr|^2 = \sum_{i,j\in\mathbb{Z}}\bigl|\hat{\psi}_1(2^{s+i}\omega_1)\bigr|^2 \frac{\bigl|\hat{\psi}_2\bigl(2^i\frac{\omega_2}{\omega_1}+j\bigr)\bigr|^2}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(2^i\frac{\omega_2}{\omega_1}+m+j\bigr)\bigr|^2}}$$
$$= \sum_{i\in\mathbb{Z}}\bigl|\hat{\psi}_1(2^{s+i}\omega_1)\bigr|^2 \sum_{j\in\mathbb{Z}} \frac{\bigl|\hat{\psi}_2\bigl(2^i\frac{\omega_2}{\omega_1}+j\bigr)\bigr|^2}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(2^i\frac{\omega_2}{\omega_1}+m+j\bigr)\bigr|^2}} = 1$$

for a.e. $\omega \in \hat{\mathbb{R}}^2$. By (5.6), it follows that $\operatorname{supp}\hat{\psi} \subset [-1/2, 1/2]^2$. Thus, using Proposition 5.4 as in Section 5.2.1, it follows that ψ is a PF AB-wavelet for $L^2(\mathbb{R}^2)$, where $A = \{a^i\colon i \in \mathbb{Z}\}$ and $B = \{b^j\colon j \in \mathbb{Z}\}$. Furthermore, it follows by the construction that $\hat{\psi} \in C^N(\hat{\mathbb{R}}^2)$, so $|\psi(x)| \leqslant K_N(1+|x|)^{1-N}$ for some $K_N > 0$.

In addition, unlike the example in Section 5.2.1, we can show that ψ is a PF *MRA* AB-wavelet. In order to show this, let $V_0 = \overline{\operatorname{span}}\{D_bT_m\phi\colon b \in B,\ m \in \mathbb{Z}^2\}$ and $V_j = D_a^{-j}V_0$, $j \in \mathbb{Z}$. Then, using the computation we made before, the following observation shows that $V_0 \subset V_1$:

$$m_0(2^{s-1}\omega_1)\sum_{k=0}^{N+1} d_k^{(N)} M_0(\omega a^{-1})\hat{\phi}(\omega a^{-1}b^{-k}) = m_0(2^{s-1}\omega_1)\hat{\phi}_1(2^{s-1}\omega_1)\frac{\hat{\psi}_2\bigl(2\frac{\omega_2}{\omega_1}\bigr)}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(\frac{\omega_2}{\omega_1}+m\bigr)\bigr|^2}}$$
$$= \hat{\phi}_1(2^s\omega_1)\frac{\hat{\psi}_2\bigl(\frac{\omega_2}{\omega_1}\bigr)}{\sqrt{\sum_{m\in\mathbb{Z}}\bigl|\hat{\psi}_2\bigl(\frac{\omega_2}{\omega_1}+m\bigr)\bigr|^2}} = \hat{\phi}(\omega).$$

By induction, we have that $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$. Observe, however, that this MRA system is somewhat different from those defined in Section 4, since the spaces V_0 and $W_0 = \overline{\operatorname{span}}\{D_bT_m\psi\colon b \in B,\ m \in \mathbb{Z}^2\}$ are *not* mutually orthogonal.

5.3. Characterization equations

An application of Theorem 2.1 in [13] gives the following complete characterization of all functions $\Psi = \{\psi^1, \ldots, \psi^L\}$ such that the system Ψ_{AB}, given by (1.1), is a PF AB-multiwavelet.

Theorem 5.5. *Let* $A = \{a^k\colon k \in \mathbb{Z}\} \subset GL_n(\mathbb{Z})$, $B \subset \widetilde{SL}_n(\mathbb{Z}) = \{b \in GL_n(\mathbb{R})\colon |\det b| = 1\}$ *and* $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. *Suppose that*

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}} \sum_{b\in B} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat f} \left|\hat f(\xi + mba^k)\right|^2 \left|\hat\psi^\ell(\xi a^{-k} b^{-1})\right|^2 \mathrm{d}\xi < \infty \tag{5.7}$$

for all $f \in \mathcal{D}$, *where* $\mathcal{D}$ *is a dense subspace of* $L^2(\mathbb{R}^n)$ *contained in the set*

$$\left\{f \in L^2(\mathbb{R}^n)\colon\ \hat f \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp}\hat f \text{ is compact}\right\}.$$

Then Ψ_{AB}, *given by* (1.1), *is a PF for* $L^2(\mathbb{R}^n)$ *if and only if*

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}} \sum_{b\in B} \left|\hat\psi^\ell(\xi a^k b)\right|^2 = 1, \tag{5.8}$$

$$\sum_{\ell=1}^{L} \sum_{k\geqslant 0} \sum_{b\in B} \hat\psi^\ell(\xi a^k b)\overline{\hat\psi^\ell\big((\xi+q)a^k b\big)} = 0 \quad \text{if } q \in \hat{\mathbb{Z}}^n \setminus (\hat{\mathbb{Z}}^n a), \tag{5.9}$$

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}} \sum_{b\in B} \hat\psi^\ell(\xi a^k b)\overline{\hat\psi^\ell\big((\xi+q)a^k b\big)} = 0 \quad \text{if } q \in \bigcap_{k\in\mathbb{Z}} (\hat{\mathbb{Z}}^n a^k) \setminus \{0\}. \tag{5.10}$$

Hypothesis (5.7) is the LIC referred to in Section 3.1. For all examples of AB-multiwavelets discussed in this paper, one can show by lengthy computations that (5.7) is satisfied. Note that (5.8) is the Calderòn equation to which we have often referred above. Equation (5.9) is the analogue of the so-called t_q equation for classic dyadic wavelets (cf. [14]). However, (5.10) has a different character. The striking differences between Eqs. (5.9) and (5.10) and characterization equations for the classical dyadic wavelets were part of the motivation that led us to formulate our first examples of AB-multiwavelets and subsequently develop the theory presented in this paper.

6. *AB*-wavelet sets

In this section, we will show how to construct singly generated ON AB-wavelets. When A and B satisfy the hypotheses of Theorem 5.2 with $L > 1$, these singly generated ON AB-wavelets cannot be of MRA type. Below, we will carry out the demanding technical details for the example of Section 2, where $a = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. A much easier construction applies when a is replaced by $a = \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$, and is presented in [11]. In both cases, the AB-wavelets are inverse Fourier transform of characteristic functions of fractal-like sets. Our point of view is that such ON non-MRA AB-wavelets are "pathological" and far less useful than the Parseval frame wavelets such as those in Section 5.

Let $\mathcal{C} \subset GL_n(\mathbb{R})$ be an admissible dilation set (cf. Section 3). A *$\mathcal{C}$-wavelet set* is a measurable set $W \subset \hat{\mathbb{R}}^n$ such that $\psi = (\chi_W)^\vee$ is an orthonormal $\mathcal{C}$-wavelet.

It is easy to verify (cf. [12]) that W is a wavelet set if and only if W is both a tiling set for $\mathbb{Z}^n$ translations and a tiling set for $\mathcal{C}^{-1}$ dilations. There are several examples of $\mathcal{C}$-wavelet sets in the literature for $\mathcal{C} = \{a^i\colon i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$ [1,2,12,13,17,18]. Many such constructions use a technique introduced in [6] that modifies a set T for which $(\chi_T)^\vee$ is a Parseval frame $\mathcal{C}$-wavelet to produce a wavelet set W of the form $W = (T \setminus P) \cup Q$, where the union is disjoint, $P \subset T$ and $Q \subset \hat{\mathbb{R}}^n$ are measurable. For a general $\mathcal{C} \in GL_n(\mathbb{R})$, the conditions on P and Q are the following:

(i) $Q = \bigcup_{\xi \in P} Q_\xi$ is a disjoint union, where Q_ξ is chosen so that $\xi\mathcal{C}^{-1} = \bigcup_{\eta \in Q_\xi} \eta\mathcal{C}^{-1}$.
(ii) Let π be the projection $\pi(\xi) = \xi + \mathbb{Z}^n$ from $\hat{\mathbb{R}}^n$ into $\mathbb{T}^n$; then $\pi_{|Q}$, i.e., the restriction of π to Q, is one-to-one with image $\pi(P) \cup (\mathbb{T}^n \setminus \pi(T))$.

In fact, since $(\chi_T)^\vee$ is a Parseval frame $\mathcal{C}$-wavelet, T is a tiling set for $\mathcal{C}^{-1}$ dilations and

$$\hat{\mathbb{R}}^n = \bigcup_{\xi \in T} \xi\mathcal{C}^{-1} = \left(\bigcup_{\xi \in T \setminus P} \xi\mathcal{C}^{-1}\right) \cup \left(\bigcup_{\xi \in P} \xi\mathcal{C}^{-1}\right),$$

where the union is disjoint. By (i), $\bigcup_{\xi \in P} \xi\mathcal{C}^{-1} = \bigcup_{\eta \in Q_\xi} \eta\mathcal{C}^{-1}$. Thus (i) implies that W is a tiling set for $\mathcal{C}^{-1}$ dilations. Next, since $(\chi_T)^\vee$ is a Parseval frame $\mathcal{C}$-wavelet, T is a packing set for $\mathbb{Z}^n$ translations and so $\pi_{|T}$ is one-to-one. Also,

$$\mathbb{T}^n = \pi(T) \cup \big(\mathbb{T}^n \setminus \pi(T)\big) = \pi(T \setminus P) \cup \pi(P) \cup \big(\mathbb{T}^n \setminus \pi(T)\big),$$

where the union is disjoint. Thus, W is a tiling set for $\mathbb{Z}^n$ translations iff π maps Q one-to-one onto $\pi(P) \cup (\mathbb{T}^n \setminus \pi(T))$.

In [6], $\mathcal{C}$ is assumed to contain an expanding matrix $a \in GL_n(\mathbb{R})$ for which $a\mathcal{C}^{-1} = \mathcal{C}^{-1}$. Since a is expanding, then there is a tiling wavelet $(\chi_T)^\vee$, where $T \subset \hat{\mathbb{R}}^n$ is measurable and bounded (cf. Section 3), and a measurable set $U \subset \hat{\mathbb{R}}^n$ such that $T \subseteq U$, U is a tiling set for $\mathbb{Z}^n$ translations and $Ua \cap U = \varnothing$. Since $\xi\mathcal{C}^{-1} = \xi a\mathcal{C}^{-1}$ for all ξ, then for any $P \subseteq T$, condition (i) is satisfied by $Q = Pa$. Using the fact that $|\det a| > 1$, one can obtain a set $P \subseteq T$ for which (ii) is also satisfied, and so $W = (T \setminus P) \cup Pa$ is a wavelet set. This construction applies, for example, to $\mathcal{C} = AB = \{a^i b\colon\ i \in \mathbb{Z}, b \in B\}$, where $a \in GL_n(\mathbb{R})$ is expanding and $B \subset GL_n(\mathbb{R})$ satisfies $aBa^{-1} = B$. The orthogonal and the hyperbolic AB-wavelets described in Sections 3.3.1 and 3.3.2, respectively, are in this class.

More generally, let us consider the case $\mathcal{C} = AB = \{a^i b\colon\ i \in \mathbb{Z},\ b \in B\}$, where B is a subgroup of $\widetilde{SL}_n(\mathbb{Z})$, $a \in GL_n(\mathbb{R})$ is not necessarily expanding, and $aBa^{-1} \subsetneq B$. These assumptions imply that $a\mathcal{C}^{-1} = aBA = (aBa^{-1})(aA) \subsetneq BA = \mathcal{C}^{-1}$. Let $N = \operatorname{card}(B/(aBa^{-1}))$ and $\{b_1, \ldots, b_N\}$ be a complete set of coset representatives of $B/(aBa^{-1})$. Then we have that $BA = \bigcup_{j=1}^N b_j aBa^{-1}A = \bigcup_{j=1}^N b_j aBA$. Let U_0 be a tiling set for $\mathbb{Z}^n$ translations for which $S_0 = \bigcup_{b \in B} U_0 b$ is contained in $S_1 = S_0 a$, and let $T_0 \subset U_0$ be a tiling set for $(AB)^{-1}$ dilations. Thus, given $P \subset T_0$, we can satisfy condition (i) by setting $Q = \bigcup_{\xi \in P} Q_\xi$, where $Q_\xi = \{\xi b_j(\xi) a\colon\ 1 \leqslant j \leqslant N\}$, and $\{b_1(\xi), \ldots, b_N(\xi)\}$ is a complete set of coset representatives of $B/(aBa^{-1})$. The dependence of the coset representatives $b_j(\xi)$ on ξ will be clarified in the proof of the following theorem, where we will show the details for this construction for the example of Section 2. A similar construction holds for more general shear group matrices B. In these constructions, the coset representatives $b_j(\xi)$ are not bounded and, as a consequence, the wavelet set is unbounded.

Theorem 6.1. *Let $A = \{a^i\colon\ i \in \mathbb{Z}\}$, $B = \{b^j\colon\ j \in \mathbb{Z}\}$ where $a = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then AB-wavelets exist.*

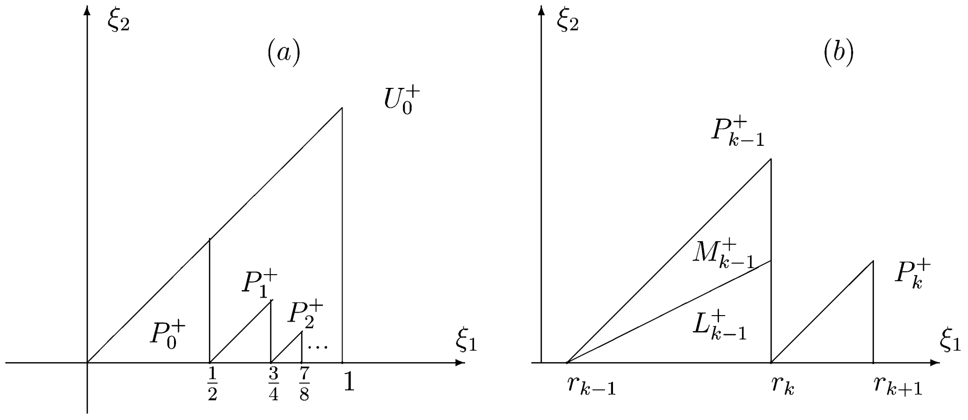

Fig. 5. (a) Construction of the sets $P_0^+, P_1^+, P_2^+ \subset U_0^+$. (b) The triangle projection π maps $P_k^+ a$ into $P_{k-1}^+ a' \subset P_{k-1}^+$.

Proof. The set $U_0 = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ 0 < |\xi_1| \leqslant 1 \text{ and } 0 \leqslant \xi_2/\xi_1 \leqslant 1\}$ is both a tiling set for $\mathbb{Z}^2$ translations and an S_0-tiling set for B dilations, where $S_0 = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ |\xi_1| \leqslant 1\}$. Let $T_0 = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ \frac{1}{2} \leqslant |\xi_1| \leqslant 1\} \subset U_0$. Then T_0 is a tiling set for BA dilations and, thus, $(\chi_{T_0})^\vee$ is a PF AB-wavelet. Let $T_1 = T_0 a$, $U_1 = U_0 a$.

As in the general construction outlined before, we will construct a wavelet set of the form $W = (T_0 \setminus P) \cup Q$. As we did in Section 2, we shall denote $T_0 = T_0^- \cup T_0^+$, where T_0^- and T_0^+ denote the intersection of T with the half-planes $\{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ \xi_1 \geqslant 0\}$ and $\{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2\colon\ \xi_1 < 0\}$, respectively. We will use a similar notation for any other set in $\hat{\mathbb{R}}^2$. Since the construction is symmetric with respect to reflection through the origin, it will be sufficient to construct the set W^+.

Let $P_0^+ = U_0^+ \setminus T_0^+$ and, for each $k \geqslant 1$, let $P_k^+ = 2^{-k} P_0^+ + (r_k, 0)$, where $r_k = \sum_{i=1}^k 2^{-i} = 1 - 2^{-k}$ and $P^+ = \bigcup_{k \geqslant 1} P_k^+$. The triangles P_k^+ are illustrated in Fig. 5a. It is clear that $P^+ \subset T_0^+$. For each k, the line segment from $(r_{k-1}, 0)$ to $(r_k, 2^{-(k+1)})$ subdivides P_{k-1} into a lower triangle L_{k-1}^+ and an upper triangle M_{k-1}^+ of equal area (see Fig. 5b). Observe that $r_{k+1} - r_k = 2^{-(k+1)}$. It is then easy to see that $\mathrm{Area}(P_{k-1}^+) = 4\,\mathrm{Area}(P_k^+)$ and $\mathrm{Area}(L_{k-1}^+) = \mathrm{Area}(M_{k-1}^+) = 2\,\mathrm{Area}(P_k^+)$.

Observe that $ab^j a^{-1} = b^{2j}$, and so a complete set of coset representatives of the quotient group $B/(aBa^{-1})$ has the form $\{b^{j_1}, b^{j_2}\}$, where j_1 is an even integer and j_2 is an odd integer. For simplicity, let $j_1 = 0$ and $j_2 = 2j + 1$ for some $j \in \mathbb{Z}$. Thus, for any $\xi \in \hat{\mathbb{R}}^2$, we can choose any $j(\xi) \in \mathbb{Z}$ such that $\xi BA = \xi aBA \cup \xi b^{2j(\xi)+1} aBA$. Define $Q^+ = P^+ a \cup \{\xi b^{2j(\xi)+1} a\colon\ \xi \in P^+\} = \bigcup_{k \geqslant 1} Q_k^+$, where $Q_k^+ = P_k^+ a \cup \{\xi b^{2j(\xi)+1} a\colon\ \xi \in P_k^+\}$, and the integers $j(\xi)$ for $\xi \in P^+$ will be specified later. This shows that condition (i) is satisfied.

Next we have to show that condition (ii) is also satisfied. We shall identify $\hat{\mathbb{T}}^2$ with $[0,1]^2 = U_0^+ \cup (U_0^- + (1,1))$. Then the projection mapping $\pi : \hat{\mathbb{R}}^2 \mapsto \hat{\mathbb{T}}^2$ is given by $\xi \mapsto [\xi]$, where $[\xi] = ([\xi_1], [\xi_2])$ and $[\xi_j]$ is the fractional part of ξ_j. In particular, if $\xi \in U_0^+$, then $[\xi] = \xi$. A simple computation shows that, for $k \geqslant 1$, $\xi \in P_k^+$ if and only if $\pi(\xi a) \in L_{k-1}^+$. Indeed, for $\xi = (\xi_1, \xi_2) \in P_k^+$, we have $r_k \leqslant \xi_1 \leqslant r_{k+1}$ and $0 \leqslant \xi_2 \leqslant (\xi_1 - r_k)$. Then $\pi(\xi a) = (2\xi_1 - 1, \xi_2)$ and, in view of $r_{k-1} = 2r_k - 1$, we have that $r_{k-1} \leqslant 2\xi_1 - 1 \leqslant r_k$ with $0 \leqslant \xi_2 \leqslant \xi_1 - r_k = \frac{1}{2}((2\xi_1 - 1) - r_{k-1})$.

We shall now construct a measurable map $\xi \mapsto j(\xi)$ from P^+ to $\mathbb{Z}$ such that $\pi(\xi b^{2j(\xi)+1} a)$ maps P_k^+ onto M_{k-1}^+ for each $k \geqslant 1$ modulo null sets. Note that, for each $j \in \mathbb{Z}$ and $\xi = (\xi_1, \xi_2) \in P_k^+$, the map

$\pi(\xi b^{2j(\xi)+1}a)$ has the form $(2\xi_1, \xi_2 + (2j+1)\xi_1 - m)$ for some $m \in \mathbb{Z}$. Once we construct such a map, then it follows that $\pi(Q^+) = \pi(\bigcup_{k\geqslant 1} Q_k^+) = (U_0^+ \setminus T_0^+) \cup P^+$, and, as a consequence, (ii) is satisfied. This fact, together with the previous part of the proof, implies that $U_0 = P_0 \cup T_0$ is a disjoint union and $W = (T_0 \setminus P) \cup Q$ is an AB-wavelet set. Thus, it only remains to construct the measurable map that we have described.

Fix $k \geqslant 1$. For $j, m \in \mathbb{Z}$, let $\pi_{j,m}(\xi_1, \xi_2) = (2\xi_1 - 1, \xi_2 + (2j+1)\xi_1 - m)$ and let $T_{j,m} = \{\xi \in$ interior of P_k^+: $\pi_{j,m}(\xi) \in$ interior of $M_{k-1}^+\}$. Let $J = \{(j,m) \in \mathbb{Z}^2\colon\ T_{j,m} \neq \varnothing\}$. For $(j,m) \in J$, the set $T_{j,m}$ is an open triangle or an open quadrilateral in P_k^+, and $S_{j,m} = \pi_{j,m}(T_{j,m})$ is an open subset of similar shape in M_{k-1}^+, with $\operatorname{Area}(S_{j,m}) = 2\operatorname{Area}(T_{j,m})$ since $\pi_{j,k}$ has Jacobian 2. For $\xi \notin \mathbb{Q}$, the set $\{(2j+1)\xi_1 - m\colon\ j, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. It follows that the open set $\bigcup_{(j,m)\in J} T_{j,m}$ is dense in P_K^+ and similarly the set $\bigcup_{(j,m)\in J} S_{j,m}$ is dense in M_{k-1}^+. It is clear that, for $\xi \in T_{j,m}$, we have that $\pi(\xi b^{2j(\xi)+1}a) = \pi_{j,m}(\xi)$. Let $\{(j_i, m_i)\colon\ i \geqslant 1\}$ be an enumeration of the countable set J and let $T_1 = T_{j_1,m_1}$ with $j(\xi) = j_1$ on T_1. Then let $T_2 = T_1 \cup \{\xi \in T_{j_2,m_2} \setminus T_1\colon\ \pi_{j_2,m_2}(\xi) \notin \pi_{j_1,m_1}(T_1)\}$ and $j(\xi) = j_2$ on $T_2 \setminus T_1$. We proceed inductively, with T_n constructed so that $T_n = \bigcup_{i=1}^n T_n \cap T_{j_i,m_i}$ and $S_n = \bigcup_{i=1}^n \pi_{j_i,m_i}(T_n \cap T_{j_i,m_i})$ have disjoint unions in P_k^+ and M_{k-1}^+, respectively. Then we define $T_{n+1} = T_n \cup \{\xi \in T_{j_{n+1},m_{n+1}} \setminus T_n\colon\ \pi_{j_{n+1},m_{n+1}}(\xi) \notin S_n\}$ and let $j(\xi) = j_i$ on $T_{n+1} \cap T_{j_i,m_i}$. The sets T_n and S_n are unions of open polygons with $\operatorname{Area}(S_n) = 2\operatorname{Area}(T_n)$. For each $c \in (r_k, r_{k+1})$, each of the maps $\pi_{j,m}$ sends the vertical line $\xi_1 = c$ to the vertical line $\eta_1 = 2c - 1$. Hence for $T = \bigcup_{n=1}^\infty T_n = \lim_{n\to\infty} T_n$ and $S = \bigcup_{n=1}^\infty S_n = \lim_{n\to\infty} S_n$, the segment $T \cap \{(\xi_1,\xi_2)\colon\ \xi_1 = c\}$ is a union of open intervals whose total length $\ell(c)$ coincides with the length of the segment $S \cap \{(\xi_1,\xi_2)\colon\ \xi_1 = 2c-1\}$. If $\ell(c) = c - r_k$ and thus is equal to the length of the segment $P_k^+ \cap \{(\xi_1,\xi_2)\colon\ \xi_1 = c\}$ for a.e. $c \in (r_k, r_{k+1})$, then clearly T has full measure in P_k^+. Otherwise, arguing by contradiction, let us suppose that $\ell(c) < c - r_k$ for some $c \notin \mathbb{Q}$. Then $P_k^+ \cap \{(\xi_1,\xi_2)\colon\ \xi_1 = c\}$ contains an open interval I_c of points (c, ξ_2) not in T and, as a consequence, $M_{k-1}^+ \cap \{(\xi_1,\xi_2)\colon\ \xi_1 = 2c-1\}$ contains an open interval J_c of points $(2c-1, \xi_2)$ not in S. By our comments before, it follows that $\pi_{j,m}(I_c) \subset J_c$ for some $(j,m) \in J$. However, this contradicts the definition of T since $(j,m) = (j_i, m_i)$ for some i, and I_c would have been included in T_i. It follows that T has full measure in P_k^+ and necessarily S has full measure in M_{k-1}^+. Observe that the map $\xi \mapsto j(\xi)$ defined in the construction of T is constant on polygonal sets and hence is measurable. This completes the proof that condition (ii) is satisfied. □

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