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# Lambda-determinants and domino-tilings

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Dedicated to the memory of David Robbins

**Abstract**

Consider the $2n$-by-$2n$ matrix $M = (m_{i,j})_{i,j=1}^{2n}$ with $m_{i,j} = 1$ for $i, j$ satisfying $|2i - 2n - 1| + |2j - 2n - 1| \leqslant 2n$ and $m_{i,j} = 0$ for all other $i, j$, consisting of a central diamond of 1's surrounded by 0's. When $n \geqslant 4$, the $\lambda$-determinant of the matrix $M$ (as introduced by Robbins and Rumsey [Adv. Math. 62 (1986) 169–184]) is not well defined. However, if we replace the 0's by $t$'s, we get a matrix whose $\lambda$-determinant is well defined and is a polynomial in $\lambda$ and $t$. The limit of this polynomial as $t \to 0$ is a polynomial in $\lambda$ whose value at $\lambda = 1$ is the number of domino-tilings of a $2n$-by-$2n$ square.


## 1. Lambda-determinants...

In Section 5 of their article [7], David Robbins and Howard Rumsey Jr. defined a generalization of the determinant of a matrix, which they dubbed the $\lambda$-determinant. It is a rational function of the entries of the matrix along with an extra parameter, $\lambda$; when $\lambda$ is set equal to $-1$, one obtains the ordinary determinant of the matrix, at least in the case where all matrix entries are non-zero. (For more details, see [2].)

In this article I will consider certain matrices with many vanishing entries. For these matrices, one cannot apply Robbins and Rumsey's definition literally, but there is still a natural way to attempt to compute the $\lambda$-determinant, by replacing the zeroes by an

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indeterminate $t$, taking the $\lambda$-determinant of the $t$-perturbed matrix, and then taking the limit as $t \to 0$. In particular, I will give a one-parameter family of 0,1-matrices whose $n$th member is a $2n$-by-$2n$ matrix whose $\lambda$-determinant, defined by this continuity method and then specialized to $\lambda = 1$, is the number of domino-tilings of a $2n$-by-$2n$ square.

Let us start by recalling Robbins and Rumsey's first, recursive definition of the $\lambda$-determinant. If $M$ is a 1-by-1 matrix, its $\lambda$-determinant is its sole entry. Now suppose $M$ is an $n$-by-$n$ matrix, with $n > 1$. Let $M_{\mathrm{NW}}$, $M_{\mathrm{NE}}$, $M_{\mathrm{SW}}$, and $M_{\mathrm{SE}}$ denote the $\lambda$-determinants of the $(n-1)$-by-$(n-1)$ connected submatrices in the northwest, northeast, southwest, and southeast corners of $M$, and let $M_{\mathrm{C}}$ be the $\lambda$-determinant of the central connected $(n-2)$-by-$(n-2)$ submatrix of $M$ (we take $M_{\mathrm{C}} = 1$ in the case $n = 2$). As long as $M_{\mathrm{C}}$ is non-zero, we define the $\lambda$-determinant of $M$ as

$$\det_\lambda M = (M_{\mathrm{NW}} M_{\mathrm{SE}} + \lambda M_{\mathrm{NE}} M_{\mathrm{SW}})/M_{\mathrm{C}}. \tag{1}$$

Using this definition, we can calculate the $\lambda$-determinant of the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

as $1 + \lambda$ and the $\lambda$-determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

as $((1+\lambda)(1+\lambda) + \lambda(1+\lambda)(1+\lambda))/(1) = (1+\lambda)^3$. As was pointed out by Robbins and Rumsey (and is easy to check by induction), the $\lambda$-determinant of the $n$-by-$n$ all-ones matrix is $(1+\lambda)^{n(n+1)/2}$.

When one is using (1) to calculate $\lambda$-determinants, a subtle distinction becomes important, namely, the distinction between working in $\mathbf{Q}(\lambda)$ throughout the recursion and substituting a particular value of $\lambda$ at the end (on the one hand), and using that particular value of $\lambda$ when performing the recursion (on the other). Consider, for instance, the 4-by-4 matrix whose entries are all 1's. Its $\lambda$-determinant is $(1+\lambda)^6$, and if we put $\lambda = -1$, we get 0. However, if we were to use $\lambda = -1$ in carrying out the recurrence, we would run into trouble, since $M_{\mathrm{NW}}$, $M_{\mathrm{NE}}$, $M_{\mathrm{SW}}$, $M_{\mathrm{SE}}$, and $M_{\mathrm{C}}$ all vanish; the $(-1)$-determinant of the matrix is given by the indeterminate expression $((0)(0) - (0)(0))/0$. If our goal is to be make sense of the $\lambda$-determinant over as broad a class of matrices as possible, clearly we should work in $\mathbf{Q}(\lambda)$ whenever we can.

Robbins and Rumsey give another, non-recursive formula for the $\lambda$-determinant of an $n$-by-$n$ matrix $M = (m_{i,j})_{i,j=1}^n$:

$$\det_\lambda(M) = \sum_{B \in \mathcal{A}_n} \lambda^{P(B)} (1+\lambda)^{N(B)} M^B. \tag{2}$$

Here $\mathcal{A}_n$ is the set of $n$-by-$n$ alternating-sign matrices $B = (b_{i,j})_{i,j=1}^n$, $P(\cdot)$ and $N(\cdot)$ are integer-valued functions on $\mathcal{A}_n$, and

$$M^B = \prod_{i=1}^{n} \prod_{j=1}^{n} m_{i,j}^{b_{i,j}}.$$

In more detail: An alternating-sign matrix is a matrix of $+1$'s, $-1$'s, and 0's such that in each row and column, the non-zero entries alternate in sign, beginning and ending with a $+1$ (which may be the same entry). If $B$ is an $n$-by-$n$ alternating-sign matrix, we define its inversion number as $I(B) = \sum b_{i,j} b_{r,s}$ where the sum is over all $1 \leqslant i, j, r, s \leqslant n$ with $i < r$ and $j > s$ (if $B$ is a permutation matrix, this coincides with the ordinary inversion statistic). We also define $N(B)$ as the number of negative entries in $B$, and $P(B)$ as $I(B) - N(B)$.

The summation formula (1) for $\det_\lambda M$ has exactly the same domain of applicability as the recursive formula (2), if we work over $\mathbf{Q}(\lambda)$ (and if the matrix entries themselves do not depend on $\lambda$); specifically, both formulas apply and give the same answer as long as the entries of $M$ that are in the central $(n-2)$-by-$(n-2)$ submatrix are all non-zero (these are precisely the positions in an $n$-by-$n$ alternating-sign matrix where the entry $-1$ can occur). However, if we use specific values of $\lambda$ in the recursion, the recursive formula can run into problems where the summation formula does not. For instance, consider the 4-by-4 matrix with all entries equal to 1; as we have seen, if we try to compute its $(-1)$-determinant using the specialization of the recurrence to $\lambda = -1$, we get an indeterminate result, whereas the formula in terms of alternating-sign matrices gives 0.

Unfortunately, neither of Robbins and Rumsey's two definitions works when central entries of $M$ are equal to zero. Nor is a straightforward appeal to continuity going to help us to define $\det_\lambda M$ for every $M$. Consider for instance the family of matrices

$$M_c(t) = \begin{pmatrix} t & t & t \\ t & t^4/c & t \\ t & t & t \end{pmatrix}.$$

For $c$ and $t$ non-zero, $\det_\lambda M_c(t) = (c\lambda + c\lambda^2) + (2\lambda + 2\lambda^2)t^3 + (1/c + \lambda^3/c)t^6$, which converges to $c\lambda + c\lambda^2$ as $t \to 0$. This limit depends on the value $c$. Hence, if we attempt to define the $\lambda$-determinant of the three-by-three all-zeroes matrix by taking a trajectory through that matrix in the space of three-by-three matrices and invoking continuity, the limit will depend on the trajectory we choose, and may even fail to exist.

Clearly the principled thing to do would be to study continuity properties of the $\lambda$-determinant, and I hope others will adopt this approach and undertake a more systematic study of what happens when different trajectories through a bad matrix are taken; this may have some bearing on the issue of how Dodgson condensation can be applied to matrices with vanishing connected minors (i.e., with singular connected submatrices). However, in this article I will take an easier path and restrict attention to the trajectory through $M$ in a particular direction. Specifically, I will replace the zeroes in $M$ by a new variable, $t$, and see what happens to the $\lambda$-determinant of the perturbed matrix (the "$t$-perturbation of $M$") as $t \to 0$. This may be unprincipled, but it is easy to compute. Moreover, for many matrices

$M$, the resulting rational function in $t$ is not just a Laurent polynomial in $t$ (as it must be, because of the summation formula), but is in fact an ordinary polynomial in $t$. In this case, the $\lambda$-determinant of $M$ can be defined as the constant term of this polynomial.

For example, consider the eight-by-eight matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It has interior zeroes, so its $\lambda$-determinant cannot be computed by the original formulas (1) and (2). However, if we replace all 0's by $t$, we get a polynomial in $\lambda$ and $t$ with 191 terms. Replacing $t$ by 0, we get a polynomial in $\lambda$ with a mere 17 terms, which evaluates to 12 988 816 when we set $\lambda = 1$.

## 2. ... and domino-tilings

12 988 816 is also the number of ways to cover an 8-by-8 square with 32 1-by-2 rectangles, commonly known as dominos. More generally, the number of ways to cover a $2n$-by-$2n$ square with $2n^2$ dominos was computed by Temperley and Fisher [9] and (simultaneously) by Kasteleyn [4], and is given by the double product

$$\prod_{j=1}^{n}\prod_{k=1}^{n}\left(4\cos^2\frac{\pi j}{2n+1}+4\cos^2\frac{\pi k}{2n+1}\right).$$

It turns out that if one considers the sub-region of the $2n$-by-$2n$ square that consists of the 2 central cell in the first and last rows, the 4 central cells in the second and second-from-last rows, the 6 central cells in the third and third-from-last rows, etc., one gets a region whose domino-tilings are enumerated by a much simpler expression, namely

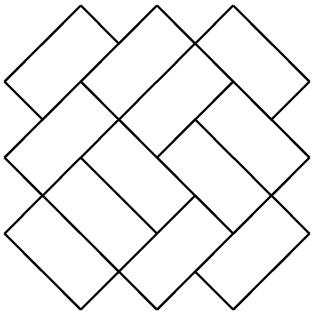
$$2^{n(n+1)/2}.$$

This region is called the Aztec diamond of order 8, and was first studied in detail in [3], although earlier occurrences of the shape appear in the literature. As an example, consider the case $n = 4$. If we associate the 64 cells of the 8-by-8 square with the entries of the 8-by-8 matrix considered at the end of the previous section, then the entries that contain 1's correspond to the cells that belong to the Aztec diamond of order 4.

It turns out that the domino-tilings of the Aztec diamond of order $n$ are intimately related to the $\lambda$-determinant of the generic $n$-by-$n$ matrix. (If the meaning of "generic" is unclear, then the reader should imagine that we are working over the ring $\mathbf{Q}(\lambda, a, b, c, \ldots)$, where

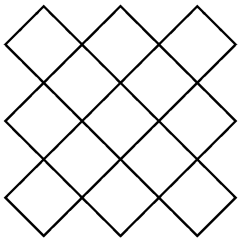
$a, b, c, \ldots$ are the entries of the matrix; then the entries of the matrix are perforce non-zero, so the summation formula for the $\lambda$-determinant is unproblematical in this context.) Specifically, if we apply the distributive rule to the Robbins–Rumsey summation formula for $\det_\lambda$, and ignore the fact that multiplication is commutative, each term $\lambda^{P(B)}(1+\lambda)^{N(B)}M^B$ expands to a sum of $2^{N(B)}$ monomials of the form $\lambda^k M^B$. The resulting sum has $2^{n(n+1)/2}$ terms, and these terms can be put into 1–1 correspondence with the $2^{n(n+1)/2}$ domino-tilings of the Aztec diamond of order $n$. (See [3] for details.)

If it seems paradoxical that the $\lambda$-determinant of a square matrix of 1's counts domino-tilings of an Aztec diamond, whereas the $\lambda$-determinant of the 0,1-matrix whose 1's form an Aztec diamond counts domino-tilings of a square, it may be helpful to imagine rotating the Aztec diamond by 45 degrees. Here is one of the 64 domino-tilings of the Aztec diamond of order 3:

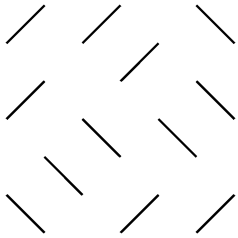


We can replace the tiling problem by the dual matching problem. We define an Aztec diamond graph whose vertices correspond to the cells of the Aztec diamond, with an edge between two vertices when the two corresponding cells are adjacent. Then a tiling of the Aztec diamond of order $n$ corresponds to a perfect matching of the Aztec diamond graph, that is, a collection of edges with the property that each vertex of the graph belongs to exactly one of the edges in the collection.

To illustrate, here is the Aztec diamond graph of order 3:

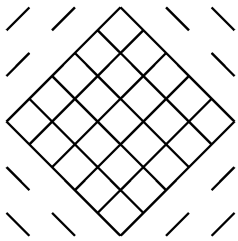


And here is the perfect matching of the Aztec diamond graph of order 3 that corresponds to the domino-tiling of the Aztec diamond shown in the first figure:



Looking back at the original figure, note that if the northwest, northeast, southwest, and southeast tiles are removed, what's left is a domino-tiling of the 4-by-4 square. More generally, there is a one-to-one correspondence between domino-tilings of the $2n$-by-$2n$ square and domino-tilings of the Aztec diamond of order $2n$ in which there are $n(n-1)/2$ forced tiles in each of the four corners (slanting from southwest to northeast in the northwest and southeast corners, and slanting from northwest to southeast in the southwest and northeast corners). The latter in turn correspond to perfect matchings of the Aztec diamond graph of order $2n$ in which there are $n(n-1)/2$ forced edges in each corner.

One way we can force some edges to be present is to force other edges to be absent, and one way to force edges to be absent is to work in the setting of weighted enumeration. Given an assignment of non-negative weights to the edges of a graph, we define the weight of an individual perfect matching as the product of the weights of its edges. In the article [6], I showed how the method of [3] could be adapted to the general problem of finding the sum of the weights of all the perfect matchings of an edge-weighted Aztec diamond graph. I applied this to the case of $2n$-by-$2n$ squares, setting some edge-weights equal to 1 and the rest equal to 0, in the fashion shown below for an Aztec diamond graph of order 5 (the edges shown get weight 1, and the rest get weight 0):



However, what was missing from that account was an explanation of the link with $\lambda$-determinants.

A good way to understand this link, without using all the machinery of generalized domino-shuffling, is to directly rely on Kuo's method of graphical condensation [5]. Let $G$ be a weighted Aztec diamond graph of order $n$. Let $G_{\mathrm{NW}}$ be the weighted Aztec diamond graph of order $n-1$ derived from $G$ by eliminating the southernmost $n$ vertices, the southernmost $2n$ edges, the easternmost $n$ vertices, and the easternmost $2n$ edges. Define $G_{\mathrm{NE}}$, $G_{\mathrm{SW}}$, and $G_{\mathrm{SE}}$ analogously. Define $G_{\mathrm{C}}$ to be the weighted Aztec diamond graph of order $n-2$ derived from $G$ by eliminating all of the aforementioned vertices and edges.

Lastly, define $w_{\mathrm{NW}}$ to be the weight of the northwesternmost edge, and define $w_{\mathrm{NE}}$, $w_{\mathrm{SW}}$, and $w_{\mathrm{SE}}$ analogously. Then Kuo's formula can be stated as

$$W(G)W(G_{\mathrm{C}}) = w_{\mathrm{NE}}w_{\mathrm{SW}}W(G_{\mathrm{NW}})W(G_{\mathrm{SE}}) + w_{\mathrm{NW}}w_{\mathrm{SE}}W(G_{\mathrm{NE}})W(G_{\mathrm{SW}}),$$

where $W(\cdot)$ denotes the sum of the weights of the perfect matchings of the graph in question. In our application, the edge-weights $w_{\mathrm{NW}}$, $w_{\mathrm{NE}}$, $w_{\mathrm{SW}}$, and $w_{\mathrm{SE}}$ are all 0's and 1's. This formula becomes a recurrence relation if one divides both sides by $W(G_{\mathrm{C}})$ (though this is only sensible if $W(G_{\mathrm{C}})$ is non-zero). If one runs the recurrence for the case at hand, the prefactors $w_{\mathrm{NE}}w_{\mathrm{SW}}$ and $w_{\mathrm{NW}}w_{\mathrm{SE}}$ are always equal to 1. Hence the recurrence takes the simplified form

$$W(G) = \big(W(G_{\mathrm{NW}})W(G_{\mathrm{SE}}) + W(G_{\mathrm{NE}})W(G_{\mathrm{SW}})\big)/W(G_{\mathrm{C}}),$$

which is equation (1) in the special case $\lambda = 1$. This gives us a combinatorial proof of the main claim of this paper, namely, that the number of domino-tilings of the $2n$-by-$2n$ square can be calculated as the $(+1)$-determinant of the $2n$-by-$2n$ matrix whose $i, j$th entry (for $1 \leqslant i, j \leqslant 2n$) is 1 if $|2i - 2n - 1| + |2j - 2n - 1| \leqslant 2n$ and is 0 otherwise.

Kuo's method also gives us a combinatorial interpretation to all the numbers that occur in the course of evaluating the $(+1)$-determinant of the matrix (by applying (1) with $\lambda = 1$): these numbers count domino-tilings of regions obtained from the Aztec diamond by eliminating the cells that lie in certain bands bordering the boundaries of the region.

For instance, consider the process of computing the $(+1)$-determinant of the 4-by-4 matrix

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The quantities that turn up in the recursive application of (1) are precisely the $(+1)$-determinants of all the connected submatrices of $M$. $M$ itself does double-duty as the matrix of $(+1)$-determinants of all the 1-by-1 submatrices of $M$. The $(+1)$-determinants of the connected 2-by-2 submatrices of $M$ form the 3-by-3 matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

whose entries are given by the $\lambda$-condensation formula (1) in the special case $\lambda = +1$. Turning the crank again gives the 2-by-2 matrix

$$\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$$

and turning it a final time gives the 1-by-1 matrix

(36)

whose sole entry is the number of tilings of the 4-by-4 square. The 6's count the domino-tilings of the region obtained from the 4-by-4 square by removing three cells from one corner and three cells from an adjoining corner.

This combinatorial interpretation of the $(+1)$-determinants of the connected submatrices of $M$ makes it clear that Robbins and Rumsey's original recursive formula for the $\lambda$-determinant can be applied to all of the matrices in our one-parameter family, provided we interpret the indeterminate expression $0/0$ as 0 whenever it crops up; each such occurrence corresponds to a subgraph of the weighted Aztec diamond graph whose perfect matchings all have weight 0.

The one thing missing from this explanation is an explicit discussion of the behavior of the $(+1)$-determinant of the $t$-perturbed matrix. We need to know that it is a polynomial in $t$. But this follows from the fact that it is equal to the sum of the weights of all the perfect matchings of the weighted graph. Indeed, all the rational functions of $t$ that occur during the recursion are polynomials in $t$, for the same reason.

This takes care of the case $\lambda = +1$: the inclusion of $t$ has solved the indeterminacy problem. It follows a fortiori that for generic $\lambda$, inclusion of $t$ also lets us carry out the recurrence (1) without encountering indeterminacy. (However, if one puts $\lambda = -1$, one still encounters indeterminacy, on account of the cancellations that occur.)

One consequence of the main result of this paper is that if one takes any $2n$-by-$2n$ alternating-sign matrix $A = (a_{i,j})_{i,j=1}^{2n}$ and sums those entries $a_{i,j}$ for which $|2i-2n-1|+|2j-2n-1| \leqslant 2n$, one gets a non-negative sum. A similar claim holds for odd-by-odd alternating-sign matrices: if one takes any alternating-sign matrix $A = (a_{i,j})_{i,j=1}^{2n+1}$ and sums those entries $a_{i,j}$ for which $|i-n-1|+|j-n-1| \leqslant n$, one gets a non-negative sum. This allows one to use the $t$-perturbation trick to extend the definition of the $\lambda$-determinant to the odd-by-odd matrix $M = (m_{i,j})_{i,j=1}^{2n+1}$ with $m_{i,j} = 1$ for those $i, j$ for which $|i-n-1|+|j-n-1| \leqslant n$ and $m_{i,j} = 0$ for all other $i, j$. (In fact, the $(+1)$-determinant of this matrix is also equal to the number of domino-tilings of the $2n$-by-$2n$ square. This was proved by Trevor Bass and Kezia Charles [1].)

Moreover, if one takes the intersection of either the odd-by-odd or even-by-even diamond pattern with any connected square submatrix of $M$, one gets a smaller pattern which also has the property that sum of the corresponding entries of an alternating-sign matrix must be non-negative. It would be interesting to have a classification of those partial sums of the entries of an $n$-by-$n$ matrix that are non-negative for all choices of an alternating-sign matrix. It would also be desirable to extend the results of this paper to the context of the octahedron recurrence with general initial conditions, as considered in [8].

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