Computing Fixed Densities of Markov Operators Defined by Stochastic Kernels

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Abstract—We use the method of Markov finite approximations to numerically compute a fixed density of a Markov operator defined by a stochastic kernel. The convergence and the error estimate are also presented.

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1. INTRODUCTION

The purpose of this paper is to propose a new numerical method for solving the fixed point problem of a Markov operator defined by a stochastic kernel. The usual method for solving an integral equation is the collocation method, which may not preserve useful properties of the original operator. Because our integral equation is defined by a Markov operator, it is beneficial to approximate it with finite-dimensional Markov operators.

Let $L^1(0,1)$ be the space of Lebesgue integrable functions $f$ with $\|f\| = \int_0^1 |f(x)| \, dx$. Let $D$ denote the set of densities, that is, all nonnegative $f \in L^1(0,1)$ such that $\|f\| = 1$. A linear operator $P : L^1(0,1) \to L^1(0,1)$ is called a Markov operator if $PD \subset D$. In this paper, we study a class of Markov operators that are defined by stochastic kernels. In other words,

$$ Pf(x) = \int_0^1 K(x,y)f(y) \, dy, \quad f \in L^1(0,1), \quad (1) $$

where the kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ is nonnegative and satisfies

$$ \int_0^1 K(x,y) \, dx = 1, \quad y \in [0,1] \text{ a.e.} $$

$P$ is indeed a Markov operator since $P$ is clearly a linear positive operator and for any $f \in L^1(0,1)$, by the Fubini Theorem,

$$ \int_0^1 Pf(x) \, dx = \int_0^1 \int_0^1 K(x,y)f(y) \, dy \, dx = \int_0^1 f(y) \, dy \int_0^1 K(x,y) \, dx = \int_0^1 f(y) \, dy. $$

Since $P$ is a positive integral operator of norm 1, there is a fixed density of $P$. In the next section, we prove this fact and prove the convergence of a class of numerical methods for computing the fixed density.
2. MARKOV FINITE APPROXIMATIONS

We assume that $K$ in (1) satisfies the condition

$$K'(x,y) \in \mathcal{L}(0,1) \text{ for } x \in [0,1] \text{ and there is a constant } b \text{ such that}$$

$$|K'(x,y)| \ dy \leq b, \quad \forall x \in [0,1]. \quad (2)$$

**Theorem 2.1.** Under Assumption (I), there is a density $f^* \in L^1(0,1)$ of bounded variation such that $Pf^* = f^*$.

**Proof.** Given $f \in D$. Since

$$\frac{1}{n} Pf = \int_0^1 |(Pf)'(x)| \ dx = \int_0^1 \int_0^1 K'_x(x,y) f(y) \ dy \ dx$$

$$\leq \int_0^1 f(y) \ dy \int_0^1 |K'_x(x,y)| \ dx \leq b \int_0^1 f(y) \ dy = b,$$

$$\{\sqrt{n} P^n f\}$$ is uniformly bounded by $b$. By Helly’s Theorem, $\{P^n f\}$ is relatively compact, which implies, by the Kakutani-Yosida Theorem [1], that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} P^n f = f^*$$

for some $f^* \in L^1(0,1)$. Of course, $f^* \in D$ and $f^*$ is of bounded variation.

**Remark 2.1.** In fact, we have proved the following inequality:

$$\frac{1}{n} Pf \leq \|f\|, \quad \forall f \in L^1(0,1). \quad (3)$$

Now we propose a numerical scheme for computing $f^*$. The method is based on the idea of Markov finite approximations of a Markov operator. Divide $[0,1]$ into $n$ equal subintervals $I_i = [x_{i-1}, x_i]$ with the length $h = 1/n$. Denote by $\Delta^0_n, \Delta^1_n, \Delta^2_n$, respectively, the corresponding subspaces of (discontinuous) piecewise constant, continuous piecewise linear, and continuous piecewise quadratic functions of $L^1(0,1)$. Bases for $\Delta^0_n, \Delta^1_n, \Delta^2_n$ are

$$e^0_i(x) = \chi_{I_i}(x), \quad i = 1, \ldots, n,$$

$$e^1_i(x) = w \left( \frac{x-x_i}{h} \right), \quad i = 0, 1, \ldots, n,$$

$$e^2_i(x) = u \left( \frac{x-x_i}{h} \right), \quad i = 0, 1, \ldots, n, \text{ and}$$

$$e^2_{i-1}(x) = v \left( \frac{x-x_i}{h} \right), \quad i = 1, 2, \ldots, n,$$

where $\chi_A$ is the characteristic function of $A$, $w(x) = (1-|x|)\chi_{[-1,1]}(x)$, $u(x) = (1-|x|)^2\chi_{[-1,1]}(x)$, and $u(x) = 2x(1-x)\chi_{[0,1]}(x)$.

Let $f_i = (1/h) \int_{I_i} f \ dm$. Define $Q^0_n : L^1(0,1) \to \Delta^0_n, Q^1_n : L^1(0,1) \to \Delta^1_n$, and $Q^2_n : L^1(0,1) \to \Delta^2_n$ by

$$Q^0_n f = \sum_{i=1}^n f_i e^0_i,$$

$$Q^1_n f = f_0 e^0_0 + \sum_{i=1}^{n-1} \frac{f_i + f_{i+1}}{2} e^1_i + f_n e^1_n,$$

$$Q^2_n f = f_0 e^0_0 + \sum_{i=1}^{n-1} \frac{f_i + f_{i+1}}{2} e^2_i + \sum_{i=1}^n f_i e^2_{i-1} + f_n e^2_n.$$
Now for \( j = 0,1,2 \), define \( P_n^j = Q_n^j \circ P \), and we solve numerically
\[
P_n^j f = f, \quad f \in \Delta_n^j,
\]
to get an approximate solution \( f_n^j \) to the original fixed point problem \( Pf = f \).

**Remark 2.2.** \( Q_n^0 \) is related to Ulam's original piecewise constant approximation scheme [2], while \( Q_n^1 \) and \( Q_n^2 \) were constructed in [3] initially for computing invariant measures of piecewise \( C^2 \) and stretching mappings by solving corresponding Frobenius-Perron operator equations.

It was shown in [2,3] that

1. \( Q_n^j \) are Markov operators of finite rank,
2. \( \lim_{n \to \infty} Q_n^j f = f \) strongly in \( L^1(0,1) \),
3. \( \forall \theta \), \( Q_n^j f \leq \theta f \), and
4. \( P_n^j \) has a fixed density \( f_n^j \in \Delta_n^j \).

Assumption (V) makes our numerical scheme well posed. And from (IV),
\[
\frac{1}{0} \int_0^1 P_n f \leq b \|f\|, \quad \forall f \in L^1(0,1).
\]

Let \( BV(0,1) = \{ f \in L^1(0,1) : \int_0^1 f < \infty \} \) with the norm
\[
\|f\|_{BV} = \|f\| + \frac{1}{0} \int_0^1 f, \quad \forall f \in BV(0,1).
\]

Then \( BV(0,1) \) is a Banach space, and any closed bounded subset of \( BV(0,1) \) is compact in \( L^1(0,1) \), due to the Helly Theorem.

**Theorem 2.2.** Suppose \( f^* \) is the unique fixed density of \( P \). Then for any sequence \( f_n^j \in \Delta_n^j \) of \( P_n^j \),
\[
\lim_{n \to \infty} \|f_n^j - f^*\| = 0, \quad j = 0,1,2, \quad \text{and}
\]
\[
\lim_{n \to \infty} \|f_n^j - f^*\|_{BV} = 0, \quad j = 1,2.
\]

**Proof.** (8) is from the fact that
\[
\frac{1}{0} \int_0^1 f_n^j = \frac{1}{0} \int_0^1 P_n^j f_n^j \leq b \|f_n^j\| = b,
\]
and conditions (II), (III) and the uniqueness of the fixed density of \( P \). To prove (9), for \( j = 1,2 \), from (3) and
\[
f^* - f_n^j = f^* - P_n^j f^* + P_n^j f^* - P_n^j f_n^j = P_n^j (f^* - f_n^j) + f^* - Q_n^j f^*,
\]
\[
\frac{1}{0} \int_0^1 (f^* - f_n^j) \leq \frac{1}{0} \int_0^1 P_n^j (f^* - f_n^j) + \frac{1}{0} \int_0^1 (f^* - Q_n^j f^*)
\]
\[
\leq b \|f^* - f_n^j\| + \frac{1}{0} \int_0^1 (f^* - Q_n^j f^*).
\]
Since \( \lim_{n \to \infty} \int_0^1 (f^* - Q_n^j f^*) = 0 \) by a direct computation (also see [4]), (9) follows from (8) and the definition of the BV-norm.

Now using the general result in [4], we have

**Theorem 2.3.** Let \( f_n^j \) be the fixed density of \( P_n^j \) in \( \Delta_n^j \). Then for \( j = 1,2 \),
\[
\|f_n^j - f^*\|_{BV} = O \left( \|f^* - Q_n^j f^*\|_{BV} \right) = O \left( \frac{1}{n} \right).
\]
REFERENCES