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Hopf Bifurcation of the Three-Dimensional Navier–Stokes Equations

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This paper is concerned with the three-dimensional Navier–Stokes flows excited by a unidirectional external force and gives an explicit example of Hopf bifurcation phenomenon occurring in the Navier–Stokes problem. Complete rigorous analysis on the existence of this instability behavior is provided. © 1999 Academic Press

1. INTRODUCTION

Transition of a fluid flow from the laminar to the turbulent state is a matter of everyday experience. From the well-known criteria of Landau [12] and Ruelle and Takens [22], the process involved is understood through the notions of bifurcation theory. In particular, the initial stage of the transition is commonly supposed to be the Hopf bifurcation. That is, time-dependent periodic flows branch off a basic steady flow at a critical Reynolds number.

Hopf bifurcation is the inherent phenomenon to be expected in nonlinear dynamical systems. Mathematical theory on the appearance of such a phenomenon for a system of ordinary differential equations was established by Hopf [7] under the assumption that the associated non-zero critical eigenvalue of the linear part is simple and transversal to the imaginary axis of the complex plane (see, for example, [6, 14]).

This result has been extended to general incompressible viscous fluid motions by Joseph and Sattinger [9]. Nevertheless, it seems difficult to examine the eigenvalue simplicity and the eigenvalue transversality conditions with respect to a linearized Navier–Stokes problem. In his review of bifurcation theory on hydrodynamics appeared in 1975, Kirchgässner [10]

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commented that although the possibility of Hopf bifurcation was well established if the certain conditions on the linearization were satisfied, they had not been realized for a single model. This comment was also mentioned in [11], where an explicit example of Hopf bifurcation in a rotating Bénard problem was presented. However, rigorous mathematical analysis on the Hopf bifurcation of the Navier–Stokes equations seems still in its early stage, and there exists a large gap in understanding the eigenvalue conditions. Inspired by the study of Meshalkin and Sinai [16], the gap was narrowed by [4] in connection with the two-dimensional Navier–Stokes equations,

$$\begin{aligned} \partial_t u - \Delta u + \lambda(u\cdot\nabla)u + \lambda\nabla p &= k'^2(\sin k'y, 0),\\ \nabla\cdot u &= 0,\\ u(t,x,y,z) = u(t,x+2\pi,y) &= u(t,x,y+2\pi), \qquad (1)\\ \int_0^{2\pi}\int_0^{2\pi} u\,dx\,dy &= 0, \end{aligned}$$

when $k' > 1$ is even. The simplicity condition with respect to this model can be verified based on its flow invariance structure found in [4], where, however, no complete proof is provided for the existence of Hopf bifurcation phenomenon for the lack of the verification on the eigenvalue transversality condition. Hence Hopf bifurcation was only examined in [4] by truncating (1) reduced to a flow invariant subspace into an ordinary differential system and by numerical experiments.

The model (1) was first formulated by Kolmogorov (see [2]). The pioneering work on (1) is due to Meshalkin and Sinai [16] which proves the global stability of the basic flow $(\sin y, 0)$ when $k' = 1$, and, in fact, Hopf bifurcation is prohibited there. With the use of the approach from Meshalkin and Sinai [16], instability of (1) with $k' \geq 2$ was found to occur by Iudovich [8] on the study of steady-state bifurcation of (1). Following the work of Iudovich [8], the further steady-state bifurcation phenomena are examined in [1, 15, 18, 19]. The relation between the above-mentioned viscous fluid motion and the associated inviscid fluid motion is investigated in [5, 17, 18]. One can also refer to [20] for the computational study on the instability of the Kolmogorov's model.

In this paper we consider the space-periodic fluid motions in R^3 excited by the unidirectional external force $4k^2(\sin 2kz, 0, 0)$ with $k \geq 1$ an integer. This forcing term gives rise to the unidirectional steady flow $u_0 = (\sin 2kz, 0, 0)$. The dynamical behavior of this fluid flow system defined in terms of velocity u and pressure p is described by the three-dimensional

Navier–Stokes equations,

$$\begin{aligned} \partial_t u - \Delta u + \lambda(u\cdot\nabla)u + \lambda\nabla p &= 4k^2(\sin 2kz, 0, 0), \\ \nabla\cdot u &= 0, \end{aligned} \tag{2}$$

where $\lambda > 0$ represents the Reynolds number characterizing the viscous fluid motion, and u satisfies the space-periodic condition

$$\begin{aligned} u(t, x+2\pi, y, z) &= u(t, x, y+2\pi, z) = u(t, x, y, z+2\pi) \\ &= u(t, x, y, z). \end{aligned} \tag{3}$$

The average velocity condition

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} u\,dx\,dy\,dz = 0 \tag{4}$$

is required to ensure the uniqueness of the associated Stokes problem. This viscous shear fluid flow system is the generalization of the two-dimensional Navier–Stokes system (1).

Based on the Hopf bifurcation theorem from [9], we obtain the existence of the time-dependent periodic solutions to (2)–(4) branching off the basic flow u_0 when the Reynolds number varies around some critical values. Thus an explicit example of Hopf bifurcation phenomenon with respect to the Navier–Stokes problem is presented from the viewpoint of rigorous analysis.

To state our results more precisely, we introduce the following Hilbert space

$$H^2 = \left\{u | u, \Delta u \in L^2\left([0, 2\pi]^3; R^3\right), \nabla\cdot u = 0, u \text{ satisfies } (3)\text{–}(4)\right\},$$

and its subspaces

$$\begin{aligned} H^2_{l,j,k} = \Bigg\{ u \in H^2 | u = &\sum_{n=1}^{\infty} (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}) \sin 2nkz \\ &+ \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} (\eta_{m,n1}, \eta_{m,n,2}, \eta_{m,n,3}) \\ &\times \sin(mlx + mjy + mkz + 2knz) \Bigg\}, \end{aligned}$$

with $l \geq 0$ and $j \in Z$. Here Z denotes the integer set.

Removing the pressure p by using the divergence free condition $\nabla \cdot u = 0$, we rewrite (2)–(4) in the following form,

$$\partial_t u - \Delta u - \lambda B(u) = 4k^2(\sin kz, 0, 0), \qquad u(t, \cdot, \cdot, \cdot) \in H^2, \quad (5)$$

where

$$B(u) = u \cdot \nabla u - \nabla\Delta^{-1}\big(\partial_x(u \cdot \nabla u_1) + \partial_y(u \cdot \nabla u_2) + \partial_z(u \cdot \nabla u_3)\big).$$

The definition of the subspace $H^2_{l,j,k}$ ensures the invariance condition:

$$B(u) \in H^2_{l,j,k} \text{ whenever } u \in H^2_{l,j,k} \cap W^{3,2}\big([0, 2\pi]^3; R^3\big).$$

Thus it is easy to see that the solution of (5) locally exists within $H^2_{l,j,k}$ (see the proof of [4, Lemma 2.1] for details) or there holds the following flow invariance result:

LEMMA 1.1. *Let the integers $l \geq 0$, $k \geq 1$, and $j \in Z$. Then for every $\hat{u} \in H^2_{l,j,k}$, there exists a constant $T > 0$ such that* (5) *admits a unique solution*

$$u \in C\big([0, T); H^2_{l,j,k}\big) \quad \textit{with } u(0) = \hat{u}.$$

In order to obtain the simplicity property, this lemma ensures us to reduce (5) to each of $H^2_{l,j,k}$ as

$$\partial_t u - \Delta u - \lambda B(u) = 4k^2(\sin kz, 0, 0), \qquad u(t, \cdot, \cdot, \cdot) \in H^2_{l,j,k}. \quad (6)$$

We are now in the position to state our result on the existence of Hopf bifurcations.

THEOREM 1.1. *Let the integer value (l, j, k) satisfy the condition,*

$$l = 1 \quad \textit{and} \quad j = 0, 1, -1 \quad \textit{when } k = 1,$$

$$\tfrac{1}{3}k^2 < l^2 + j^2 \leq \tfrac{1}{2}k^2, \qquad 1 \leq l \quad \textit{and} \quad j \in Z \quad \textit{when } k \geq 2.$$

Then there exists a critical Reynolds number $\lambda_{l,j,k} > 0$ of (6) *such that $(\lambda_{l,j,k}, u_0)$ is a Hopf bifurcation point of* (6).

In approaching this result, we use the Hopf bifurcation theorem due to Joseph and Sattinger [9]. For the reader's convenience, we simplify this theorem concerning (6) as

THEOREM 1.2 (Joseph and Sattinger [9]). *Let (l, j, k) be an integer vector. Assume that there exists a positive number $\lambda_{l,j,k} > 0$, and assume that the spectral problem, the linearization of* (6) *around u_0,*

$$\rho u = \Delta u - \lambda(u_0 \cdot \nabla u + u \cdot \nabla u_0 + \nabla p), \qquad u \in H^2_{l,j,k} \qquad (7)$$

admits a simple eigenvalue $\rho = \rho_{l,j,k}$ *subject to the transversal crossing condition*:

$$\operatorname{Re} \rho(\lambda_{l,j,k}) = 0, \qquad \operatorname{Im} \rho(\lambda_{l,j,k}) \neq 0 \quad \textit{and} \quad \operatorname{Re} \frac{d\rho(\lambda_{l,j,k})}{d\lambda} \neq 0.$$

Then there exist continuous functions $\nu_{l,j,k,\epsilon}$ *and* $\lambda_{l,j,k,\epsilon} \neq \lambda_{l,j,k}$ *for small* $\epsilon > 0$ *such that* (6) *with* $\lambda = \lambda_{l,j,k,\epsilon}$ *admits a solution*

$$u = u_{l,j,k,\epsilon} \in C\big([0,\infty); H^2_{l,j,k}\big)$$

satisfying

$$u_{l,j,k,\epsilon}(t,\cdot,\cdot,\cdot) = u_{l,j,k,\epsilon}\left(t + \frac{2\pi}{\lambda_{l,j,k,\epsilon}\nu_{l,j,k,\epsilon}}, \cdot,\cdot,\cdot\right) \quad \textit{for small } \epsilon > 0,$$

$$u_{l,j,k,\epsilon} \to u_0, \qquad \lambda_{l,j,k,\epsilon} \to \lambda_{l,j,k} \quad \textit{and}$$

$$\nu_{l,j,k,\epsilon} \to \operatorname{Im} \rho(\lambda_{l,j,k}) \quad \textit{as } \epsilon \to 0.$$

Here $2\pi(\lambda_{l,j,k,\epsilon}\nu_{l,k,j,\epsilon})^{-1}$ *is the period of* $u_{l,j,k,\epsilon}$.

Without loss of generality, we may suppose that the spaces H^2 and $H^2_{l,j,k}$ are complex when they are concerned with the spectral problem.

Taking Lemma 1.1 and Theorem 1.2 into account, we see that it remains to show the existence of the eigenvalue $\rho_{l,j,k}$ subject to the simplicity and transversal crossing conditions in Theorem 1.2. Thus we are led to the proof of the following result on spectral problem (7).

THEOREM 1.3. *Let* (l,j,k) *satisfy the condition in Theorem* 1.1. *Then the following assertions hold true*:

(i) *Equation* (7) *admits an eigenvalue* $\rho = \rho_{l,j,k}(\lambda)$ *such that*

$$\operatorname{Im} \rho_{l,j,k}(\lambda) < 0 \qquad (\lambda > 0) \quad \textit{and} \quad \lim_{\lambda \to 0} \rho_{l,j,k}(\lambda) = -l^2 - j^2 - k^2. \tag{8}$$

(ii) $\rho_{l,j,k}(\lambda)$ *is smooth and is a unique eigenvalue in a neighborhood of* $\rho_{l,j,k}(\lambda)$ *for* $\lambda > 0$. *Moreover* $\rho_{l,j,k}$ *satisfies the monotonicity property*

$$\operatorname{Re} \frac{d\rho_{l,j,k}(\lambda)}{d\lambda} > 0, \qquad \lambda > 0.$$

(iii) $\rho_{l,j,k}$ *satisfies positivity property*

$$\lim_{\lambda \to \infty} \operatorname{Re} \rho_{l,j,k}(\lambda) > 0.$$

(iv) $\rho_{l,j,k}$ *is simple in the following sense*

$$1 = \dim\left\{u \in H^2_{l,j,k} | A_\lambda u = \rho_{l,j,k}(\lambda)u\right\}, \qquad \lambda > 0,$$

where $A_\lambda u$ denotes the right-hand side of (7).

This result gives the existence of the critical Reynolds number $\lambda_{l,j,k}$ and eigenvalue $\rho_{L,j,k}$ satisfying the assumption of Theorem 1.2, and so (6) admits a time-dependent periodic solution $u_{l,j,k,\epsilon}$ branching off u_0 as λ varies across $\lambda_{l,j,k}$. Hence Theorem 1.1 is an immediate consequence of Lemma 1.1, Theorems 1.2 and 1.3. Consequently, it is the purpose of this paper to show Theorem 1.3, which is to be obtained by combining the examinations in remaining sections. Section 2 contains a result on a formulation of the spectral problem (7). With the use of this result, the proofs of the assertions (i)–(iv) are to be given, respectively, in Sections 3–6. The main difficulty of this paper is to show assertions (ii) and (iii).

It should be mentioned that the fluid motions governed by (6) becomes those satisfying the two-dimensional problem (1) with $k' = 2k$ whenever the integer $j = 0$. Thus the three-dimensional problem (2)–(4) admits the extra Hopf bifurcation points $(\lambda_{l,j,k}, u_0)$ and $(\lambda_{l,-j,k}, u_0)$ with $j \neq 0$. Theorem 1.1 implies that much more time-dependent periodic solutions branch off the basic flow u_0 for the three-dimensional Navier–Stokes problem, and provides an explicit example supporting the Prandtl's criterion (see [21]) on the possibility of greater instability of three-dimensional disturbances compared with the two-dimensional disturbance in parallel shear fluid motions. Moreover, Theorem 1.1 shows the global existence of regular solutions starting from some large initial data, although the global existence of regular and large solutions to a three-dimensional Navier–Stokes problem still remains to be an open question, which dates back to Leray [13] in 1934.

Finally, we note that, in Section 2, we apply the Riesz–Schauder theory to transform the three-dimensional spectral problem (7) to algebraic equations (9)–(10), which are essentially determined by the integer functions

$$\beta_n(l,j,k) - 4k^2 = l^2 + j^2 + (2nk + k)^2 - 4k^2.$$

If $j = 0$, (9)–(10) become the formulation of the associated two-dimensional spectral problem examined in [4]. For the two-dimensional Navier–Stokes problem (1) with $k' = 2k$, the eigenvalue transversal crossing condition is verified in [3] with the use of an approach similar to the present paper, and thus the rigorous analysis on the existence of the Hopf bifurcation is shown in [3] under the assumption

$$\beta_0(l,0,k) - 4k^2 = l^2 - 3k^2 < 0 \quad \text{and} \quad l \geq 1.$$

However, for the three-dimensional Navier–Stokes problem (6) under the acceptable condition

$$\beta_0(l,j,k) - 4k^2 = l^2 + j^2 - 3k^2 < 0, \qquad 1 \leq l \quad \text{and} \quad j \in Z,$$

a difficulty arises in examining (9)–(10) to show eigenvalue transversal crossing condition as k increases. We thus impose the assumption in Theorem 1.1,

$$\tfrac{1}{3}k^2 < l^2 + j^2 \leq \tfrac{1}{2}k^2, \qquad 1 \leq l \quad \text{and} \quad j \in Z,$$

when $k \geq 2$.

2. FORMULATION OF THE SPECTRAL PROBLEM

Let us introduce the following invariant subspace of the operator A_λ:

$$E_{l,j,k} = \left\{ u \in H^2 | u = \sum_{n=-\infty}^{\infty} (\xi_n, \eta_n, \zeta_n) \sin(lx + jy + kz + 2nkz) \right\}.$$

In connection with (1), it is, in fact, implied in [16] that its linearization can be reduced to algebraic equations (see also [4] for details). In this section, we extend this result to the three-dimensional spectral problem (5) in the following form.

THEOREM 2.1. *For the integers* $l \geq 1$, $j \in Z$, $k \geq 1$, *and the complex number* $\operatorname{Re} \rho > -l^2 - j^2 - k^2$, *the spectral problem* (7) *has the equivalent formulation,*

$$2\beta_n(\beta_n + \rho)\zeta_n + \lambda l(\beta_{n-1} - 4k^2)\zeta_{n-1} - \lambda l(\beta_{n+1} - 4k^2)\zeta_{n+1} = 0, \qquad n \in Z, \quad (9)$$

where and in what follows $\beta_n = \beta_n(l, j, k) = j^2 + l^2 + (2nk + k)^2$. *Additionally*, (9) *is equivalent to the equation*

$$\frac{2\beta_0(\beta_0 + \rho)}{\lambda l(\beta_0 - 4k^2)} + \cfrac{1}{\cfrac{2\beta_1(\beta_1 + \rho)}{\lambda l(\beta_1 - 4k^2)} + \cfrac{1}{\cfrac{2\beta_2(\beta_2 + \rho)}{\lambda l(\beta_2 - 4k^2)} + \cfrac{1}{\ddots}}} = i, \quad (10)$$

when $\{\zeta_n\}_{n \in Z}$ *is represented explicitly as*

$$\begin{aligned} (-1)^n \zeta_n &= \zeta_0 \gamma_1 \cdots \gamma_n \frac{\beta_0 - 4k^2}{\beta_n - 4k^2}, & n \geq 1, \\ \zeta_0 &= \zeta_0, & \quad (11) \\ \zeta_{-1} &= i\zeta_0, & \\ (-1)^{n-1} \zeta_{-n} &= i\zeta_{n-1}, & n \geq 2, \end{aligned}$$

where ζ_0 *is an arbitrary complex number, and*

$$\gamma_n = \cfrac{1}{\cfrac{2\beta_n(\beta_n + \rho)}{\lambda l(\beta_n - 4k^2)} + \cfrac{1}{\cfrac{2\beta_{n+1}(\beta_{n+1} + \rho)}{\lambda l(\beta_{n+1} - 4k^2)} + \cfrac{1}{\ddots}}}, \qquad n \geq 0.$$

Proof. First, we note from (7) that

$$-\Delta u + \rho u + \lambda \sin 2kz\, \partial_x u + \lambda u_3\, \partial_z u_0 + \lambda \nabla p = 0$$

for $u = (u_1, u_2, u_3)$. On substitution of

$$u = \sum_{n=-\infty}^{\infty} (\xi_n, \eta_n, \zeta_n)\phi_n \quad \text{with } \phi_n = \sin(lx + jy + kz + 2knz),$$

into this equation and after elementary calculations, we obtain

$$\sum_{n=-\infty}^{\infty} \left((\beta_n + \rho)\xi_n + \tfrac{1}{2}\lambda l(\xi_{n-1} - \xi_{n+1}) + \lambda k(\zeta_{n-1} + \zeta_{n+1})\right)\phi_n$$
$$= -\lambda\, \partial_x p, \quad (12)$$

$$\sum_{n=-\infty}^{\infty} \left((\beta_n + \rho)\eta_n + \tfrac{1}{2}\lambda l(\eta_{n-1} - \eta_{n+1})\right)\phi_n = -\lambda\, \partial_y p, \quad (13)$$

$$\sum_{n=-\infty}^{\infty} \left((\beta_n + \rho)\zeta_n + \tfrac{1}{2}\lambda l(\zeta_{n-1} - \zeta_{n+1}) \right) \phi_n = -\lambda\, \partial_z p, \tag{14}$$

$$\sum_{n=-\infty}^{\infty} (l\xi_n + j\eta_n + (k + 2nk)\zeta_n)\phi_n = 0. \tag{15}$$

By (12)–(13),

$$\sum_{n=-\infty}^{\infty} \Big((\beta_n + \rho)(l\xi_n + j\eta_n) + \tfrac{1}{2}\lambda l(l\xi_{n-1} + j\eta_{n-1} - l\xi_{n+1} - j\eta_{n+1}) + \lambda lk(\zeta_{n-1} + \zeta_{n+1}) \Big)\phi_n$$
$$= -l\lambda\, \partial_x p - j\lambda\, \partial_y p.$$

Applying the operators ∂_z and $-l\partial_x - j\partial_y$ to this equation and (14), respectively, and summing the resultant equations, we have

$$\Big((\beta_n + \rho)(l\xi_n + j\eta_n) + \tfrac{1}{2}\lambda l(l\xi_{n-1} + j\eta_{n-1} - l\xi_{n+1} - j\eta_{n+1}) + \lambda lk(\zeta_{n-1} + \zeta_{n+1}) \Big)(2nk + k)$$
$$= \left((\beta_n + \rho)\zeta_n + \tfrac{1}{2}\lambda l(\zeta_{n-1} - \zeta_{n+1}) \right)(l^2 + j^2), \qquad n \in Z.$$

Likewise, applying the operators ∂_y and $-\partial_x$ to Eqs. (12) and (13), respectively, and summing the resultant equations, we also have the equations without the pressure p involved

$$\left((\beta_n + \rho)\xi_n + \tfrac{1}{2}\lambda l(\xi_{n-1} - \xi_{n+1}) + \lambda k(\zeta_{n-1} + \zeta_{n+1}) \right) j$$
$$= \left((\beta_n + \rho)\eta_n + \tfrac{1}{2}\lambda l(\eta_{n-1} - \eta_{n+1}) \right) l, \qquad n \in Z.$$

Thus by (15), (7) becomes the coupled set of algebraic equations, $n \in Z$,

$$\frac{2\beta_n(\beta_n + \rho)}{\lambda l}\zeta_n + (\beta_{n-1} - 4k^2)\zeta_{n-1} - (\beta_{n+1} - 4k^2)\zeta_{n+1} = 0,$$

$$\frac{2(\beta_n + \rho)}{\lambda}(j\xi_n - l\eta_n) + l(j\xi_{n-1} - l\eta_{n-1}) - l(j\xi_{n+1} - l\eta_{n+1}) \tag{16}$$
$$= -2kj(\zeta_{n-1} + \zeta_{n+1}),$$

$$l\xi_n + j\eta_n = -(k + 2nk)\zeta_n. \tag{17}$$

Next, we borrow a technique of [8, 16] to show that $\{\xi_n\}_{n \in Z}$ and $\{\eta_n\}_{n \in Z}$ are uniquely determined by $\{\zeta_n\}_{n \in Z}$ when $\operatorname{Re} \rho > -l^2 - j^2 - k^2$.

Let $\{\tau_n\}_{n\in Z}$ with $\sum_{n=-\infty}^{\infty}|\tau_n|^2 < \infty$ satisfy the coupled set of algebraic equations

$$\frac{2(\beta_n + \rho)}{\lambda l}\tau_n + \tau_{n-1} - \tau_{n+1} = 0, \qquad n \in Z. \tag{18}$$

From the reasoning of [16], we may suppose that $\tau_n \neq 0$ for all $n \in Z$. Dividing (18) by τ_n yields

$$\frac{\tau_{\pm n}}{\tau_{\mp(n-1)}} = \frac{\mp 1}{2(\beta_{\pm n} + \rho)/\lambda l \mp \tau_{\pm(n+1)}/\tau_{\pm n}}, \qquad n \geq 0. \tag{19}$$

This together with the boundedness of $|\tau_{\pm n}|/|\tau_{\pm(n-1)}|$ implies

$$\left|\frac{\tau_{\pm n}}{\tau_{\pm(n-1)}}\right| \to 0 \quad \text{as } n \to \infty.$$

Applying (19) repeatedly gives

$$\frac{\tau_{\pm n}}{\tau_{\pm(n-1)}} = \cfrac{\mp 1}{\cfrac{2(\beta_{\pm n} + \rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_{\pm(n+1)} + \rho)}{\lambda l} + \cfrac{1}{\ddots}}}, \qquad n \geq 0,$$

which together with the simple identity $(\tau_0/\tau_{-1})^{-1} = \tau_{-1}/\tau_0$ implies

$$\frac{2(\beta_0 + \rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_1 + \rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_2 + \rho)}{\lambda l} + \cfrac{1}{\ddots}}}$$

$$= \cfrac{-1}{\cfrac{2(\beta_{-1} + \rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_{-2} + \rho)}{\lambda l} + \cfrac{1}{\ddots}}}.$$

Since $\beta_{-n-1} = \beta_n$, this equation becomes

$$\cfrac{2(\beta_0 + \rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_1 + \rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_2 + \rho)}{\lambda l} + \cfrac{1}{\ddots}}} = i. \tag{20}$$

Observing that

$$\operatorname{Re} \rho > -l^2 - j^2 - k^2 = -\beta_0 \geq -\beta_n,$$

we see that this equation is not true, since the real part of the left-hand side term is positive. Thus (18) has no nontrivial solution. Using the Riesz–Schauder theory, we see that for every $\{\zeta_n\}_{n \in Z}$ with $\sum_{n=-\infty}^{\infty} |\zeta_n|^2 < \infty$, (16) admits a unique solution $\{j\xi_n - l\eta_n\}_{n \in Z}$ with $\sum_{n=-\infty}^{\infty} |j\xi_n - l\eta_n|^2 < \infty$. This together with (17) implies that $\{\xi_n\}_{n \in Z}$ and $\{\eta_n\}_{n \in Z}$ are uniquely determined by $\{\zeta_n\}_{n \in Z}$. Consequently, (7) is determined by (9).

Finally, (9) becomes

$$\frac{2\beta_n(\beta_n + \rho)}{\lambda l(\beta_n - 4k^2)} \sigma_n + \sigma_{n-1} - \sigma_{n+1} = 0, \qquad n \in Z,$$

by setting $\sigma_n = \lambda l(\beta_n - 4k^2)\zeta_n$. Thus the derivation of (20) from (18) implies the equivalence of (9) and (10) for $\{\zeta_n\}_{n \in Z}$ subject to the condition

$$\frac{(\beta_{\pm n} - 4k^2)\zeta_{\pm n}}{(\beta_{\pm(n-1)} - 4k^2)\zeta_{\pm(n-1)}} = \cfrac{\mp 1}{\cfrac{2\beta_{\pm n}(\beta_{\pm n} + \rho)}{\lambda l(\beta_{\pm n} - 4k^2)} + \cfrac{1}{\cfrac{2\beta_{\pm(n+1)}(\beta_{\pm(n+1)} + \rho)}{\lambda l(\beta_{\pm(n+1)} - 4k^2)} + \cfrac{1}{\ddots}}}, \qquad n \geq 0,$$

which gives (10) and (11). The proof is complete.

3. PROOF THEOREM 1.3(i)

Let us note that, in what follows, the integer vector (l, j, k) is always supposed to satisfy the assumption of Theorem 1.1. Thus we have

$$\beta_0 - 4k^2 = l^2 + j^2 - 3k^2 < 0 \quad \text{and} \quad \beta_n - 4k^2 > 0 \quad \text{for } n \geq 1.$$

With the use of Theorem 2.1, the assertion (i) of Theorem 1.3 is in fact from [4] by using the Brouwer fixed-point theorem. However, for the reader's convenience, we provide a complete proof for this assertion.

It follows from Theorem 2.1 that it suffices to seek the function $\rho = \rho_{l,j,k}$ satisfying (8) and (10). By (10), we have

$$-\beta_0 - i\frac{\lambda l(4k^2-\beta_0)}{2\beta_0} + \frac{4k^2-\beta_0}{2\beta_0}\,\cfrac{\lambda l}{\cfrac{2\beta_1(\beta_1+\rho)}{(\beta_1-4k^2)\lambda l} + \cfrac{1}{\cfrac{2\beta_2(\beta_2+\rho)}{(\beta_2-4k^2)\lambda l} + \cfrac{1}{\ddots}}} = \rho.$$

Denoting by $\Psi_\lambda(\rho)$ the left-hand side of this equation, we have $\operatorname{Re}\Psi_\lambda(\rho) > -\beta_0$ and

$$\begin{aligned}\left|\Psi_\lambda(\rho) + \beta_0\right| &\le \frac{(4k^2-\beta_0)l\lambda}{2\beta_0} + \frac{(4k^2-\beta_0)(\beta_1-4k^2)l^2\lambda^2}{4\beta_0\beta_1(\beta_1+\operatorname{Re}\rho)}\\ &\le (4k^2-\beta_0)(\lambda+\lambda^2)\end{aligned}$$

for all $\operatorname{Re}\rho \ge -\beta_0$ and $\lambda > 0$. Hence Ψ_λ maps G into itself, provided that

$$G = \left\{c \in C \,|\operatorname{Re} c \ge -\beta_0, |c| \le \beta_0 + (4k^2-\beta_0)(\lambda+\lambda^2)\right\}.$$

Note that $\Psi_\lambda(\rho)$ is continuous with respect to (ρ, λ). By the Brouwer fixed-point theorem, Ψ_λ admits a fixed point $\rho(\lambda) = \rho_{l,j,k}(\lambda) \in G$.

Furthermore, noting

$$\begin{aligned}\left|\rho(\lambda) + \beta_0 + i\frac{l(4k^2-\beta_0)}{2\beta_0}\lambda\right| &\le \frac{(4k^2-\beta_0)(\beta_1-4k^2)l^2\lambda^2}{4\beta_0\beta_1(\beta_1+\operatorname{Re}\rho)}\\ &\le \frac{4k^2-\beta_0}{4\beta_0}\lambda^2,\end{aligned}$$

we have

$$\rho(\lambda) = -\beta_0 - i\frac{\lambda l(4k^2-\beta_0)}{2\beta_0} + O(\lambda^2).$$

This implies $\operatorname{Im}\rho(\lambda) < 0$, since $4k^2 - \beta_0 > 0$ and (10) gives $\operatorname{Im}\rho(\lambda) \ne 0$ for all $\lambda > 0$. The proof is complete.

4. PROOF OF THEOREM 1.3(ii)

First, we use the implicit function theorem to show the smoothness and uniqueness of $\rho_{l,j,k}$.

Recall from Theorem 2.1 that the eigenvalue $\rho(\lambda) = \rho_{l,j,k}(\lambda)$ satisfies the coupled set of the algebraic equations

$$\begin{aligned} 2\beta_n(\beta_n + \rho(\lambda))\zeta_n + \lambda l(\beta_{n-1} - 4k^2)\zeta_{n-1} \\ - \lambda l(\beta_{n+1} - 4k^2)\zeta_{n+1} = 0, \qquad n \in Z, \end{aligned} \tag{21}$$

where the sequence $\{\zeta_n\}_{n\in Z}$ is defined by (11). Introducing the function

$$\begin{aligned} F(\rho, \lambda) = &\sum_{n=-\infty}^{\infty} (-1)^n \beta_n(\beta_n + \rho)(\beta_n - 4k^2)\zeta_n^2 \\ &+ \lambda l \sum_{n=-\infty}^{\infty} (-1)^{n+1}(\beta_n - 4k^2)(\beta_{n+1} - 4k^2)\zeta_n\zeta_{n+1}, \end{aligned}$$

and using (21), we have $F(\rho(\lambda), \lambda) = 0$. Thus the smoothness and the uniqueness of $\rho(\lambda)$ follow from the implicit function theorem, whenever $\partial F(\rho, \lambda)/\partial\rho \neq 0$ holds true for all $\operatorname{Re}\rho > -\beta_0$ and $\lambda > 0$.

Indeed, by (11) and (21),

$$\begin{aligned} \frac{\partial F}{\partial \rho} = &\sum_{n=-\infty}^{\infty} (-1)^n \beta_n(\beta_n - 4k^2)\zeta_n^2 \\ &+ \sum_{n=-\infty}^{\infty} (-1)^n 2\beta_n(\beta_n + \rho)(\beta_n - 4k^2)\zeta_n \frac{\partial \zeta_n}{\partial \rho} \\ &+ \lambda l \sum_{n=-\infty}^{\infty} (-1)^{n+1}(\beta_n - 4k^2)(\beta_{n+1} - 4k^2) \\ &\times \left(\frac{\partial \zeta_n}{\partial \rho}\zeta_{n+1} + \zeta_n \frac{\partial \zeta_{n+1}}{\partial \rho} \right) \\ = &\sum_{n=-\infty}^{\infty} (-1)^n \beta_n(\beta_n - 4k^2)\zeta_n^2 + \sum_{n=-\infty}^{\infty} (-1)^n (\beta_n - 4k^2)\frac{\partial \zeta_n}{\partial \rho} \\ &\cdot\left(2\beta_n(\beta_n + \rho)\zeta_n + \lambda l(\beta_{n-1} - 4k^2)\zeta_{n-1} - \lambda l(\beta_{n+1} - 4k^2)\zeta_{n+1}\right) \\ = &\sum_{n=-\infty}^{\infty} (-1)^n \beta_n(\beta_n - 4k^2)\zeta_n^2 \\ = &\, 2\sum_{n=0}^{\infty} (-1)^n \beta_n(\beta_n - 4k^2)\zeta_n^2. \end{aligned} \tag{22}$$

On the other hand, multiplying the nth equation of (21) by $(\beta_n - 4k^2)\bar{\zeta}_n$, and summing the resultant equations, we have

$$\sum_{n=-\infty}^{\infty} 2\beta_n(\beta_n + \rho)\left(\beta_n - 4k^2\right)|\zeta_n|^2 + \lambda l \sum_{n=-\infty}^{\infty} \left(\beta_n - 4k^2\right)\left(\beta_{n+1} - 4k^2\right)\left(\zeta_n \bar{\zeta}_{n+1} - \zeta_{n+1}\bar{\zeta}_n\right) = 0,$$

which yields, by (11),

$$\sum_{n=-\infty}^{\infty} \beta_n(\beta_n + \operatorname{Re}\rho)\left(\beta_n - 4k^2\right)|\zeta_n|^2 = \sum_{n=0}^{\infty} 2\beta_n(\beta_n + \operatorname{Re}\rho)\left(\beta_n - 4k^2\right)|\zeta_n|^2 = 0. \tag{23}$$

Using this equation, we have, for $\operatorname{Re}\rho + \beta_0 > 0$,

$$\begin{aligned}
\left|\frac{\partial F}{\partial \rho}\right| &\geq 2\beta_0\left(4k^2 - \beta_0\right)|\zeta_0|^2 - 2\left|\sum_{n=1}^{\infty} (-1)^n \beta_n\left(\beta_n - 4k^2\right)\zeta_n^2\right| \\
&= \frac{2}{\beta_0 + \operatorname{Re}\rho} \sum_{n=1}^{\infty} \beta_n(\beta_n + \operatorname{Re}\rho)\left(\beta_n - 4k^2\right)|\zeta_n|^2 \\
&\quad - 2\left|\sum_{n=1}^{\infty} (-1)^n \beta_n\left(\beta_n - 4k^2\right)\zeta_n^2\right| \\
&\geq \frac{2}{\beta_0 + \operatorname{Re}\rho} \sum_{n=1}^{\infty} \beta_n(\beta_n + \operatorname{Re}\rho)\left(\beta_n - 4k^2\right)|\zeta_n|^2 \\
&\quad - 2\sum_{n=1}^{\infty} \beta_n\left(\beta_n - 4k^2\right)|\zeta_n|^2 \\
&= \frac{2}{\beta_0 + \operatorname{Re}\rho} \sum_{n=1}^{\infty} \beta_n(\beta_n - \beta_0)\left(\beta_n - 4k^2\right)|\zeta_n|^2 \\
&> 0,
\end{aligned}$$

where the use is made of Theorem 2.1 on the property $|\zeta_n| > 0$ for $\lambda > 0$. The proof is complete.

Next, we verify the validity of the monotonicity property.

Note that $\{\zeta_n\}_{n\in Z}$ together with $\rho = \rho_{l,j,k}$ is defined by (10)–(11). From (22), we see

$$\sum_{n=0}^{\infty} (-1)^n \beta_n\left(\beta_n - 4k^2\right)\zeta_n^2 \neq 0.$$

Take a suitable choice of ζ_0 to give

$$\sum_{n=-\infty}^{\infty}(-1)^n\beta_n(\beta_n-4k^2)\zeta_n^2=2\sum_{n=0}^{\infty}(-1)^n\beta_n(\beta_n-4k^2)\zeta_n^2=2. \quad (24)$$

In what follows, we need (23), that is,

$$\sum_{n=0}^{\infty}\beta_n(\beta_n-4k^2)|\zeta_n|^2 = -\frac{1}{\beta_0+\operatorname{Re}\rho}\sum_{n=1}^{\infty}\beta_n(\beta_n-\beta_0)(\beta_n-4k^2)|\zeta_n|^2<0. \quad (25)$$

Recall that

$$0=\sum_{n=-\infty}^{\infty}(-1)^n\beta_n(\beta_n+\rho(\lambda))(\beta_n-4k^2)\zeta_n^2 + \lambda l\sum_{n=-\infty}^{\infty}(-1)^{n+1}(\beta_n-4k^2)(\beta_{n+1}-4k^2)\zeta_n\zeta_{n+1}. \quad (26)$$

Differentiating this equation with respect to λ yields

$$\begin{aligned}0&=2\sum_{n=-\infty}^{\infty}(-1)^n\beta_n(\beta_n+\rho(\lambda))(\beta_n-4k^2)\zeta_n\frac{d\zeta_n}{d\lambda}\\&\quad+\sum_{n=-\infty}^{\infty}(-1)^n\beta_n(\beta_n-4k^2)\zeta_n^2\frac{d\rho}{d\lambda}\\&\quad+l\sum_{n=-\infty}^{\infty}(-1)^{n+1}(\beta_n-4k^2)(\beta_{n+1}-4k^2)\zeta_n\zeta_{n+1}\\&\quad+\lambda l\sum_{n=-\infty}^{\infty}(-1)^{n+1}(\beta_n-4k^2)(\beta_{n+1}-4k^2)\left(\frac{d\zeta_n}{d\lambda}\zeta_{n+1}+\zeta_n\frac{d\zeta_{n+1}}{d\lambda}\right)\\&=2\frac{d\rho}{d\lambda}+l\sum_{n=-\infty}^{\infty}(-1)^{n+1}(\beta_n-4k^2)(\beta_{n+1}-4k^2)\zeta_n\zeta_{n+1}\\&\quad+2\sum_{n=-\infty}^{\infty}(-1)^n(\beta_n-4k^2)\frac{d\zeta_n}{d\lambda}\\&\quad\cdot\left(2\beta_n(\beta_n+\rho)\zeta_n+\lambda l(\beta_{n-1}-4k^2)\zeta_{n-1}-\lambda l(\beta_{n+1}-4k^2)\zeta_{n+1}\right)\end{aligned}$$

$$= 2\frac{d\rho}{d\lambda} - \frac{1}{\lambda}\sum_{n=-\infty}^{\infty}(-1)^n\beta_n(\beta_n+\rho)(\beta_n-4k^2)\zeta_n^2$$
$$= 2\frac{d\rho}{d\lambda} - \frac{2}{\lambda}\sum_{n=0}^{\infty}(-1)^n\beta_n(\beta_n+\rho)(\beta_n-4k^2)\zeta_n^2$$
$$= 2\frac{d\rho}{d\lambda} - \frac{2}{\lambda}(\beta_0+\rho) - \frac{2}{\lambda}\sum_{n=1}^{\infty}(-1)^n\beta_n(\beta_n-\beta_0)(\beta_n-4k^2)\zeta_n^2,$$

where the use is made of (9), (10), (24), and (26). Therefore, by (25),

$$\lambda\,\mathrm{Re}\frac{d\rho}{d\lambda} = \beta_0 + \mathrm{Re}\,\rho + \sum_{n=1}^{\infty}(-1)^n\beta_n(\beta_n-\beta_0)(\beta_n-4k^2)\mathrm{Re}(\zeta_n^2)$$
$$\geq \beta_0 + \mathrm{Re}\,\rho - \sum_{n=1}^{\infty}\beta_n(\beta_n-\beta_0)(\beta_n-4k^2)|\zeta_n|^2$$
$$= (\beta_0+\mathrm{Re}\,\rho)\left(1 + \sum_{n=0}^{\infty}\beta_n(\beta_n-4k^2)|\zeta_n|^2\right).$$

Now we show

$$1 + \sum_{n=0}^{\infty}\beta_n(\beta_n-4k^2)|\zeta_n|^2 > 0.$$

Indeed, by (25), we denote by c the positive constant such that

$$c = -\sum_{n=0}^{\infty}\beta_n(\beta_n-4k^2)|\zeta_n|^2$$
$$= \beta_0(4k^2-\beta_0)|\zeta_0|^2 - \sum_{n=1}^{\infty}\beta_n(\beta_n-4k^2)|\zeta_n|^2.$$

Using (24), we have

$$\beta_0(4k^2-\beta_0)\left(|\zeta_0|^2 + \mathrm{Im}(\zeta_0^2)\right)$$
$$= c + \sum_{n=1}^{\infty}\beta_n(\beta_n-4k^2)\left(|\zeta_n|^2 + (-1)^n\,\mathrm{Im}(\zeta_n^2)\right),$$

and

$$\beta_0(4k^2-\beta_0)\left(|\zeta_0|^2 - \mathrm{Im}(\zeta_0^2)\right)$$
$$= c + \sum_{n=1}^{\infty}\beta_n(\beta_n-4k^2)\left(|\zeta_n|^2 - (-1)^n\,\mathrm{Im}(\zeta_n^2)\right).$$

Taking the Schwartz inequality into account, we have

$$
\begin{aligned}
&\beta_0^2\left(4k^2-\beta_0\right)^2\left|\operatorname{Re}\left(\zeta_0^2\right)\right|^2\\
&\quad=\beta_0^2\left(4k^2-\beta_0\right)^2\left(|\zeta_n|^2-\operatorname{Im}\left(\zeta_n^2\right)\right)\left(|\zeta_n|^2+\operatorname{Im}\left(\zeta_n^2\right)\right)\\
&\quad=\left(c+\sum_{n=1}^{\infty}\beta_n\left(\beta_n-4k^2\right)\left(|\zeta_n|^2+(-1)^n\operatorname{Im}\left(\zeta_n^2\right)\right)\right)\\
&\qquad\cdot\left(c+\sum_{n=1}^{\infty}\beta_n\left(\beta_n-4k^2\right)\left(|\zeta_n|^2-(-1)^n\operatorname{Im}\left(\zeta_n^2\right)\right)\right)\\
&\quad\geq\left(c+\sum_{n=1}^{\infty}\beta_n\left(\beta_n-4k^2\right)\left|\operatorname{Re}\left(\zeta_n^2\right)\right|\right)^2.
\end{aligned}
$$

This implies, by (24),

$$
\begin{aligned}
c&\leq\beta_0\left(4k^2-\beta_0\right)\left|\operatorname{Re}\left(\zeta_0^2\right)\right|-\sum_{n=1}^{\infty}\beta_n\left(\beta_n-4k^2\right)\left|\operatorname{Re}\left(\zeta_n^2\right)\right|\\
&\leq 1+\left|\sum_{n=1}^{\infty}(-1)^n\beta_n\left(\beta_n-4k^2\right)\operatorname{Re}\left(\zeta_n^2\right)\right|-\sum_{n=1}^{\infty}\beta_n\left(\beta_n-4k^2\right)\left|\operatorname{Re}\left(\zeta_n^2\right)\right|\\
&<1,
\end{aligned}
$$

since the representation of ζ_n in Theorem 2.1 implies $\zeta_n^2\neq 0$ for all $n\geq 0$. The proof is complete.

5. PROOF OF THEOREM 1.3(iii)

For $\rho=\rho_{l,j,k}$, we set the real functions $\mu=\operatorname{Re}\rho$ and $\nu=\operatorname{Im}\rho$. Let us start with two technical lemmas.

LEMMA 5.1. *There holds the estimate*

$$
-1\leq\frac{\nu(\lambda)}{\lambda l}<0\quad\text{for }\lambda>0.
$$

Proof. Recalling from Theorems 2.1 and the assertion (i) of Theorem 1.3 that

$$
\gamma_n(\lambda)=\cfrac{1}{\cfrac{2\beta_n\left(\beta_n+\rho(\lambda)\right)}{\lambda l\left(\beta_n-4k^2\right)}+\cfrac{1}{\cfrac{2\beta_{n+1}\left(\beta_{n+1}+\rho(\lambda)\right)}{\lambda l\left(\beta_{n+1}-4k^2\right)}+\cfrac{1}{\ddots}}},\qquad n\geq 0
$$

and $\nu(\lambda) < 0$, we see that

$$|\gamma_n(\lambda)| \le \frac{\lambda l(\beta_n - 4k^2)}{2\beta_n(\beta_n + \mu(\lambda))} \le \frac{\lambda l(\beta_n - 4k^2)}{2\beta_n(\beta_n - \beta_0)} \to 0 \quad \text{as } n \to \infty, \quad (27)$$

and

$$\begin{aligned} \operatorname{Im} \gamma_{n+1}(\lambda) &= \frac{2\beta_n|\nu(\lambda)|}{l(\beta_n - 4k^2)\lambda} + \operatorname{Im}\frac{1}{\gamma_n(\lambda)}, \qquad n \ge 0, \\ &\ge \frac{2|\nu(\lambda)|}{l\lambda} + \operatorname{Im}\frac{1}{\gamma_n(\lambda)}, \qquad n \ge 1. \end{aligned}$$

Moreover, from Theorem 2.1 we see $\gamma_0(\lambda) = -i$, which yields

$$\operatorname{Im} \gamma_1(\lambda) = 1 + \frac{2(l^2 + j^2 + k^2)\nu(\lambda)}{l(3k^2 - l^2 - j^2)\lambda} \le 1 + \frac{\nu(\lambda)}{l\lambda},$$

where the assumption on the integer vector (l, j, k) given in Theorem 1.3 is used. If $\nu(\lambda)/(\lambda l) < -1$ for some $\lambda > 0$, we see that $\operatorname{Im} \gamma_1(\lambda) < 0$, and thus $\operatorname{Im} \gamma_n(\lambda) > 1$ for $n \ge 2$. This contradicts to (27), and completes the proof.

Without loss of generality, from this lemma we may introduce the constant ν_0 such that

$$\nu_0 = \lim_{\lambda\to\infty} \frac{2\nu(\lambda)}{\lambda l}.$$

LEMMA 5.2. *Assume that* $\mu_0 = \lim_{\lambda\to\infty} \mu(\lambda) < \infty$. *Then there holds the estimate*

$$\begin{aligned} \frac{\beta_0(\beta_0 + \mu_0)}{4k^2 - \beta_0} &\ge \frac{\beta_1(\beta_1 + \mu_0)}{\beta_1 - 4k^2}\left(\frac{\beta_0\nu_0}{4k^2 - \beta_0} + 1\right)^2 \\ &\quad + \frac{\beta_2(\beta_2 + \mu_0)}{\beta_2 - 4k^2}\left(\frac{\beta_1\nu_0}{\beta_1 - 4k^2}\left(\frac{\beta_0\nu_0}{4k^2 - \beta_0} + 1\right) + 1\right)^2. \end{aligned} \quad (28)$$

Proof. For convenience, we set the real functions $h = h(\lambda)$ and $g = g(\lambda)$ such that

$$\frac{h(\lambda) + ig(\lambda)}{\lambda} = \frac{2\beta_2(\beta_2 + \rho(\lambda))}{l(\beta_2 - 4k^2)\lambda} + \cfrac{1}{\cfrac{2\beta_3(\beta_3 + \rho(\lambda))}{l(\beta_3 - 4k^2)\lambda} + \cfrac{1}{\ddots}},$$

which implies obviously

$$h(\lambda) > \frac{2\beta_2(\beta_2 + \mu(\lambda))}{l(\beta_2 - 4k^2)}.$$

Let us begin with the proof on the case

$$\frac{\beta_0 \nu_0}{4k^2 - \beta_0} + 1 = 0.$$

By (10), we see that

$$\frac{h + ig}{\lambda} = \frac{\dfrac{2\beta_0(\beta_0 + \rho)}{l(4k^2 - \beta_0)\lambda} + i}{-\dfrac{2\beta_1(\beta_1 + \rho)}{l(\beta_1 - 4k^2)\lambda}\left(\dfrac{2\beta_0(\beta_0 + \rho)}{l(4k^2 - \beta_0)\lambda} + i\right) + 1},$$

which together with the assumption $\mu_0 < \infty$ implies

$$\lim_{\lambda \to \infty} \frac{h(\lambda) + ig(\lambda)}{\lambda} = i\frac{\beta_0 \nu_0}{4k^2 - \beta_0} + i = 0. \tag{29}$$

On the other hand, by (10),

$$\begin{aligned}
&\frac{2\beta_0(\beta_0 + \mu_0)}{4k^2 - \beta_0} \\
&\quad = \operatorname{Re} \frac{1}{\dfrac{2\beta_1(\beta_1 + \rho(\lambda))}{l(\beta_1 - 4k^2)\lambda^2} + \dfrac{1}{h(\lambda) + ig(\lambda)}} \\
&\quad = \frac{\dfrac{2\beta_1(\beta_1 + \mu(\lambda))}{l(\beta_1 - 4k^2)\lambda^2} + \dfrac{h}{h^2 + g^2}}{\left(\dfrac{2\beta_1(\beta_1 + \mu(\lambda))}{l(\beta_1 - 4k^2)\lambda^2} + \dfrac{h}{h^2 + g^2}\right)^2 + \left(\dfrac{2\beta_1 \nu(\lambda)}{l(\beta_1 - 4k^2)\lambda^2} - \dfrac{g}{h^2 + g^2}\right)^2} \\
&\quad = \frac{1}{\dfrac{2\beta_1(\beta_1 + \mu(\lambda))}{l(\beta_1 - 4k^2)\lambda^2} + \dfrac{h}{h^2 + g^2} + \dfrac{\left(\dfrac{2\beta_1 \nu(\lambda)}{l(\beta_1 - 4k^2)\lambda^2} - \dfrac{g}{h^2 + g^2}\right)^2}{\dfrac{2\beta_1(\beta_1 + \mu(\lambda))}{l(\beta_1 - 4k^2)\lambda^2} + \dfrac{h}{h^2 + g^2}}}
\end{aligned}$$

$$\geq \left(\frac{2\beta_1(\beta_1 + \mu(\lambda))}{l(\beta_1 - 4k^2)\lambda^2} + \frac{h}{h^2 + g^2} + \frac{\left(\frac{2\beta_1 \nu(\lambda)}{l(\beta_1 - 4k^2)\lambda^2} - \frac{g}{h^2 + g^2} \right)^2 (h^2 + g^2)}{h} \right)^{-1}$$

$$= \frac{1}{\frac{2\beta_1(\beta_1 + \mu(\lambda))}{l(\beta_1 - 4k^2)\lambda^2} + \frac{(2\beta_1\nu(\lambda))^2 h}{l^2(\beta_1 - 4k^2)^2\lambda^4} + \left(\frac{2\beta_1\nu(\lambda) g}{l(\beta_1 - 4k^2)\lambda^2} - 1 \right)^2 \frac{1}{h}}.$$

This together with Lemma 5.1 and (29) implies, by setting $\lambda \to \infty$,

$$\frac{2\beta_0(\beta_0 + \mu_0)}{4k^2 - \beta_0} \geq \lim_{\lambda \to \infty} h(\lambda) \geq \frac{2\beta_2(\beta_2 + \mu_0)}{l(\beta_2 - 4k^2)},$$

and hence (28) is valid.

Next, we consider another case

$$\frac{\beta_0 \nu_0}{4k^2 - \beta_0} + 1 \neq 0.$$

By (10), we see that

$$h(\lambda) = \mathrm{Re} \frac{\lambda}{-\frac{2\beta_1(\beta_1 + \rho)}{l(\beta_1 - 4k^2)\lambda} + \frac{1}{\frac{2\beta_0(\beta_0 + \rho)}{l(4k^2 - \beta_0)\lambda} + i}}$$

$$= \mathrm{Re} \frac{\lambda}{-\frac{2\beta_1(\beta_1 + \rho)}{l(\beta_1 - 4k^2)\lambda} + \frac{\frac{2\beta_0(\beta_0 + \rho)}{l(4k^2 - \beta_0)\lambda} - i}{\left(\frac{2\beta_0(\beta_0 + \mu)}{l(4k^2 - \beta_0)\lambda} \right)^2 + \left(\frac{2\beta_0 \nu}{l(4k^2 - \beta_0)\lambda} + 1 \right)^2}}$$

$$= \frac{\lambda a(\lambda)}{a(\lambda)^2 + b(\lambda)^2},$$

by setting

$$a(\lambda) = -\frac{2\beta_1(\beta_1 + \mu)}{l(\beta_1 - 4k^2)\lambda} + \frac{\dfrac{2\beta_0(\beta_0 + \mu)}{l(4k^2 - \beta_0)\lambda}}{\left(\dfrac{2\beta_0(\beta_0 + \mu)}{l(4k^2 - \beta_0)\lambda}\right)^2 + \left(\dfrac{2\beta_0 \nu}{l(4k^2 - \beta_0)\lambda} + 1\right)^2},$$

and

$$b(\lambda) = \frac{2\beta_1 \nu}{l(\beta_1 - 4k^2)\lambda} + \frac{\dfrac{2\beta_0 \nu}{l(4k^2 - \beta_0)\lambda} + 1}{\left(\dfrac{2\beta_0(\beta_0 + \mu)}{l(4k^2 - \beta_0)\lambda}\right)^2 + \left(\dfrac{2\beta_0 \nu}{l(4k^2 - \beta_0)\lambda} + 1\right)^2}.$$

We thus have

$$\lambda a(\lambda) = h(\lambda)\left(a(\lambda)^2 + b(\lambda)^2\right).$$

Passing to the limit as $\lambda \to \infty$ in this equation yields

$$\begin{aligned}
-\frac{2\beta_1(\beta_1 + \mu_0)}{l(\beta_1 - 4k^2)} + \frac{\dfrac{2\beta_0(\beta_0 + \mu_0)}{l(4k^2 - \beta_0)}}{\left(\dfrac{\beta_0 \nu_0}{4k^2 - \beta_0} + 1\right)^2}
&= \left(\frac{\beta_1 \nu_0}{\beta_1 - 4k^2} + \frac{1}{\dfrac{\beta_0 \nu_0}{4k^2 - \beta_0} + 1}\right)^2 \lim_{\lambda \to \infty} h(\lambda) \\
&\geq \left(\frac{\beta_1 \nu_0}{\beta_2 - 4k^2} + \frac{1}{\dfrac{\beta_0 \nu_0}{4k^2 - \beta_0} + 1}\right)^2 \frac{2\beta_2(\beta_2 + \mu_0)}{l(\beta_2 - 4k^2)},
\end{aligned}$$

which gives (28) and completes the proof.

With the aid of the above lemmas, we can now carry out the proof for the desired assertion.

Proof of Theorem 1.3(*iii*). We argue by contradiction. If there holds

$$\mu_0 = \lim_{\lambda \to \infty} \mu(\lambda) \le 0,$$

it follows from Lemma 5.2 that

$$\frac{\beta_0^2}{4k^2 - \beta_0} \ge \frac{\beta_1^2}{\beta_1 - 4k^2}\left(\frac{\beta_0 \nu_0}{4k^2 - \beta_0} + 1\right)^2 + \frac{\beta_2^2}{\beta_2 - 4k^2}\left(\frac{\beta_1 \nu_0}{\beta_1 - 4k^2}\left(\frac{\beta_0 \nu_0}{4k^2 - \beta_0} + 1\right) + 1\right)^2. \quad (30)$$

Recalling that $l = 1$ and $j = 0, \pm 1$ when $k = 1$, we thus have, for $k = 1$,

$$\frac{(2 + j^2)^2}{2 - j^2} \ge \frac{(26 + j^2)^2}{22 + j^2}\left(\frac{(10 + j^2)\nu_0}{6 + j^2}\left(\frac{(2 + j^2)\nu_0}{2 - j^2} + 1\right) + 1\right)^2, \quad j = 0, \pm 1.$$

Note that, for any ν_0,

$$\frac{(10 + j^2)\nu_0}{6 + j^2}\left(\frac{(2 + j^2)\nu_0}{2 - j^2} + 1\right) + 1 > 0, \qquad j = 0, \pm 1.$$

We have

$$\frac{(2 + j^2)^2}{2 - j^2} \ge \frac{(26 + j^2)^2}{22 + j^2}\left(1 - \frac{(10 + j^2)(2 - j^2)}{4(6 + j^2)(2 + j^2)}\right)^2,$$

and hence

$$2 \ge \frac{26^2}{22}\left(1 - \frac{5}{12}\right)^2 > 2 \quad \text{when } j = 0,$$

$$9 \ge \frac{27^2}{23}\left(1 - \frac{11}{84}\right)^2 > 9 \quad \text{when } j = \pm 1.$$

This leads to a contradiction and thus $\mu_0 > 0$ is true when $k = 1$.

For another case of $k \ge 2$, the similar contradiction is to be deduced from (30). For convenience, we introduce the number r such that $l^2 + j^2 = rk^2$. Thus the assumption $k^2/3 < l^2 + j^2 \le k^2/2$ implies $\frac{1}{3} < r < \frac{1}{2}$ and

(30) becomes

$$\frac{(r+1)^2}{3-r} \geq \frac{(r+9)^2}{r+5}\left(\frac{(r+1)\nu_0}{3-r}+1\right)^2$$
$$+\frac{(r+25)^2}{r+21}\left(\frac{(r+9)\nu_0}{r+5}\left(\frac{(r+1)\nu_0}{3-r}+1\right)+1\right)^2. \quad (31)$$

When $(r+1)\nu_0/(3-r)+1 > \frac{1}{4}$, this inequality implies,

$$\begin{aligned}\frac{3}{5}(r+1) &\geq \frac{(r+1)^2}{3-r}\\ &\geq \frac{(r+9)^2}{r+5}\left(\frac{(r+1)\nu_0}{3-r}+1\right)^2\\ &> \frac{(r+9)^2}{16(r+5)}\\ &> \frac{1}{10}(r+9) > \frac{3}{5}(r+1).\end{aligned}$$

When $(r+1)\nu_0/(3-r)+1 \leq \frac{1}{4}$, we see that

$$\begin{aligned}\frac{(r+9)\nu_0}{r+5}\left(\frac{(r+1)\nu_0}{3-r}+1\right)+1 &\geq 1-\frac{3(r+9)(r+1)}{16(3-r)(r+5)}\\ &\geq 1-\frac{9(r+9)}{80(r+5)} \geq 1-\frac{63}{320}.\end{aligned}$$

This together with (31) implies

$$\begin{aligned}\frac{3}{5}(r+1) &\geq \frac{(r+1)^2}{3-r}\\ &\geq \frac{(r+25)^2}{r+21}\left(\frac{(r+9)\nu_0}{r+5}\left(\frac{(r+1)\nu_0}{3-r}+1\right)+1\right)^2\\ &\geq \frac{(r+25)^2}{r+21}\left(1-\frac{63}{320}\right)^2\\ &\geq \frac{51}{43}(r+25)\left(1-\frac{63}{320}\right)^2 > \frac{3}{5}(r+1).\end{aligned}$$

We thus conclude that $\mu_0 > 0$ and complete the proof.

6. PROOF OF THEOREM 1.3(iv)

Let us note that $H^2_{l,j,k}$ is the orthogonal sum of the following three subspaces

$$\left\{u \in H^2 | u = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} (\xi_{m,n}, \eta_{m,n} \zeta_{m,n}) \sin(2mlx + 2mjy + 2nkz)\right\},$$

$$\left\{u \in H^2 | u = \sum_{m=2}^{\infty} \sum_{n=-\infty}^{\infty} (\xi_{m,n}, \eta_{m,n}, \zeta_{m,n}) \sin((2m-1)(lx + jy) + kz + 2nkz)\right\},$$

$$\left\{u \in H^2 | u = \sum_{n=-\infty}^{\infty} (\xi_n, \eta_n, \zeta_n) \sin(lx + jy + kz + 2nkz)\right\},$$

and each of these subspaces is invariant with respect to the operator A_λ.

If the spectral problem $A_\lambda u = \rho(\lambda) u$ is reduced to the first subspace, by taking the proof of Theorem 2.1 into account, it is readily obtained as in Iudovich [8] that only real eigenvalues possibly exist in the half complex plane $\{c \in C | \operatorname{Re} c > -k^2\}$.

Note that the second subspace is the orthogonal sum of $E_{(2m-1)l,(2m-1)j,k}$, $m \geq 2$, the subspaces defined in Section 2. By Theorem 2.1 together with its proof, the spectral problem $A_\lambda u = \rho(\lambda) u$ reduced to $E_{(2m-1)l,(2m-1)j,k}$ is equivalent to the algebraic equation

$$\frac{2\beta_{m,0}(\beta_{m,0}+\rho)}{(\beta_{m,0}+4k^2)\lambda l} + \cfrac{1}{\cfrac{2\beta_{m,1}(\beta_{m,1}+\rho)}{(\beta_{m,1}-4k^2)\lambda l} + \cfrac{1}{\cfrac{2\beta_{m,2}(\beta_{m,2}+\rho)}{(\beta_{m,2}-4k^2)\lambda l} + \cfrac{1}{\ddots}}} = i \quad \text{for } m \geq 2, \tag{32}$$

where $\beta_{m,n} = (2m-1)^2(l^2+j^2) + (k+2nk)^2$. It is readily seen that the real part of the left-hand side of this equation is positive for any ρ with $\operatorname{Re}\rho > -k^2$, since the assumption $l^2 + j^2 > k^2/3$ implies $\beta_{m,n} - 4k^2 > 0$ for all $m \geq 2$ and $n \geq 0$. This shows the absence of the complex eigenvalue $\rho(\lambda)$ in the half complex plane $\{c \in C | \operatorname{Re} c > -k^2\}$.

However, as far as the third subspace is concerned, we see that this subspace is nothing more than $E_{l,j,k}$ and thus the eigenvalue $\rho_{l,j,k}(\lambda)$ exists. From Theorem 1.3(ii), Theorem 2.1, and the above observation on the first and the second subspaces it follows that

$$1 = \dim\left\{u \in E_{l,j,k} | A_\lambda u = \rho_{l,j,k}(\lambda)u\right\}$$
$$= \dim\left\{u \in H^2_{l,j,k} | A_\lambda u = \rho_{l,j,k}(\lambda)u\right\}.$$

The proof is complete.

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