JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 111, 177–187 (1985)

A Separating Problem on Function Spaces

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1. INTRODUCTION

In his dissertation [2], the first author has studied the following problem. Let f be a conformal (or an analytic) mapping defined on some domain G in the complex plane $\mathbb{C}$. G is usually assumed to be simply connected with its boundary a rectifiable Jordan curve. Let $\{z_n : n = 1, 2,...\}$ be a sequence of points in G. After observations are made along the sequence $\{z_n\}$, can one reconstruct this f? Or, at least, is the data good enough to determine f uniquely? To be a little more precise, let F be a space of analytic functions on G. Given the observations $\{f(z_n) + e_n, n = 1, 2,...\}$ where $f \in F$ and the random variables e_n are always assumed to be independent and identically Gaussian distributed (abbreviated as i.i.d. with $N(0, \sigma^2 I)$), we denote by P_f the probability measure generated on $\mathbb{C}^\infty$ by the observations $\{f(z_n) + e_n; n = 1, 2,...\}$. Is it true that P_{f_1} and P_{f_2} are mutually singular to each other for any two distinct functions f_1 and f_2 in F? Here we refer the readers to [4] for all the statistical terminology.

This problem originates from biology. In the growth process of an organism, for example, a unicellular paramecium, its shape changes with time. It would not deform very drastically in a short period of time, however. In order to know how the paramecium changes its shape, we are bound to make some estimates. It would be nice if estimates on a small portion would tell the whole story. By the uniqueness theorem for analytic functions, it is therefore reasonable to assume that the deformation is a conformal transformation, or an analytic transformation, to say the least.

We call the sequence $\{z_n\}$ F-separating, if the answer to the above uniqueness question is affirmative.

0022-247X/85 $3.00

In the case that $F = H(D)$, the space of all analytic functions in the open unit disk D, we need only consider the interesting situation that the infinite sequence $\{z_n\}$ approaches boundary:

$$|z_n| < 1 \qquad \text{and} \qquad \lim |z_n| = 1. \tag{1.1}$$

It is proved in [2, 3] that P_{f_1} and P_{f_2} are mutually singular to each other for any two distinct f_1 and f_2 in $H(D)$ if and only if $\sum_n |f_1(z_n) - f_2(z_n)|^2 = \infty$ and we have the following

THEOREM. *Let F be a subspace of $H(D)$. Then $\{z_n\}$ is F-separating if and only if*

$$\sum_n |f(z_n)|^2 = \infty \tag{1.2}$$

for any nonzero function f in F.

Therefore, from now on, we simply say that $\{z_n\}$ is *F-separating* if $\sum |f(z_n)|^2 = \infty$ for all nonzero functions f in F. When F is an algebra, this is equivalent to the condition $\sum |f(z_n)| = \infty$ for all $f \neq 0$ in F. Using the language of the function theory, this means $\{z_n\}$ determines f in a "strong" sense: the differences at all points z_n cannot be summable for any two different functions.

A general characterization for a separating sequence seems difficult even if we assume that $\{z_n\}$ satisfies some simple geometric conditions. One such reason is that the divergence condition (1.2) is not preserved under any kind of closure.

It is well known from complex interpolation theory that for any sequence $\{z_n\}$ satisfying (1.1) and any sequence $\{w_n\}$ of complex numbers, there exists an analytic function f in $H(D)$ such that $f(z_n) = w_n$, $n = 1, 2, \ldots$. This implies no sequence with property (1.1) can be $H(D)$-separating.

On the other hand, we may consider the separating problem for $F = H(\bar{D})$, the space of analytic functions in a neighborhood of the closed unit disk. The observation points z_n accumulate to some point. This can be viewed as observing the evolution of the organism at some particular point. Without loss of generality, we may assume this point is the origin. Then the above theorem gives a complete description of such a separating sequence which says $\{z_n\}$ is of infinite convergence exponent; namely, $\sum |z_n|^p = \infty$ for any $p > 0$, because any $f \in F$ has a Taylor series expansion about the origin (see [3], also).

A reasonable and interesting candidate between thee two function spaces for the separating problem would be $F = H^\infty$, the space of bounded analytic functions on the open unit disk. In Section 2, Theorem 1 gives a sufficient condition for an H^∞-separating sequence in terms of the measure of its nontangential accumulation points. We then give an application to the case of equidistant observations. It was hoped that this sufficient condition would also be necessary. In Section 3 it turns out, however, that this is not so. Theorem 3 gives an example of an H^∞-separating sequence which accumulates at only one point. This example, by far the simplest of its kind, should deserve a further and thorough examination. In Section 4, we give some examples of $\{z_n\}$ which are not H^∞-separating other than the obvious ones such as the Blaschke sequences.

In the last section, we give some remarks and examples on other function spaces, such as meromorphic functions and entire functions as well as the Nevanlinna class and the disk algebra, for the saparating problem.

2. A Sufficient Condition for H^∞-Separating

A sequence $\{z_n\}$ is a *Blaschke sequence* if $\sum (1-|z_n|)<\infty$. It is clear that any two different functions in H^∞ have different values at some z_n if $\{z_n\}$ is not Blaschke, and vice versa. Therefore we see immediately that a Blaschke sequence (in particular, an interpolating sequence for H^∞) cannot be H^∞-separating. This indicates that if the points are too "few," then $\{z_n\}$ fails to be H^∞-separating. On the other hand, if there are very "many" points, then $\{z_n\}$ may well be H^∞-separating. The following is one way to "meaure" the sequence $\{z_n\}$. We denote by E the set of all points on the unit circle which are the nontangential accumulation points of $\{z_n\}$. It is classical (see [6]) that for any $f \in H^\infty$, the nontangential limit exists almost everywhere and log $|f|$ is integrable on the unit circle unless f is identically zero. Therefore we get an easy sufficient condition for H^∞-separating in terms of $m(E)$, where m is the linear Lebesgue measure on the unit circle.

THEOREM 1. *The sequence $\{z_n\}$ is H^∞-separating if $m(E)>0$.*

One such sequence is the dominating sequence which appeared in [1] and [5]. For this sequence, $m(E)=2\pi$.

We will have further remarks on Theorem 1 in Sections 3 and 4. We present an application to the case of equidistant observations at this moment.

It is natural to place the points $\{z_n\}$ equidistant on concentric circles. Thus one can make the observations

$$f(z_k^n)+e_k^n, \qquad k=1, 2,..., n; n=1, 2,...,$$

where $z_k^n = (1-\delta_n)\exp(2\pi ik/n)$ with δ_n decreasing to 0 and e_k^n are i.i.d., $N(0, \sigma^2 I)$, and hope that f can be constructed from this data.

If $\sum n\,\delta_n < \infty$, $\{z_k^n\}$ is a Blaschke sequence and therefore cannot be H^∞-separating. This means in order to determine f from these observations, δ_n cannot go to 0 too fast. We do not know whether $\sum n\,\delta_n = \infty$ is enough for $\{z_k^n\}$ to be H^∞-separating. If we choose δ_n suitably, however, we can take $\exp(2\pi i\theta) \in E$ for every irrational θ, and thus $m(E) = 2\pi$.

Fix an irrational $\theta \in [0, 1]$. In the following we use the symbol c to denote a positive constant, depending only on θ, which may differ at each occurence. It is not difficult to see that $\exp(2\pi i\theta) \in E$ if and only if there is a constant c such that

$$1-(1-\delta_n)\cos(2\pi \min_{0\leqslant k\leqslant n} |\theta - k/n|) \geqslant c\sin(2\pi \min_{0\leqslant k\leqslant n} |\theta - k/n|)$$

holds for infinitely many n, which is equivalent to the existence of a constant c such that

$$\begin{aligned} \delta_n &\geqslant c \min_{0\leqslant k\leqslant n} |\theta - k/n| \\ &= cn^{-1} \min_{0\leqslant k\leqslant n} (|n\theta - k| \pmod 1) \end{aligned} \tag{2.1}$$

holds for infinitely many n. Thus the distribution of the sequence $n\theta \pmod 1$ needs to be investigated. In this aspect, the following lemma in [7, p. 3] is available.

LEMMA. *Given any irrational number θ, there are infinitely many rational numbers h/k such that*

$$|\theta - h/k| \leqslant k^{-2}.$$

With this lemma in mind, (2.1) leads to the following result.

THEOREM 2. *If there exists a constant c such that $\delta_n \geqslant cn^{-2}$, $n = 1, 2,...$, then the sequence $\{z_k^n\} = \{(1-\delta_n)\exp(2\pi ik/n),\ 1 \leqslant k \leqslant n,\ n = 1, 2,...\}$ is H^∞-separating.*

We also have the following interesting corollary which should surprise no one.

COROLLARY. *Let $\varepsilon > 0$ be a constant. Then for almost all $\theta \in [0, 1]$, there exists no constant $c(\theta)$ such that*

$$\min_{0\leqslant h\leqslant k} |\theta - h/k| \leqslant c(\theta)\, k^{-(2+\varepsilon)}$$

holds for infinitely many k.

Proof. Otherwise $m(E) > 0$ for the sequence $z_k^n = (1 - n^{-(2+\varepsilon)}) \exp(2\pi i k/n)$, $1 \leqslant k \leqslant n$, $n \geqslant 1$, and $\{z_k^n\}$ is H^∞-separating by Theorem 1. But this contradicts the fact that $\{z_k^n\}$ is a Blaschke sequence.

3. A Class of H^∞-Separating Sequences

From what we have discussed in the previous section, one may guess the set E of an H^∞-separating sequence must be necessarily big. It had been conjectured that the set E of an H^∞-separating sequence is at the least infinite. This is, as it turns out, not true. In this section, we present an H^∞-separating sequence which accumulate at only one point. In the proof, the symbol c again stands for a positive constant depending only on the given function f, and not necessarily the same at each occurence.

Theorem 3. *Let $z_n = 1 - (\log \log n)^{-1}$. Then $\{z_n\}$ is H^∞-separating.*

Proof. For any nonzero f in H, we express $f = FSB$, where F is the outer part, S is the singular inner part, and B is a Blaschke product (see [6]). We shall estimate each of the outer and inner parts.

For a point z in D, $P_z(\theta)$ is the Poisson kernel for z. It is clear that

$$\frac{1-|z|}{1+|z|} \leqslant P_z(\theta) \leqslant \frac{1+|z|}{1-|z|}.$$

Since

$$F(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log |f| \, d\theta\right],$$

we have

$$\begin{aligned} |F(z)| &\geqslant \exp\left[\frac{1}{2\pi}\int P_z(\theta) \log |f| \, d\theta\right] \\ &\geqslant \exp\left[-\frac{1}{2\pi}\int \frac{1+|z|}{1-|z|} |\log |f|| \, d\theta\right] \\ &\geqslant \exp[-c/(1-|z|)]. \end{aligned}$$

Consequently $\sum |F(z_n)| \geqslant \sum (\log \log n)^{-c} = \infty$.

Suppose

$$S(z) = \exp\left[-\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \, d\mu(\theta)\right],$$

where μ is a singular positive measure on the unit circle. Then

$$\begin{aligned}|S(z)| &= \exp\left[-\int P_z(\theta)\,d\mu(\theta)\right]\\ &\geqslant \exp\left[-\int \frac{1+|z|}{1-|z|}\,d\mu(\theta)\right]\\ &\geqslant \exp[-c/(1-|z|)]\end{aligned}$$

and thus again $\sum |S(z_n)| \geqslant \sum (\log\log n)^{-c} = \infty$.

Let

$$B(z) = \prod \frac{-\bar{\alpha}_m}{|\alpha_m|}\,\frac{z-\alpha_m}{1-\bar{\alpha}_m z},$$

where $\{\alpha_m\}$ is the sequence of zeros of B in the open unit disk. We may assume that $|\alpha_m|$ is increasing to 1. For a fixed $z \in D$, express $B = B_1 B_2$ as a factorization where B_1 and B_2 have zeros $\{\alpha_m\}$ with $|\alpha_m| \leqslant |z|$ and $|\alpha_m| > |z|$, respectively. Then

$$\begin{aligned}|B_1(z)| &= \prod \left|\frac{z-\alpha_m}{1-\bar{\alpha}_m z}\right| \geqslant \prod \frac{|z|-|\alpha_m|}{1-|\alpha_m|\,|z|}\\ &= \exp\left[-\sum \log \frac{1-|\alpha_m|\,|z|}{|z|-|\alpha_m|}\right]\\ &\geqslant \exp\left[-\sum \frac{(1+|z|)(1-|\alpha_m|)}{|z|-|\alpha_m|}\right]\\ &\geqslant \exp[-c/d(|z|, \{|\alpha_m|\})]\end{aligned}$$

where d stands for the distance from a point to a set. Here we have used the obvious inequality $\log(x-1) \leqslant x$, $x>1$, and the fact $\sum (1-|\alpha_m|) < \infty$. Similarly,

$$\begin{aligned}|B_2(z)| &= \prod \left|\frac{z-\alpha_m}{1-\bar{\alpha}_m z}\right| \geqslant \prod \frac{|\alpha_m|-|z|}{1-|\alpha_m|\,|z|}\\ &= \exp\left[-\sum \log \frac{1-|\alpha_m|\,|z|}{|\alpha_m|-|z|}\right]\\ &\geqslant \exp\left[-\sum \frac{(1+|z|)(1-|\alpha_m|)}{|\alpha_m|-|z|}\right]\\ &\geqslant \exp[-c/d(|z|, \{|\alpha_m|\})].\end{aligned}$$

The inequality we used here is $\log(1+x) \leqslant x$, $x > 0$. Thus

$$|B(z)| \geqslant \exp[-c/d(|z|, \{|\alpha_m|\})].$$

We need only to find a lower bound for $\sum_n [\exp -c/d_n]$ with $d_n = d(|z_n|, \{|\alpha_m|\})$.

Let $t_m = 1 - |\alpha_m|$ so that t_m is positive and decreases to 0. Then $\sum t_m < \infty$ and $d_n = d((\log\log n)^{-1}, \{t_m\})$ for each n. Let $b_N = (\log\log 2^{2^N})^{-1} = (N\log 2 + \log\log 2)^{-1}$. Then $\{b_N\}$ is a subsequence of $\{(\log\log n)^{-1}\}$ and $\sum b_N$ is a divergent series with positive terms decreasing to 0.

We shall need the following elementary lemma.

LEMMA. *Let $\sum b_N$ and $\sum t_m$ be two series with positive terms decreasing to 0. Suppose $\sum b_N = \infty$ and $\sum t_m < \infty$. Then there exists an infinite collection of intervals $\{(b_{N_k+1}, b_{N_k}), k = 1, 2,...\}$ such that $\{t_m\} \cap \bigcup_k (b_{N_k+1}, b_{N_k}) = \varnothing$.*

Proof of Lemma. Suppose not. Then all but finite many intervals (b_{N+1}, b_N) would contain at least one point in $\{t_m\}$. Thus $\sum t_m \geqslant \sum b_N = \infty$ which is a contradiction.

We now choose the infinite collection of intervals $\{(b_{N_k+1}, b_{N_k})\}$ which has the property stated in Lemma. There are

$$2^{2^{N_k+1}} - 2^{2^{N_k}} = 2^{2^{N_k}}(2^{2^{N_k}} - 1)$$

many points of $\{(\log\log n)^{-1}\}$ lying in the interval (b_{N_k+1}, b_{N_k}) for each $k = 1, 2,...$. Taking I_k to be the middle third of (b_{N_k+1}, b_{N_k}), we see easily that I_k should contain at least

$$2^{2^{N_k}},$$

say, many points of $\{(\log\log n)^{-1}\}$.

For each point of $\{(\log\log n)^{-1}\}$ which belongs to I_k, we have

$$\begin{aligned} d_n &\geqslant \tfrac{1}{3}(b_{N_k} - b_{N_k+1}) \\ &\geqslant cN_k^{-2}. \end{aligned}$$

Thus $\exp[-c/d_n] \geqslant \exp[-cN_k^2]$ and $\sum_n |B(z_n)| \geqslant \sum_n \exp[-c/d_n] \geqslant \sum_k 2^{2^{N_k}} \exp[-cN_k^2] = \infty$ since the general term of the last series even diverges to ∞.

Finally,

$$\begin{aligned}\sum |f(z_n)| &= \sum_n |F(z_n)|\,|S(z_n)|\,|B(z_n)| \\ &\geqslant \sum_k \Bigg(\sum_{\substack{n \\ (\log\log n)^{-1} \in I_k}} |F(z_n)|\,|S(z_n)|\,|B(z_n)| \Bigg) \\ &\geqslant \sum_k \left[c_1 2^{2^{N_k}} (\log N_k)^{-c_2} \exp(-c_3 N_k^2) \right] \\ &= \infty,\end{aligned}$$

and therefore the proof of Theorem 3 is complete.

The above proof actually works for any sequence $\{z_n\}$ with $|z_n| = 1-(\log\log n)^{-1}$, so we have the following extension of Theorem 3.

THEOREM 4. *Let* $|z_n| = 1-(\log\log n)^{-1}$. *Then* $\{z_n\}$ *is* H^∞*-separating.*

With a slight modification of the proof of Theorem 3, we can construct an H^∞-separating sequence $\{z_n\}$ which approaches the boundary with a faster rate. For instance, $|z_n| = 1-(\log\log n)^{-p}$, $p>1$, or $|z_n| = 1-(\log n)^{-p}$, $0<p\leqslant\frac{1}{2}$. We do not know whether $\{z_n\}$ is still H^∞-separating when $|z_n| = 1-(\log n)^{-p}$, $\frac{1}{2}<p<1$.

We remark that in the proof of Theorem 3, one can take $\{b_N\}$ to be any subsequence such that $\{b_N\}$ is in some sense "regularly" spaced and $\sum b_N = \infty$, to begin with. For example, $b_N = (N\log N)^{-1}$.

4. SEQUENCES WHICH ARE NOT H^∞-SEPARATING

In this section, we present some examples which are not H^∞-separating, other than the Blaschke sequences or interpolating sequences for H^∞.

EXAMPLE 1. If $z_n = 1-n^{-1}$, then $\{z_n\}$ is not H^∞-separating. We can take $f(z) = (1-z)^2$. Then $\sum |f(z_n)| = \sum n^{-2} < \infty$.

EXAMPLE 2. If $z_n = 1-(\log n)^{-1}$, then $\{z_n\}$ is not H^∞-separating. We can take f to be the singular inner function $\exp[-(1+z)/(1-z)]$. Then $\sum |f(z_n)| = \sum \exp[-(2-(\log n)^{-1})\log n] \leqslant \sum \exp[-c\log n] = \sum n^{-c} < \infty$ with some constant $c>1$. The power -1 cannot be replaced by a greater number $-p$, $0<p<1$, say, in this argument, since then $\sum \exp[-c(\log n)^p]$ would be divergent with any constant c.

EXAMPLE 3. If $\{z_n\}$ is a sequence approaching 1 nontangentially with $|z_n| = 1 - n^{-1}$ or $|z_n| = 1 - (\log n)^{-1}$, then $\{z_n\}$ is not H^∞-separating. The function f can be chosen the same one as in Example 1 or Example 2.

EXAMPLE 4. In certain cases, $\{z_n\}$ is still not H^∞-separating even when $\{z_n\}$ approaches 1 tangentially with $|z_n| = 1 - n^{-1}$. For instance, we may choose $\{z_n\}$ so that $|z_n| = 1 - n^{-1}$ and z_n is on the orocycle

$$C_\alpha = \{1 - |z|^2 = \alpha |1 - z|^2\}, \qquad \alpha > 0.$$

We can take f to be the same singular inner function in Example 2.

It is not just how "many" of the points in $\{z_n\}$ matter in characterizing the H^∞-separating sequences. This can be illustrated by the following example.

EXAMPLE 5. Let $f \in H^\infty$ with infinitely many distinct zeros $\{\alpha_m\}$ in D. Of course $\{\alpha_m\}$ is a Blaschke sequence. We first choose $\varepsilon_m > 0$, $m = 1, 2,...$ such that $\sum \varepsilon_m < \infty$. Let N_m be an arbitrary positive integer with each $m = 1, 2,...$. Since f is continuous at each α_m, we can find an open disk D_m about α_m such that $|f(z)| \leqslant \varepsilon_m/N_m$ for all $z \in D_m$. In D_m, we can choose N_m (different) points z_n. Then the total collection $\{z_n\}$ is not H^∞-separating since $\sum |f(z_n)| \leqslant \sum_m N_m(\varepsilon_m/N_m) = \sum \varepsilon_m < \infty$. We can have as "many" points as we want in $\{z_n\}$ since $\{N_m\}$ is arbitrary.

Taken together, Theorems 1 and 3 leave one wondering whether considering the set E is really the right way to study H^∞-separating sequences. We do not know whether there is a characterization of H^∞-separating sequences in terms of the rate at which $|z_n| \to 1$, or in terms of anything else.

It is clear that nonseparating sequences for H^∞ must bear some relationship to Blaschke sequences, but the exact relationship remains to be determined. One might try to resolve the following attempt at a characterization. Let $\rho(z, \omega) = |z - \omega|/|1 - \bar{\omega} z|$ be the pseudohyperbolic distance on D [6], and if S is a subset of D then let $\rho(z, S) = \inf\{\rho(z, \omega), \omega \in S\}$. One can speculate that a sequence $\{z_n\}$ is not H^∞-separating if and only if there exists a Blaschke sequence $\{\alpha_m\}$ such that $\sum \rho(z_n, \{\alpha_m\}) < \infty$. The sufficiency of this condition is easily shown. But the necessity fails. Taking $z_n = 1 - (\log n)^{-1}$, which is not H^∞-separating in Example 3, we can apply a similar argument in Theorem 3 to show that $\sum \rho(z_n, \{\alpha_m\}) = \infty$ for all Blaschke sequences $\{\alpha_m\}$. Thus one is led to conjecture that a sequence $\{z_n\}$ is not H^∞-separating if and only if there exists a Blaschke sequence $\{\alpha_m\}$ such that $\sum_n [\prod_m \rho(z_n, \alpha_m)] < \infty$. We do not know whether this condition is necessary.

5. Some Remarks

In this section we shall be investigating the separating problem for other spaces.

(1) First we consider the disk algebra A of continuous functions on the closed unit disk which are analytic on the open disk. Given a sequence $\{z_n\}$ in D, we denote by E' the set of all accumulation points of $\{z_n\}$ on the unit circle. Then results in previous sections can be extended easily to the case of disk algebra.

Theorem 1′. *If $m(E')>0$, then $\{z_n\}$ is A-separating.*

The sequence $\{z_k^n\}$ in Theorem 2 is always A-separating regardless of the rate δ_n approaches 0, since $m(E')=2\pi$. The sequence $\{z_n\}$ in Theorem 3 and Theorem 4 is obviously A-separating since $A\subset H^\infty$. The sequence $\{z_n\}$ in the first four examples of Section 4 are again not A-separating. In Example 2 and Example 4 we need to take f to be the function $(1-z)\exp[-(1+z)/(1-z)]$. Similar to Example 5 we can also construct an arbitrarily "many points" sequence $\{z_n\}$ which is not A-separating if we start with a function $f\in A$.

However, there does exist a sequence $\{z_n\}$ which is A-separating but not H^∞-separating. For example, one can take a Blaschke sequence $\{z_n\}$ which is dense on the unit circle.

(2) Let N be the Nevanlinna class. Then all the results in previous sections still hold true with H^∞ replaced by N. We do not know whether there is a sequence which is H^∞-separating but not N-separating.

(3) Let S be the space of analytic functions in D except possibly with a pole at 0. For any $f\in S$, there exists an integer $N\geqslant 0$ such that

$$f(z)=\sum_{-N}^{\infty} a_k z^k.$$

The following shows that the nonregular part of f, namely, $\sum_{-N}^{0} a_k z^k$, can be determined uniquely by observing $f(z_n)+e_n$, where $\{z_n\}$ is any sequence converging to 0 and e_n, as before, are i.i.d., $N(0,\sigma^2 I)$.

Proposition. *Any sequence $\{z_n\}$ with $z_n\to 0$ is $S/H(D)$-separating.*

Proof. Let $f(z)=\sum_{-N}^{\infty} a_k z^k$, with $N>0$, $a_{-N}\neq 0$. Then $|f(z_n)|\geqslant c\,|z_n|^{-N}$ for n sufficiently large and thus $\sum |f(z_n)|^2\geqslant c\sum |z_n|^{-2N}=\infty$.

(4) Let M be the space of the entire functions which are of at most order 1 and minimum type (i.e., $\log(\max_{|z|\leqslant r}|f(z)|=o(r)$ as $r\to\infty$). Then the sequence $\{z_n\} = \{0, \pm 1, \pm 2, \ldots\}$ is M-separating as a consequence of a theorem of Polya [8, p. 81].

Acknowledgment

The third author wishes to thank the Academia Sinica for their hospitality during his visit there in 1982–1983.

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