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# On the degree of convergence of lemniscates in finite connected domains☆

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**Abstract**

For an arbitrary bounded closed set $E$ in the complex plane with complement $\Omega$ of finite connectivity, we study the degree of convergence of the lemniscates in $\Omega$.


*Keywords:* Domains of finite connectivity; Dini-smooth contour; Lemniscates; Hilbert's theorem

## 1. Introduction

Use $C$ for the complex plane. Let $\infty \in \Omega$ be an unbounded domain of $q$ connectivity which complement $E = \hat{C} \backslash \Omega$ consists of the mutually exterior closed bounded simply connected domains $E_1, E_2, \ldots, E_q$ with respect to the extended complex plane $\hat{C} = C \bigcup \{\infty\}$. Let $\Gamma_j$ be the boundary of $E_j$, $j = 1, 2, \ldots, q$, $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_q$ be the boundary of $\Omega$.

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For analytic Jordan contour $\Gamma$ which consists of a finite number of finite mutually exterior analytic Jordan curves, in 1897, D. Hilbert proved that $\Gamma$ can be approximated by lemniscates which lie in $\Omega$ and consist of one component only in the case $q = 1$ (see [5]). In the general case, the corresponding results have been obtained by many writers, for instance, Faber in 1922, Pólya and Scegö in 1931, Fekete in 1933, and Walsh and Russell in 1934 (for details, see [5]).

Recently, Dolzhenko (cf. [2, p. 21]) raised the problem of estimating the rate of approximation of a closed Jordan curve by lemniscates in the Hausdorff metric in terms of properties of this curve. In 2000, Andrievskii in [1] estimated the rate of approximation of $\Gamma$ from outside by lemniscates in terms of level lines of a conformal mapping of $\Omega$ onto the exterior of the unit disk in the case $q = 1$.

In the present paper, we will estimate the rate of approximation of $\Gamma$ from outside by lemniscates in terms of level lines of the Green's function of $\Omega$ in the case $q \geqslant 1$.

## 2. Main definitions and results

Let $\Gamma_j$, $j = 1, 2, \ldots, q$, be arbitrary mutually exterior Jordan curves of the finite complex plane, and let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_q$ be the boundary of $\Omega$. In this situation, $\Omega$ possesses a Green's function $G(z)$ with pole at infinity, which is harmonic in $\Omega$ except at infinity; which outside of some circle $\gamma$ can be expressed as $\ln |z|$ plus some function harmonic exterior to $\gamma$ and approaching finite value $-g$ at infinity; and which is continuous in the closed domain $\bar{\Omega}$ except at infinity and vanishes on the boundary $\Gamma$ (see [5]). Let $H(z)$ be conjugate to $G(z)$ in $\Omega$.

In the case $q = 1$, let $w = \Phi(z)$ map $\Omega$ conformally and univalently onto $\Delta = \{w : |w| > 1\}$, where $\Phi(z)$ is normalized by the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Then $G(z) = \ln |\Phi(z)|$ in the case $q = 1$.

But in the case $q > 1$, $H(z)$ is a multiple-valued function, therefore $F(z) = G(z) + iH(z)$ is a multiple-valued analytic function in $\Omega$ except at infinity. Moreover (see [5]), the function $\Phi(z) = \exp\{G(z) + iH(z)\}$ maps $\Omega$ conformally but not necessarily one-to-one onto $\Delta$ so that the points at infinity in the two planes correspond to each other. The Green's function $G(z)$ has precisely $q - 1$ critical points (the points for which $F'(z) = 0$), counted according to their multiplicities in $\Omega$, which do not lie on the boundary $\Gamma$ nor at infinity (for details, see [5]).

Therefore

$$\rho = \inf\{\operatorname{dist}(z, \Gamma) | z \in \Omega, \quad F'(z) = 0\} > 0. \tag{1}$$

Because $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$ are mutually exterior bounded closed sets,

$$d = \inf\{d_{jl} | d_{jl} = \operatorname{dist}(\Gamma_j, \Gamma_l),\ j \neq l; \quad j, l = 1, 2, \ldots, q\} > 0. \tag{2}$$

By the argument in [5], when all of the critical points of $G(z)$ are outside of the level line $\Gamma_{1+\delta} = \{z \in \Omega | G(z) = \ln(1+\delta)\}$, the locus $\Gamma_{1+\delta}$ consists of mutually exterior $q$ analytic Jordan curves $\Gamma_{1\delta}, \Gamma_{2\delta}, \ldots, \Gamma_{q\delta}$. Moreover, if $0 < \delta_1 < \delta_2$, then the level line $\Gamma_{1+\delta_2}$ is exterior to $\Gamma_{1+\delta_1}$. Thus there exists $\alpha_0 > 0$, depending on $\rho$ and $d$, such that if $0 < \delta \leqslant \alpha_0$, $\Gamma_{1+\delta}$ consists of mutually exterior $q$ analytic Jordan curves $\Gamma_{1\delta}, \Gamma_{2\delta}, \ldots, \Gamma_{q\delta}$, which is a contour surrounding each component $\Gamma_j$, $j = 1, 2, \ldots, q$, of $\Gamma$.

Let $P_n$, $n = 1, 2, \ldots$, be a polynomial of degree at most $n$. Denote by $J(P_n, \mu)$, $\mu > 0$, the lemniscate

$$J(P_n, \mu) = \{z \mid |P_n(z)| = \mu\}.$$

By the definition in [1], let $S_n(E)$ denote the infimum of $s > 0$ for which there exists a polynomial $P_n = P_{n,s}$ such that $J(P_n, 1)$ is a Jordan contour satisfying the condition

$$E \subset \operatorname{int} J(P_n, 1) \subset \operatorname{int} \Gamma_{1+s}, \tag{3}$$

where $E$ is the complement of $\Omega$, int $\gamma$ denotes the interior of $\gamma$.

In what follows we denote by $c, c_1, c_2, \ldots, m, M, \ldots$, positive constants (different each time, in general) that either are absolute or depend on parameters not essential for the argument.

Using the quantity $S_n(E)$ in [1], Andrievskii obtained the following estimations.

**Theorem** $A_1$**.** *In the case $q = 1$, for arbitrary Jordan curve $\Gamma$, there exists $c > 0$ such that*

$$S_n(E) \leqslant \frac{c \ln n}{n} \quad (n > 1). \tag{4}$$

**Theorem** $A_2$**.** *In the case $q = 1$, let $\Gamma$ be a Jordan curve of bounded secant variation. Then there exists $c > 0$ such that*

$$S_n(E) \leqslant \frac{c}{n}. \tag{5}$$

In the present paper, our main tool estimating $S_n(E)$ is the well-known formula (see [3] or [5])

$$V(z) = \frac{1}{2\pi} \int_\Gamma \ln |z - \xi| \frac{\partial G}{\partial \vec{n}} |d\xi|, \quad z \in \Omega, \tag{6}$$

where $e^g$ is the capacity of $E$, $V(z) = G(z) + g$, $\frac{\partial G}{\partial \vec{n}}$ is the exterior normal derivative with respect to the contour $\Gamma$.

A further remark related to the exterior normal derivative $\frac{\partial G}{\partial \vec{n}}$, which may be proved by using the extended theorem of harmonic function (for detail, see [3–5]) is as follows:

$\frac{\partial G}{\partial \vec{n}}$ exists almost everywhere on $\Gamma$ when $\Gamma$ consists of a finite number of finite mutually exterior Jordan rectifiable curves, and

$$\frac{1}{2\pi}\int_{\Gamma}\frac{\partial G}{\partial \vec{n}}|d\xi| = 1. \tag{7}$$

The partial derivatives $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ have continuous extentions to $\Gamma$ when $\Gamma$ consists of a finite number of finite mutually exterior Dini-smooth curves (cf. [4]), and for any $z = x + iy \in \Gamma$,

$$\frac{\partial G}{\partial \vec{n}} = \frac{\partial G}{\partial x}\cos(\vec{n}, x) + \frac{\partial G}{\partial y}\cos(\vec{n}, y) > 0,$$

where $\cos(\vec{n}, x)$ and $\cos(\vec{n}, y)$ are the direction cosines of the exterior normal vector $\vec{n}$ at the point $z \in \Gamma$ (cf. [5, p. 68]). So $\frac{\partial G}{\partial \vec{n}}$ is continuous on $\Gamma$ when $\Gamma$ consists of a finite number of finite mutually exterior Dini-smooth curves (cf. [4]), and there exist $M, m > 0$ such that

$$m \leqslant \frac{\partial G}{\partial \vec{n}} \leqslant M, \quad \xi \in \Gamma. \tag{8}$$

Moreover, $\sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2}$ is continuous on the bounded closed set $\bar{\Omega}_{\alpha_0}$ bounded by $\Gamma$ and $\Gamma_{1+\alpha_0}$, and there exist $M, m > 0$ such that

$$m \leqslant \sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2} \leqslant M, \quad z \in \bar{\Omega}_{\alpha_0}. \tag{9}$$

Our main results are the following.

**Theorem 1.** *For $q \geqslant 1$, let $\Gamma$ be a Dini-smooth Jordan contour which consists of a finite number of finite mutually exterior Dini-smooth curves. Then for every $c_1 > 0$ there exist $\xi_1, \xi_2, \ldots, \xi_n \in \Gamma$ and $c_2 > 0$ such that*

$$\left| V(z) - \frac{1}{n}\sum_{k=1}^{n}\ln|z - \xi_k| \right| \leqslant \frac{c_2 \ln n}{n} \tag{10}$$

*holds for $z \in \Gamma_{1+\frac{c_1}{n}}$.*

**Theorem 2.** *For $q \geqslant 1$, let $\Gamma$ be a Dini-smooth Jordan contour which consists of a finite number of finite mutually exterior Dini-smooth curves. Then there exists $c > 0$ such that*

$$S_n(E) \leqslant \frac{c \ln n}{n} \quad (n > 1). \tag{11}$$

## 3. Some auxiliary results

Let $\Gamma$ be a Dini-smooth contour which consists of $q$ mutually exterior Dini-smooth Jordan curves. By (6), there exist $n$ arcs $l_k$, $k = 1, 2, \ldots, n$ on $\Gamma$ such that

$$\frac{1}{2\pi}\int_{l_k} \frac{\partial G}{\partial \vec{n}}|d\xi| = \frac{1}{n}. \tag{12}$$

By $\xi_k$ and $\xi_{k+1}$ denote the initial point and end point of $l_k$, respectively. In the positive direction of $\Gamma$, each $l_k$ is an oriented arc. It may occur that $l_k$ consists of two subarcs $l_k{}' \subset \Gamma_{j'}, l_k{}'' \subset \Gamma_{j''}\,(j' \neq j'', 1 \leqslant j', j'' \leqslant q)$, but that can happen for at most $q$ arcs. In this situation, we have

$$\frac{1}{2\pi}\int_{l_k} \frac{\partial G}{\partial \vec{n}}|d\xi| = \frac{1}{2\pi}\int_{l_k{}'} \frac{\partial G}{\partial \vec{n}}|d\xi| + \frac{1}{2\pi}\int_{l_k{}''} \frac{\partial G}{\partial \vec{n}}|d\xi| = \frac{1}{n}.$$

$$\frac{1}{2\pi}\int_{l_k{}'} \frac{\partial G}{\partial \vec{n}}|d\xi| \leqslant \frac{1}{n},\ \frac{1}{2\pi}\int_{l_k{}''} \frac{\partial G}{\partial \vec{n}}|d\xi| \leqslant \frac{1}{n}. \tag{13}$$

If $l_n$ is on $\Gamma_j (1 \leqslant j \leqslant q)$, then its end point $\xi_{n+1}$ coincides with the initial point of the first arc $l_k$ on $\Gamma_j$.

So that

$$V(z) - \frac{1}{n}\sum_{k=1}^{n} \ln|z-\xi_k| = \frac{1}{2\pi}\sum_{k=1}^{n}\int_{l_k}\left[\ln|z-\xi| - \ln|z-\xi_k|\right]\frac{\partial G}{\partial \vec{n}}|d\xi|. \tag{14}$$

Estimating $\text{dist}(\Gamma_{1+\delta}, \Gamma)$ $(0 < \delta \leqslant \alpha_0)$, we have

**Lemma 1.** *There is a constant* $c > 0$ *such that* $\text{dist}(\Gamma_{1+\delta}, \Gamma) \geqslant c\delta$, $0 < \delta \leqslant \alpha_0$.

**Proof.** Since there exists $j$, $1 \leqslant j \leqslant q$, such that

$$\text{dist}(\Gamma_{1+\delta}, \Gamma) = \text{dist}(\Gamma_{\text{j}\delta}, \Gamma_{\text{j}}),$$

there exist $z_1 \in \Gamma_{j\delta}$, $z_2 \in \Gamma_j$ such that

$$|z_1 - z_2| = \text{dist}(\Gamma_{\text{j}\delta}, \Gamma_{\text{j}}) = \text{dist}(\Gamma_{1+\delta}, \Gamma),$$

Let $\gamma_j$ be the straight line segment from $z_1$ to $z_2$. Then $\gamma_j$ is contained in the domain bounded by $\Gamma_j$ and $\Gamma_{j\delta}$ with the exceptional points $z_1$ and $z_2$. The domain $\Omega_{j\delta}$ bounded by $\Gamma_j$, $\Gamma_{j\delta}$ and the crosscut $\gamma_j$ is a simply connected. Hence

$$\begin{aligned}|G(z_1) - G(z_2)| &= \left|\int_{\gamma_j} dG\right| = \left|\int_{\gamma_j}\left[\frac{\partial G}{\partial x}\ \cos(\vec{\tau}, x) + \frac{\partial G}{\partial y}\ \cos(\vec{\tau}, y)\right]|d\xi|\right| \\ &\leqslant M|z_1 - z_2|,\end{aligned}$$

where $M = \sup_{z\in\Omega_{j\alpha_0}} \sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2}$ ($\Omega_{j\alpha_0}$ is the domain bounded by $\Gamma_j$ and $\Gamma_{j\alpha_0}$) is independent of $\delta$, and $\cos(\vec{\tau}, x)$, $\cos(\vec{\tau}, y)$ are the direction cosines of the tangent vector $\vec{\tau}$ on $\gamma_j$. Note that

$$G(z_1) = \ln(1+\delta), G(z_2) = 0, \quad \ln(1+\delta) \geqslant \frac{\delta}{1+\delta} \geqslant \frac{\delta}{1+\delta_0},$$

we get

$$\frac{\delta}{1+\delta_0} \leqslant M|z_1 - z_2|.$$

This completes the proof of Lemma 1. □

Setting

$$A_j = \{l_k | l_k \subset \Gamma_j, 1 \leqslant k \leqslant n\} \cup \{l_k{}' | l_k{}' \subset \Gamma_j, l_k{}'' \subset \Gamma_{j+1}, 1 \leqslant k \leqslant n-1\}$$
$$\cup \{l_k{}'' | l_k{}'' \subset \Gamma_j, l_k{}' \subset \Gamma_{j-1}, 1 < k \leqslant n\}, \quad j = 1, 2, \ldots, q,$$

we have

**Lemma 2.** *For every $c_1 > 0$, suppose that $z \in \Gamma_{j\delta_n}, \alpha_0 > \delta_n \geqslant \frac{c_1}{n}, \xi, \xi_k \in l_k$, or $\xi, \xi_k \in l_k{}'$, or $\xi, \xi_{k+1} \in l_k{}''$, $l_k, l_k{}', l_{k_j}{}''$. Then there exist $c_2, c_3 > 0$ such that*

$$c_3|z - \xi_k| \leqslant |z - \xi| \leqslant c_2|z - \xi_k|$$

*or*

$$c_3|z - \xi_{k+1}| \leqslant |z - \xi| \leqslant c_2|z - \xi_{k+1}|.$$

**Proof.** Obviously, it is enough to prove this for $\xi, \xi_k \in l_k, l_k \in A_j$. By (8) and (12), there exists $c_4 > 0$ such that

$$|l_k| \leqslant \frac{c_4}{n}, \quad 1 \leqslant k \leqslant n, \tag{15}$$

where $|l_k|$ is the arclength of $l_k$. If $z \in \Gamma_{j\delta_n}$, $\xi \in l_k$, we have

$$|z - \xi| \leqslant |z - \xi_k| + |\xi - \xi_k| \leqslant |z - \xi_k| + |l_k|.$$

Lemma 1 implies that there exists $c_2 > 0$ such that

$$|z - \xi| \leqslant c_2|z - \xi_k|.$$

The same reasoning shows that there exists $c_3 > 0$ such that

$$|z - \xi_k| \leqslant \frac{1}{c_3} |z - \xi|.$$

This completes the proof of Lemma 2. □

**Lemma 3.** *Suppose that* $z \in \Gamma_{j\delta_n}$, $j = 1, 2, \ldots, q$. *Then there exists* $c_1 > 0$ *independent of z such that*

$$\int_{\Gamma_j} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant c_1 \ln n$$

*holds for* $\delta_n = \frac{c_2}{n}$ *or* $\delta_n = \frac{c_3 \ln n}{n}$ *where* $c_2, c_3 > 0$ *are arbitrary constants.*

**Proof.** Fix $z \in \Omega$, and choose $\xi_j^* \in \Gamma_j$ such that

$$|z - \xi_j^*| = \text{dist}(\text{z}, \Gamma_\text{j}).$$

Let $\xi_j^{**} \in \Gamma_j$ be the point such that the arc lengths of the subarc $\Gamma'_j$ and $\Gamma_j{}''$ of $\Gamma_j$ between $\xi_j^*$ and $\xi_j^{**}$ equal $\frac{|\Gamma_j|}{2}$.

Since $\Gamma'_j$ is a Dini-smooth arc, the arc length parametrization

$$\xi = \xi(s), \xi \in \left[0, \frac{|\Gamma_j|}{2}\right]$$

with $\xi_j^* = \xi(0)$, $\xi_j^{**} = \xi\left(\frac{|\Gamma_j|}{2}\right)$ satisfies (see [4])

$$c_5|s_1 - s_2| \leqslant |\xi(s_1) - \xi(s_2)| \leqslant c_4|s_1 - s_2|, s_1, s_2 \in \left[0, \frac{|\Gamma_j|}{2}\right] \tag{16}$$

for some $c_4, c_5 > 0$. It is easy to see from (16) that there exists $c_6 > 0$ such that

$$\int_{\Gamma'_j} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant c_6 \left|\ln |z - \xi_j^*|\right|. \tag{17}$$

The same reasoning shows that there exists $c_7 > 0$ such that

$$\int_{\Gamma_j{}''} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant c_7 \left|\ln |z - \xi_j^*|\right|. \tag{18}$$

If $z \in \Gamma_{j\frac{c_2}{n}}$, then by Lemma 1, (17) and (18) imply that there exists $c_8 > 0$ such that

$$\int_{\Gamma_j} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant c_8 \ln n. \tag{19}$$

Note that for sufficient large $n$,

$$\left|\ln \frac{\ln n}{n}\right| < \ln n.$$

Hence it follows from Lemma 1, (17) and (18) that there exists $c_9 > 0$ such that

$$\int_{\Gamma_j} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant c_9 \ln n. \tag{20}$$

holds for $z \in \Gamma_{j\frac{c_3 \ln n}{n}}$, which completes the proof. □

## 4. Proofs of theorems

**Proof of Theorem 1.** It follows from (14) that for any $z \in \Omega$, $k = 1, 2, \ldots, n$,

$$\int_{l_k} \left|\ln|z-\xi| - \ln|z-\xi_k|\right| \frac{\partial G}{\partial \vec{n}} |d\xi| = \int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}} |d\xi|.$$

If $|z-\xi| \geqslant |z-\xi_k|$, then

$$\begin{aligned}\ln\left|\frac{z-\xi}{z-\xi_k}\right| &= \ln\left|1 - \frac{\xi-\xi_k}{z-\xi_k}\right| \\ &\leqslant \ln\left(1 + \frac{|\xi-\xi_k|}{|z-\xi_k|}\right) \\ &\leqslant \frac{|\xi-\xi_k|}{|z-\xi_k|}.\end{aligned} \tag{21}$$

If $|z-\xi| < |z-\xi_k|$, then

$$\begin{aligned}\ln\left|\frac{z-\xi_k}{z-\xi}\right| &= \ln\left|1 + \frac{\xi-\xi_k}{z-\xi}\right| \\ &\leqslant \ln\left(1 + \frac{|\xi-\xi_k|}{|z-\xi|}\right) \\ &\leqslant \frac{|\xi-\xi_k|}{|z-\xi|}.\end{aligned} \tag{22}$$

For any $z \in \Omega$, $l_k \in A_j$, $1 \leqslant k \leqslant n$, $j = 1, 2, \ldots, q$, we therefore have

$$\int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant \int_{l_k} \max\left\{\frac{1}{|z-\xi_k|}, \frac{1}{|z-\xi|}\right\} |\xi-\xi_k| \frac{\partial G}{\partial \vec{n}} |d\xi|. \tag{23}$$

Then it follows from (15) that there exists $c_1 > 0$ such that

$$\int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant \frac{c_1}{n} \int_{l_k} \max\left\{\frac{1}{|z-\xi_k|}, \frac{1}{|z-\xi|}\right\} \frac{\partial G}{\partial \vec{n}} |d\xi|. \tag{24}$$

Fix $z \in \Gamma_{1+\frac{c_2}{n}}$, and let $\Gamma_{j\frac{c_2}{n}}$ be the component of $\Gamma_{1+\frac{c_2}{n}}$ which contains $z (1 \leqslant j \leqslant q)$.

Define

$$B_j = \{l_1, l_2, \ldots, l_n\}\backslash A_j, \quad j = 1, 2, \ldots, q.$$

For any $l_k$, or $l'_k$, or $l_k'' \in B_j$, let $\Gamma_{j_1}$, $j_1 \neq j$, be the curve which contains $l_k$, or $l'_k$, or $l_k''$, for $\xi_k, \xi \in l_k$, or $\xi_k, \xi \in l'_k$, or $\xi_{k+1}, \xi \in l_k''$, we have

$$|z - \xi_k| \geqslant \mathrm{dist}(\Gamma_{\mathrm{j}\alpha_0}, \Gamma_{\mathrm{j}_1}) \geqslant \mathrm{dist}(\Gamma_{1+\alpha_0}, \Gamma),$$

$$|z - \xi_{k+1}| \geqslant \mathrm{dist}(\Gamma_{\mathrm{j}\alpha_0}, \Gamma_{\mathrm{j}_1}) \geqslant \mathrm{dist}(\Gamma_{1+\alpha_0}, \Gamma),$$

$$|z - \xi| \geqslant \mathrm{dist}(\Gamma_{\mathrm{j}\alpha_0}, \Gamma_{\mathrm{j}_1}) \geqslant \mathrm{dist}(\Gamma_{1+\alpha_0}, \Gamma).$$

Application of Lemma 1 implies that there exists $c_3 > 0$ such that

$$\mathrm{dist}(\Gamma_{1+\alpha_0}, \Gamma) \geqslant c_3\alpha_0.$$

It follows from (21) and (22) that there exists $c_4 > 0$ such that

$$\begin{aligned}\sum_{l_k \in B_j} \int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}} |d\xi| &\leqslant \frac{c_4}{n} \sum_{l_k \in B_j} \int_{l_k} \frac{\partial G}{\partial \vec{n}} |d\xi| \\ &\leqslant \frac{c_4}{n} \int_{\Gamma} \frac{\partial G}{\partial \vec{n}} |d\xi| \\ &= \frac{2\pi c_4}{n}. \end{aligned} \tag{25}$$

On the other hand, in the case $l_k \in A_j$, it follows from Lemma 2 and (24) that there exists $c_5 > 0$ such that

$$\int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant \frac{c_5}{n} \int_{l_k} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi|.$$

So that

$$\begin{aligned}\sum_{l_k \in A_j} \int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}} |d\xi| &\leqslant \frac{c_5}{n} \sum_{l_k \in A_j} \int_{l_k} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi| \\ &\leqslant \frac{c_5}{n} \int_{\Gamma_j} \frac{1}{|z-\xi|} \frac{\partial G}{\partial \vec{n}} |d\xi|. \end{aligned} \tag{26}$$

By use of Lemma 3, there exists $c_6 > 0$ such that

$$\sum_{l_k \in A_j} \int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}} |d\xi| \leqslant \frac{c_6 \ln n}{n}. \tag{27}$$

In the case $l_k = l_k' + l_k''$, $l_k' \in A_j, l_k'' \in B_j$, or $l_k' \in B_j, l_k'' \in A_j$, without loss of generality, we assume $l_k' \in A_j, l_k'' \in B_j$. Then it follows from the above that there exists $c_7 > 0$ such that

$$\begin{aligned}&\int_{l_k} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}}|d\xi| \\ &\quad \leqslant \int_{l_k'} \left|\ln\left|\frac{z-\xi}{z-\xi_k}\right|\right| \frac{\partial G}{\partial \vec{n}}|d\xi| + \int_{l_k''} \left|\ln\left|\frac{z-\xi}{z-\xi_{k+1}}\right|\right| \frac{\partial G}{\partial \vec{n}}|d\xi| \\ &\qquad + \int_{l_k''} \left|\ln\left|\frac{z-\xi_k}{z-\xi_{k+1}}\right|\right| \frac{\partial G}{\partial \vec{n}}|d\xi| \\ &\quad \leqslant \frac{c_7 \ln n}{n}. \end{aligned} \tag{28}$$

Thus it follows from (14), (25), (27) and (28) that there exists $c_8 > 0$ such that for sufficient large $n$ and any $z \in \Gamma_{1+\frac{c_2}{n}}$,

$$\begin{aligned}&\left|V(z) - \frac{1}{n}\sum_{k=1}^{n} \ln|z-\xi_k|\right| \\ &\quad \leqslant \frac{c_8 \ln n}{n}. \end{aligned} \tag{29}$$

This completes the proof of Theorem 1. □

**Proof of Theorem 2.** According to (14), put

$$\omega_n(z) = V(z) - \frac{1}{n}\sum_{k=1}^{n} \ln|z-\xi_k|. \tag{30}$$

Then it follows from the properties of the Green's function $G(z)$ of $\Omega$ that the function $\omega_n(z)$ is harmonic in $\Omega$. By Theorem 1, for any $c_1, c_2 > 0$ an application of the maximum principle on $\Gamma_{1+\frac{c_1 \ln n}{n}}$ (because $\Gamma_{1+\frac{c_1 \ln n}{n}}$ is exterior to $\Gamma_{1+\frac{c_2}{n}}$) implies that there exists $c_3 > 0$ for sufficient large $n$, and for $z \in \Gamma_{1+\frac{c_1 \ln n}{n}}$,

$$\omega_n(z) \leqslant \frac{c_3 \ln n}{n}. \tag{31}$$

It follows from (30) and (31) that for $z \in \Gamma_{1+\frac{c_1 \ln n}{n}}$

$$\begin{aligned}\prod_{k=1}^{n} |z-\xi_k| &= e^{nV(z)-n\omega_n(z)} \\ &= e^{ng} \cdot e^{nG(z)-n\omega_n(z)} \\ &\leqslant e^{ng} \cdot n^{c_1+c_3}. \end{aligned} \tag{32}$$

Write

$$p_n^*(z) = \prod_{k=1}^{n} (z - \xi_k).$$

By (32) the properties of the lemniscates (see [5]) imply that

$$\text{int}\, \Gamma_{1+\frac{c_1 \ln n}{n}} \subset \text{int}\, J(p_n^*, e^{ng} \cdot n^{c_1+c_3}). \tag{33}$$

On the other hand, for $z \in \Omega$ satisfying $G(z) \geqslant \ln\left[1 + \frac{3(c_1+c_3)\ln n}{n}\right]$ and large enough $n$, the maximum principle applied on $\Gamma_{1+\frac{3(c_1+c_3)\ln n}{n}}$ for $\omega_n(z)$ shows that (31) holds. So that for $z \in \Omega$ satisfying $G(z) \geqslant \ln\left[1 + \frac{3(c_1+c_3 \ln n)}{n}\right]$ and large enough $n$

$$\begin{aligned} nG(z) - n\omega_n(z) &\geqslant n \ln\left[1 + \frac{3(c_1+c_3)\ln n}{n}\right] - c_3 \ln n \\ &\geqslant \frac{3(c_1+c_3)\ln n}{1 + \frac{3(c_1+c_3)\ln n}{n}} - c_3 \ln n \\ &> 2(c_1+c_3)\ln n - c_3 \ln n \\ &> (c_1+c_3)\ln n. \end{aligned} \tag{34}$$

It follows from (34) that

$$\begin{aligned} \prod_{k=1}^{n} |z - \xi_k| &= e^{ng} \cdot e^{nG(z) - n\omega_n(z)} \\ &\geqslant e^{ng} \cdot n^{c_1+c_3} \end{aligned} \tag{35}$$

and

$$\text{int}\, J(p_n^*, e^{ng} \cdot n^{c_1+c_3}) \subset \text{int}\, \Gamma_{1+\frac{3(c_1+c_3)\ln n}{n}}. \tag{36}$$

Comparing (33) and (36) we get (11), which completes the proof. □

## References

[1] V. Andrievskii, On the approximation of a continuum by lemniscates, J. Approx. Theory 105 (2000) 292–304.
[2] P.A. Borodin, On polynomials most divergent from zero on a domain boundary, Moscow Univ. Math. Bull. 52 (1997) 18–21.
[3] G.M. Golusin, Geometric Theory of Functions of a Complex Variable (Moscow, 1966) Engl. transl.: American Mathematical Society, Providence, RI, 1969.
[4] Ch. Pommerenke, Boundary Behavior of Conformal Maps, Springer, New York, 1992.
[5] J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Plane, fifth ed., American Mathematical Society, Providence, RI, 1969.