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Automorphisms and isomorphisms of Chevalley groups and algebras ☆

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ABSTRACT

An adjoint Chevalley group of rank at least 2 over a rational algebra (or a similar ring), its elementary subgroup, and the corresponding Lie ring have the same automorphism group. These automorphisms are explicitly described.

0. Introduction

Suppose that Φ is a reduced irreducible root system, R is an associative commutative ring with unity, $G(\Phi, R)$ is the corresponding adjoint Chevalley group, and $E(\Phi, R)$ is its elementary subgroup (see Section 5).

There are a lot of results (see, e.g., [Wat80,Pet82,GMi83,HO'M89,Abe93,Che00,Bun07], and references therein[1]) asserting that, under some conditions, all automorphisms of Chevalley (or similar) groups are standard in some sense (depending on what a particular author succeeded to prove). In this paper, we use the most universal and natural definition of standardness suggested by A.E. Zalesskii [Zal83]: an automorphism of an adjoint Chevalley group is called *standard* if it is induced by an automorphism of the corresponding Lie algebra. More precisely, this means the following. Clearly, $E(\Phi, R)$ and $G(\Phi, R)$ embed naturally into the automorphism group of the corresponding Lie algebra $L(\Phi, R)$ over R. A slightly less obvious fact is that (under some conditions, see Section 5) both

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[1] Unfortunately, some interesting papers on this subject (e.g., [Abe93]) contain errors.

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groups are normal in $\mathrm{Aut}_R\, L(\Phi, R)$ and even in the larger group $\mathrm{Aut}_{\mathbb{Z}}\, L(\Phi, R) = \mathrm{Aut}_{\mathbb{Z}}\, R \leftthreetimes \mathrm{Aut}_R\, L(\Phi, R)$ consisting of the automorphisms of this algebra considered as a Lie ring. Thus, each automorphism $f \in \mathrm{Aut}_{\mathbb{Z}}\, L(\Phi, R)$ of the Lie ring induces an automorphism $f' : g \mapsto fgf^{-1}$ of the Chevalley groups $G(\Phi, R)$ and $E(\Phi, R)$. The main results of this paper are the following theorems.

Automorphism theorem. *For any reduced irreducible root system Φ of rank $\geqslant 2$, there exists an integer m such that, for any associative commutative ring R without additive torsion, with unity and $\frac{1}{m}$, all automorphisms of the Chevalley group $G(\Phi, R)$ and its elementary subgroup $E(\Phi, R)$ are standard; the groups $\mathrm{Aut}_{\mathbb{Z}}\, L(\Phi, R)$, $\mathrm{Aut}_{\mathbb{Z}}\, R \leftthreetimes \mathrm{Aut}_R\, L(\Phi, R)$, $\mathrm{Aut}\, G(\Phi, R)$, and $\mathrm{Aut}\, E(\Phi, R)$ are isomorphic; the map $\mathrm{Aut}_{\mathbb{Z}}\, L(\Phi, R) \ni f \mapsto f' \in \mathrm{Aut}\, G(\Phi, R)$ is a group isomorphism; a similar map $\mathrm{Aut}_{\mathbb{Z}}\, L(\Phi, R) \to \mathrm{Aut}\, E(\Phi, R)$ is a group isomorphism also.*

Isomorphism theorem. *For any reduced irreducible root system Φ of rank $\geqslant 2$, there exists an integer m such that, for any associative commutative rings R and R' without additive torsion, with unity and $\frac{1}{m}$, there are natural one-to-one correspondences between the following three sets:*

$$\{\text{group isomorphisms } G(\Phi, R) \to G(\Phi, R')\}, \qquad \{\text{group isomorphisms } E(\Phi, R) \to E(\Phi, R')\}, \quad \text{and}$$
$$\{\text{Lie ring isomorphisms } L(\Phi, R) \to L(\Phi, R')\},$$

i.e., each group isomorphism $G(\Phi, R) \to G(\Phi, R')$ maps $E(\Phi, R)$ onto $E(\Phi, R')$; each group isomorphism $E(\Phi, R) \to E(\Phi, R')$ can be extended uniquely to an isomorphism $G(\Phi, R) \to G(\Phi, R')$; each ring isomorphism $f : L(\Phi, R) \to L(\Phi, R')$ induces a group isomorphism $\mathrm{Aut}_{\mathbb{Z}}\, L(\Phi, R) \supseteq G(\Phi, R) \to G(\Phi, R') \subseteq \mathrm{Aut}_{\mathbb{Z}}\, L(\Phi, R')$ by the formula $\varphi \mapsto f\varphi f^{-1}$; each group isomorphism $G(\Phi, R) \to G(\Phi, R')$ is induced by such a way by a unique ring isomorphism.

Each ring isomorphism $f : L(\Phi, R) \to L(\Phi, R')$ is semilinear, i.e., $f(rx) = \alpha(r) f(x)$ for some ring isomorphism $\alpha : R \to R'$ uniquely determined by f.

In particular, these theorems allow us to describe the automorphisms of all adjoint Chevalley groups of rank at least 2 over any commutative $\mathbb{Q}$-algebras. Similar results were obtained by Yu Chen [Che95,Che96] (see also [Che00]), but he assumed additionally that R is an algebra over $\mathbb{Q}$ without zero divisors.

The idea of describing the automorphisms of linear groups by the passage to the Lie algebras was first introduced and applied by V.M. Levchuk [Lev83] and E.I. Zelmanov [Zel85]. We use this general idea, but our approach is quite different.

The above theorems reduce the problem of finding automorphisms/isomorphisms of Chevalley groups to an (easier) analogous problem for Chevalley algebras. The automorphisms of Chevalley algebra are explicitly described in Section 7. Each automorphism of $L(\Phi, R)$ is a composition of an inner automorphism (i.e., a conjugation by an element of $G(\Phi, R)$) and automorphisms induced by symmetries of the corresponding Dynkin diagram.

Our proofs are completely calculation-free and use only few properties of Chevalley groups. Thus, this approach can work in a more general setting. An *elementary group scheme* E is a subgroup of $\mathbf{SL}_n(\mathbb{Z}[z_1, z_2, \ldots])$ generated by some matrices $\{x_i(z_j);\ i \in I,\ j = 1, 2, \ldots\}$. For an elementary group scheme E, the symbol $E(R)$ denotes the subgroup of $\mathbf{SL}_n(R)$ consisting of all matrices of the form $a(r_1, r_2, \ldots)$, where $a \in E$ and $r_j \in R$. We say that $E(R)$ is an n-dimensional R-*group*. Clearly, $E(R)$ is generated by the matrices $\{x_i(r);\ i \in I,\ r \in R\}$. For an R-group $E(R)$ we consider the following conditions:

- (EX) *Exponentiality*: $x_i(z_1)x_i(z_2) = x_i(z_1 + z_2)$ for all $i \in I$.
- (AL) *Algebraicity*: $E(R[t])$ is a normal subgroup of a linear algebraic group $G \subseteq \mathbf{SL}_n(R[t])$ defined by some polynomial equations with integer coefficients. The group $E(R[t])$ is the normal closure of its subgroup $E(R)$.
- (PC_S) *Power conjugacy*: two matrices x_i and x_i^s are conjugate in $E(R)$ for each $i \in I$ and each $s \in S$, where $S \subseteq \mathbb{Z}$ is a set of integers.

Example. In Section 5, we show that, under the assumption of the above theorems, an adjoint elementary Chevalley group $E(\Phi, R)$ has Properties (EX), (AL), and (PC_S), where $S = \mathbb{Z} \cap \{a^2;\ a \in R^*\}$.

1. Nullstellensatz

Recall that an ideal is called *radical* if the corresponding factor ring has no nonzero nilpotents. We use the following form of Hilbert's Nullstellensatz.

Nullstellensatz. *Suppose that $g, f_1, \ldots, f_l \in \mathbb{Z}[y_1, \ldots, y_m]$ are some polynomials and the quasi-identity*

$$\forall r_1, \ldots, r_l \in R \quad f_1(r_1, \ldots) = 0 \,\&\, \ldots \,\&\, f_l(r_1, \ldots) = 0 \quad \Longrightarrow \quad g(r_1, \ldots) = 0$$

holds for $R = \mathbb{C}$. Then there exists positive integer b such that the quasi-identity

$$\forall r_1, \ldots, r_l \in R \quad f_1(r_1, \ldots) = 0 \,\&\, \ldots \,\&\, f_l(r_1, \ldots) = 0 \quad \Longrightarrow \quad \bigl(g(r_1, \ldots)\bigr)^b = 0$$

holds for any associative commutative ring R with unity and without additive torsion. If the ideal of $\mathbb{Z}[y_1, \ldots, y_m]$ generated by $f_1, \ldots, f_l$ is radical, then we can take $b = 1$.

2. Unipotence

A matrix A is called *unipotent* if $A - 1$ is a nilpotent matrix. We say that an automorphism of an R-group $E(R)$ is *unipotent* if it sends all $x_i(r)$ to unipotent matrices. An automorphism φ is said to be *m-unipotent* if $(\varphi(x_i(r)) - 1)^m = 0$ for all $i \in I$ and $r \in R$.

Proposition 1. *Suppose that an associative commutative ring R with unity has no additive torsion. Then, for any integers $n \geqslant 1$, $p \geqslant 2$, and $d \geqslant 1$, there exist positive integers q and m such that any automorphism of an n-dimensional R-group with Property ($PC_{\{p,q^d\}}$) is m-unipotent.*

Proof. If the R-group satisfies Property ($PC_{\{p,q^d\}}$), then, for any automorphism φ, the matrices $\varphi(x_i(r))$, $\varphi(x_i(r))^p$, and $\varphi(x_i(r))^{q^d}$ are conjugate. Thus, Proposition 1 is a corollary of the following lemma.

Lemma 1. *For any integers $n \geqslant 1$, $p \geqslant 2$, and $d \geqslant 1$, there exist positive integers q and m such that, if the characteristic polynomials of matrices $A, A^p, A^{q^d} \in \mathbf{SL}_n(R)$ over an associative commutative ring R with unity and without additive torsion coincide, then $(A - 1)^m = 0$.*

Proof. First, assume that $R = \mathbb{C}$. Then, these three matrices have the same set of eigenvalues and raising to the power p acts as a permutation of these eigenvalues. Hence, for any eigenvalue λ,

$$\lambda^{p^{n!}-1} = 1 \quad \text{and, by the same reason,} \quad \lambda^{q^{dn!}-1} = 1.$$

Clearly, these equalities imply $\lambda = 1$ if we take, e.g., $q = p^{n!} - 1$ (then $p^{n!} - 1$ and $q^{dn!} - 1$ are coprime). So, the assertion is proven for the case $R = \mathbb{C}$.

The condition

$$\det A = 1 \text{ and the characteristic polynomials of } A,\ A^p, \text{ and } A^{q^d} \text{ coincide} \qquad (*)$$

is a system of integer-coefficient polynomial equations on entries of matrix A. Each complex root $B \in M_n(\mathbb{C})$ of this system is unipotent (if q is chosen as above). By Nullstellensatz, this implies that, if a matrix A over R satisfies $(*)$, then each entry c_{ij} of the matrix $C = (A - 1)^n$ satisfies the equality

$c_{ij}^b = 0$ for some integer b. Hence, $C^{bn^2} = (A-1)^{bn^3} = 0$. This completes the proof of Lemma 1 and Proposition 1. □

3. Curves

Take an R-group $E(R)$. We say that the group $E(R[t])$ is the *group of curves on the group* $E(R)$.

Clearly, for any curve $g(t) \in E(R[t])$ and any polynomial $f(t) \in R[t]$, the curve $\mathrm{REP}_f(g) \overset{\mathrm{def}}{=} g(f(t))$ also belongs to $E(R[t])$. We say that $g(f(t))$ is the *reparametrisation of the curve g by means of the polynomial f*. Thus, $\mathrm{REP}_f : E(R[t]) \to E(R[t])$ is an endomorphism of the group of curves.

Performing fairly standard calculations

$$\left(1+tX+t^2Y+o\left(t^2\right)\right)^{-1} = 1-\left(tX+t^2Y\right)+\left(tX+t^2Y\right)^2+o\left(t^2\right) = 1-tX+t^2\left(X^2-Y\right)+o\left(t^2\right),$$

$$\begin{aligned}
&\left[\left(1+tX_1+t^2Y_1+o\left(t^2\right)\right), \left(1+tX_2+t^2Y_2+o\left(t^2\right)\right)\right]\\
&\quad = \left(1+tX_1+t^2Y_1+o\left(t^2\right)\right)\left(1+tX_2+t^2Y_2+o\left(t^2\right)\right)\\
&\qquad \times \left(1+tX_1+t^2Y_1+o\left(t^2\right)\right)^{-1}\left(1+tX_2+t^2Y_2+o\left(t^2\right)\right)^{-1}\\
&\quad = 1+t^2\left(Y_1+Y_2-Y_1-Y_2+X_1^2+X_2^2+X_1X_2-X_1^2-X_1X_2-X_2X_1-X_2^2+X_1X_2\right)+o\left(t^2\right)\\
&\quad = 1+t^2(X_1X_2-X_2X_1)+o\left(t^2\right),
\end{aligned}$$

we obtain a fairly standard formula:

$$\left[\left(1+tX_1+o(t)\right), \left(1+tX_2+o(t)\right)\right] = \mathrm{REP}_{t^2}\left(1+t[\![X_1, X_2]\!]+o(t)\right)+o\left(t^2\right), \tag{1}$$

where $[x, y] \overset{\mathrm{def}}{=} xyx^{-1}y^{-1}$ is the group commutator and $[\![x, y]\!] \overset{\mathrm{def}}{=} xy-yx$ is the ring commutator.

4. Continuity and smoothness

The set of matrices $T(E(R)) = \{X \in M_n(R) \mid 1+tX+t^2Y \in E(R[t])$ for some $Y \in M_n(R[t])\}$ is called the *tangent module* of an R-group $E(R)$. Clearly, this set is an $R[E(R)]$-module, i.e., it is closed with respect to

- addition: $(1+tX+o(t))(1+tY+o(t)) = 1+t(X+Y)+o(t)$;
- multiplication by scalars: $\mathrm{REP}_{rt}(1+tX+o(t)) = 1+tXr+o(t)$;
- the action of the group $E(R)$: $g(1+tX+o(t))g^{-1} = (1+tgXg^{-1}+o(t))$ (in what follows, we put $g \circ X \overset{\mathrm{def}}{=} gXg^{-1}$).

If the tangent module is a Lie algebra, i.e., if it is closed with respect to ring commutator $[\![A, B]\!] = AB-BA$, we call this module the *tangent algebra*. We say that an n-dimensional R-group $E(R)$ is *adjoint* if $T(E(R))$ is a Lie algebra isomorphic as an $R[E(R)]$-module to R^n (with the natural action of $E(R)$).

We say that an automorphism φ of an R-group $E(R)$ is *quasicontinuous* if it can be extended to an automorphism $\widetilde{\varphi}$ of the group of curves such that $\widetilde{\varphi}$ commutes with all integer-coefficient reparametrisations: $\widetilde{\varphi}(\mathrm{REP}_f(g)) = \mathrm{REP}_f(\widetilde{\varphi}(g))$ for all $g \in E(R[t])$ and all $f \in \mathbb{Z}[t]$. The automorphism φ is called *continuous* if it is quasicontinuous, the automorphism $\widetilde{\varphi}$ is quasicontinuous, the automorphism $\widetilde{\widetilde{\varphi}}$ is quasicontinuous, and so on (infinitely many times).

Put $E_k(R) \overset{\mathrm{def}}{=} E(R[t]) \cap (1+t^kM_n(R[t]))$. Since $\ker \mathrm{REP}_0 = E_1(R)$, we have the equality $\widetilde{\varphi}(E_1(R)) = E_1(R)$ for any continuous automorphism φ. We say that a continuous automorphism φ is *smooth* (two times differentiable) if $\widetilde{\varphi}(E_k(R)) = E_k(R)$ for $k = 1, 2, 3$. Note that the continuity [smoothness] of an automorphism implies the continuity [smoothness] of the inverse automorphism. In Section 5, we show that any continuous automorphism of a Chevalley group is smooth (under some conditions).

Proposition 2. *Any smooth automorphism φ of a group $E(R)$ induces an automorphism $d\varphi$ (the* differential of *φ) of the tangent module considered as an abelian group. In addition, we have*

$$\widetilde{\varphi}\big(1+tX+o(t)\big)=1+td\varphi(X)+o(t) \quad \text{and} \quad d\varphi(g\circ X)=\varphi(g)\circ d\varphi(X)$$
$$\text{for all } g\in E(R) \text{ and } X\in T\big(E(R)\big).$$

If $T(E(R))$ is a Lie algebra, then $d\varphi$ is an automorphism of this algebra considered as a Lie ring.

Proof. If $X \in T(E(R))$, then $1+tX+t^2Y \in E(R[t])$ for some $Y \in M_n(R[t])$ and $\widetilde{\varphi}(1+tX+t^2Y)=1+tZ+o(t)$ for some $Z \in M_n(R)$ (because $E_1(R)$ is an invariant subgroup). Put $d\varphi(X)=Z$. This is well defined, because $\widetilde{\varphi}$ leaves invariant the subgroup $E_2(R)$. The bijectivity of $d\varphi$ follows from the smoothness of φ^{-1}. The equalities

$$\begin{aligned}
&\widetilde{\varphi}\big(\big(1+tX+o(t)\big)\big(1+tY+o(t)\big)\big)=\widetilde{\varphi}\big(1+t(X+Y)+o(t)\big)=1+td\varphi(X+Y)+o(t)\\
&\qquad\qquad\qquad ||\\
&\widetilde{\varphi}\big(1+tX+o(t)\big)\widetilde{\varphi}\big(1+tY+o(t)\big)=\big(1+td\varphi(X)+o(t)\big)\big(1+td\varphi(Y)+o(t)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad =1+t\big(d\varphi(X)+d\varphi(Y)\big)+o(t)
\end{aligned}$$

show that $d\varphi$ is an endomorphism of the additive group. A similar argument

$$\begin{aligned}
&\widetilde{\varphi}\big(g\big(1+tX+o(t)\big)g^{-1}\big)=\widetilde{\varphi}\big(1+tgXg^{-1}+o(t)\big)=1+td\varphi\big(gXg^{-1}\big)+o(t)\\
&\qquad\qquad ||\\
&\varphi(g)\widetilde{\varphi}\big(1+tX+o(t)\big)\varphi(g)^{-1}=\varphi(g)\big(1+td\varphi(X)+o(t)\big)\varphi(g)^{-1}=1+t\varphi(g)d\varphi(X)\varphi(g)^{-1}+o(t)
\end{aligned}$$

proves the equality $d\varphi(g\circ X)=\varphi(g)\circ d\varphi(X)$.

The automorphism $\widetilde{\varphi}$ commutes with integer-coefficient reparametrisations, leaves invariant $E_3(R)$, and, hence, maps equality (1) to

$$\big[\widetilde{\varphi}\big(1+tX_1+o(t)\big),\widetilde{\varphi}\big(1+tX_2+o(t)\big)\big]=\mathrm{REP}_{t^2}\,\widetilde{\varphi}\big(1+t[\![X_1,X_2]\!]+o(t)\big)+o\big(t^2\big).$$

Therefore,

$$\big[\big(1+td\varphi(X_1)+o(t)\big),\big(1+td\varphi(X_2)+o(t)\big)\big]=\mathrm{REP}_{t^2}\big(1+td\varphi\big([\![X_1,X_2]\!]\big)+o(t)\big)+o\big(t^2\big).$$

Applying formula (1) to the left-hand side, we obtain

$$\begin{aligned}
\mathrm{REP}_{t^2}\big(1+t\,[\![d\varphi(X_1),d\varphi(X_2)]\!]+o(t)\big)+o\big(t^2\big)&=\big[\big(1+td\varphi(X_1)+o(t)\big),\big(1+td\varphi(X_2)+o(t)\big)\big]\\
&=\mathrm{REP}_{t^2}\big(1+td\varphi\big([\![X_1,X_2]\!]\big)+o(t)\big)+o\big(t^2\big).
\end{aligned}$$

Thus, $[\![d\varphi(X_1),d\varphi(X_2)]\!]=d\varphi([\![X_1,X_2]\!])$. This proves that $d\varphi$ is an endomorphism of the tangent algebra. □

If $E(R)$ is adjoint, then it embeds naturally into the automorphism group $\mathrm{Aut}_{\mathbb{Z}}\,T(E(R))$ of its tangent algebra considered as a Lie ring.

Proposition 3. *Any smooth automorphism of an adjoint R-group $E(R)$ is standard, i.e., it has the form $\varphi(g) = \alpha g \alpha^{-1}$, where α is an automorphism of the Lie ring $T(E(R))$ normalising the subgroup $E(R)$.*

Proof. This follows immediately from Proposition 2, we can take $\alpha = d\varphi$. □

Proposition 4. *Suppose that a commutative associative ring R with unity and $\frac{1}{q!}$ has no additive torsion, an R-group $E(R)$ has Properties* (EX) *and* (AL)*, and φ and φ^{-1} are mutually inverse q-unipotent automorphisms of $E(R)$. Then these automorphisms are continuous.*

Proof. Take a matrix $a(t) \in E(R[t])$. Clearly, $a(r) \in E(R)$ for any $r \in R$. Let us prove that

$$\textit{the matrix } \varphi\big(a(k)\big) \textit{ depends polynomially on the number } k \in \mathbb{Z},$$

i.e., there exists a matrix $b_a(t) \in \mathbf{SL}_n(R[t])$ such that $\varphi(a(k)) = b_a(k)$ for all $k \in \mathbb{Z}$. (Note that the absence of additive torsion implies the uniqueness of such matrix $b_a(t)$.)

Indeed, it is sufficient to prove this for $a(t) = x_i(rt^l)$, because these matrices generate the group $E(R[t])$. So,

$$\varphi\big(x_i\big(rk^l\big)\big) = \big(\varphi\big(x_i(r)\big)\big)^{k^l} \quad \text{by Property (EX).}$$

But $(\varphi(x_i(r)))^m$ depends polynomially on m, as the matrix $\varphi(x_i(r))$ is unipotent:

$$\big(\varphi\big(x_i(r)\big)\big)^m = (1 + A)^m = 1 + mA + \frac{m(m-1)}{2}A^2 + \cdots + \frac{m(m-1)\cdots(m-q+1)}{q!}A^q.$$

Thus, we can extend the automorphism φ to the group $E(R[t])$ putting $\widetilde{\varphi}(a(t)) \stackrel{\text{def}}{=} b_a(t)$.

Let us prove that $\widetilde{\varphi}(a(t)) = b_a(t)$ lies in $E(R[t])$. For each integer k, the matrix $b_a(k)$ belongs to $E(R)$ and, hence, belongs to the group G defined by integer-coefficient polynomial equations (see Property (AL)). Therefore, the matrix $b_a(t)$ satisfies the same equations. Thus, $b_a(t) \in G$ and we have

$$\begin{array}{ccc} E(R) = \widetilde{\varphi}(E(R)) & \subseteq & E(R[t]) \\ \cap\!| & & \cap\!| \\ \widetilde{\varphi}(E(R[t])) & \subseteq & G \end{array} \quad \text{and}$$

$$\begin{array}{c} E(R[t]) = \langle\langle E(R)\rangle\rangle_{E(R[t])} \subseteq \langle\langle E(R)\rangle\rangle_G \supseteq \langle\langle E(R)\rangle\rangle_{\widetilde{\varphi}(E(R[t]))} = \widetilde{\varphi}(E(R[t])) \\ \| \qquad\qquad \text{(by Property (AL))} \\ E(R[t]), \end{array}$$

where $\langle\langle X\rangle\rangle_H$ means the normal closure of a set X in a group H. Thus, $\widetilde{\varphi}(E(R[t])) \subseteq E(R[t])$.

The automorphism φ^{-1} also can be extended to the group of curves and $\widetilde{\varphi}\widetilde{(\varphi^{-1})}(a(k)) = \widetilde{(\varphi^{-1})}\widetilde{\varphi}(a(k)) = a(k)$ for any $k \in \mathbb{Z}$ and any $a(t) \in E(R[t])$. Clearly, this implies the equalities $\widetilde{\varphi}\widetilde{(\varphi^{-1})}(a(t)) = \widetilde{(\varphi^{-1})}\widetilde{\varphi}(a(t)) = a(t)$ (because R has no additive torsion) and the bijectivity of $\widetilde{\varphi}$.

By the construction, the automorphism $\widetilde{\varphi}$ commutes with all integer-coefficient reparametrisations. So, φ is quasicontinuous. Clearly, $\widetilde{\varphi}$ is also q-unipotent and, hence, quasicontinuous. Thus, an obvious induction argument completes the proof of the continuity of φ. □

5. Chevalley groups

Suppose that Φ is a reduced irreducible root system, $L(\Phi)$ is the corresponding simple complex Lie algebra. The algebra $L(\Phi)$ has a basis $h_1, h_2, \ldots, x_1, x_2, \ldots$ (the *Chevalley basis*) such that the structure constants are integer and the matrices of the operators $(\operatorname{ad} x_i)^k/k!$ are integer and nilpotent for all $k \in \mathbb{N}$. The *Chevalley algebra* is the Lie R-algebra $L(\Phi, R)$ with the same structure constants.

Suppose that $N(\Phi) = \operatorname{Aut}_{\mathbb{C}} L(\Phi)$ is the automorphism group of the algebra $L(\Phi)$, and $G(\Phi) = (\operatorname{Aut}_{\mathbb{C}} L(\Phi))^\circ$ is the connected component of the identity of this group. The algebraic groups $G(\Phi) \subseteq N(\Phi) \subseteq \mathbf{GL}(L(\Phi)) \subset \mathbf{SL}_n(\mathbb{C})$ are defined over $\mathbb{Z}$. Let R be an associative commutative ring with unity and let $N(\Phi, R)$ and $G(\Phi, R)$ be the groups of R-rational points of $N(\Phi)$ and $G(\Phi)$, i.e., the subgroups of $\mathbf{SL}_n(R)$ (where $n = 1 + \dim L(\Phi)$) defined by the same integer-coefficient polynomial equations as the groups $N(\Phi)$ and $G(\Phi)$, respectively (in the Chevalley basis). Note that $N(\Phi, R) = \operatorname{Aut}_R L(\Phi, R)$, because the property of being an automorphism can be written as a system of integer-coefficient polynomial equations (depending on the structure constants). The group $G(\Phi, R)$ is called the (*adjoint*) *Chevalley group*. The group $E(\Phi, R) \subseteq G(\Phi, R)$ generated by the matrices $x_i(r) = \exp(\operatorname{ad} r x_i)$, where $r \in R$, is called the *elementary subgroup* of the Chevalley group $G(\Phi, R)$.

Example. For the root system A_l, we have $L(A_l) = \mathbf{sl}_{l+1}(\mathbb{C})$ is the Lie algebra consisting of all traceless matrices, $L(A_l, R) = \mathbf{sl}_{l+1}(R)$, $G(A_l, R) = \mathbf{PGL}_{l+1}(R)$, and $E(A_l, R) = \mathbf{PE}_{l+1}(R)$ is the subgroup of $\mathbf{PGL}_{l+1}(R)$ generated by the images of the transvections $1 + rE_{ij}$, where $i \neq j$ and $r \in R$. (Note that, for some rings, this group $\mathbf{PGL}_{l+1}(R)$ can be large than the central quotient of the general linear group $\mathbf{GL}_{l+1}(R)$.)

In the following lemma, we summarise some (probably) known properties of Chevalley groups and algebras.

Lemma 2. *Let Φ be a reduced irreducible root system of rank $\geqslant 2$ and let R be an associative commutative ring without additive torsion, with unity and $\frac{1}{6}$. Then*

(i) *the group $E(\Phi, R)$ is an R-group with Properties* (EX) *and* (PC_S), *where $S = \mathbb{Z} \cap \{a^2;\ a \in R^*\}$;*
(ii) *for each subgroup H of $G(\Phi, R)$ normalised by $E(\Phi, R)$, there exists a unique ideal J of R such that H is contained in $G(\Phi, R) \cap (1 + M_n(J))$ and contains the normal closure $\langle\langle \{x_i(r);\ r \in J\} \rangle\rangle_{E(\Phi,R)}$ of the set $\{x_i(r);\ r \in J\}$;*
(iii) *$E(\Phi, R)$ is an automorphism invariant* (*i.e., characteristic*) *subgroup of $G(\Phi, R)$;*
(iv) *$E(\Phi, R)$ satisfies Property* (AL);
(v) $\operatorname{Aut}_{\mathbb{Z}} L(\Phi, R) \simeq \operatorname{Aut}_{\mathbb{Z}} R \leftthreetimes \operatorname{Aut}_R L(\Phi, R)$;
(vi) *in the group $\operatorname{Aut}_{\mathbb{Z}} L(\Phi, R)$, the subgroups $G(\Phi, R)$ and $E(\Phi, R)$ are normal and their centralisers are trivial.*

Proof. (i) Property (EX) follows immediately from the definition. Steinberg's relation R5, $h_i(s) x_i(r) h_i(s)^{-1} = x_i(s^2 r)$ (see, e.g., [VPl96]), where $r \in R$, $s \in R^*$, and $h_i(s) \in E(\Phi, R)$ are some particular matrices, shows that Property (PC_S) holds too.

(ii) Taking into account that $G(\Phi, R)$ is centreless in the adjoint case [AHu88], we see that (ii) is a slightly weakened form of the well-known theorem on subgroups of Chevalley groups normalised by the elementary subgroups [Vas86] (see also [ASu76,Abe89,Gol97,CKe99,VGN06]).

(iii) This was also proven by Vaserstein in [Vas86]. Note that, in [HaV03], it was in fact proven the endomorphism invariance of the elementary subgroup of a Chevalley group.

(iv) The normality of $E(\Phi, R[t])$ in the linear algebraic group $G(\Phi, R[t])$ defined by polynomial equations with integer coefficients follows immediately from (iii). The equality $\langle\langle E(\Phi, R) \rangle\rangle_{E(\Phi,R[t])} = E(\Phi, R[t])$ follows from (ii). Indeed, put $H = \langle\langle E(\Phi, R) \rangle\rangle_{E(\Phi,R[t])}$. The inclusion $E(\Phi, R) \subseteq H \subseteq G(\Phi, R[t]) \cap (1 + M_n(J))$ implies $J = R[t]$. Therefore, $E(\Phi, R[t]) = \langle\langle \{x_i(f);\ f \in R[t]\} \rangle\rangle_{E(\Phi,R[t])} = \langle\langle \{x_i(f);\ f \in J\} \rangle\rangle_{E(\Phi,R[t])} \subseteq H$ and $H = E(\Phi, R[t])$.

(v) Let U be the algebra $L(\Phi, R)$ considered as a left module over itself. Then

$$\operatorname{End}_{L(\Phi,R)} U = R, \quad \text{i.e., all endomorphisms are scalar multiples of the identity.} \qquad (**)$$

Indeed, this is true for $R = \mathbb{C}$, because the algebra $L(\Phi, \mathbb{C})$ is simple. Therefore, $(**)$ holds for any R without additive torsion, because both conditions on a matrix, being an endomorphism of U and being a scalar multiple of the identity, are integer-coefficient systems of linear equations on the entries of the matrix.

Note that $(**)$ remains valid if we consider U as a module over Lie ring $L(\Phi, R)$, i.e., each endomorphism f of U must be R-linear. Indeed, for any $u \in U$, there exist $y_i \in L(\Phi, R)$ and $u_i \in U$ such that $u = \sum (\operatorname{ad} y_i)(u_i)$, because $L(\Phi, R) = [L(\Phi, R), L(\Phi, R)]$. Therefore,

$$ru = \sum (\operatorname{ad} r y_i)(u_i) \quad \text{and}$$

$$f(ru) = f\Bigl(\sum (\operatorname{ad} r y_i)(u_i)\Bigr) = \sum (\operatorname{ad} r y_i) f(u_i) = r \sum (\operatorname{ad} y_i) f(u_i) = r f(u).$$

Now, take an automorphism φ of the ring $L(\Phi, R)$ and consider the algebra $L(\Phi, R)$ as an $L(\Phi, R)$-module U_φ with action $(y, u) \mapsto (\operatorname{ad} \varphi(y))u$. The mapping $u \mapsto \varphi(u)$ is an isomorphism between the modules U and U_φ over the Lie ring $L(\Phi, R)$. This isomorphism induces an isomorphism of endomorphism rings $R = \operatorname{End}_{L(\Phi,R)} U \xrightarrow{\alpha_\varphi} \operatorname{End}_{L(\Phi,R)} U_\varphi = R$. Thus, we have a homomorphism $\operatorname{Aut}_{\mathbb{Z}} L(\Phi, R) \to \operatorname{Aut}_{\mathbb{Z}} R$, $\varphi \mapsto \alpha_\varphi$, whose kernel is $\operatorname{Aut}_R L(\Phi, R)$. The right inverse homomorphism $\operatorname{Aut}_{\mathbb{Z}} R \to \operatorname{Aut}_{\mathbb{Z}} L(\Phi, R)$ maps $\alpha \in \operatorname{Aut}_{\mathbb{Z}} R$ to the obvious automorphism of the Lie ring $L(\Phi, R) = L(\Phi, \mathbb{Z}) \otimes R$ induced by α. So, we obtain the required decomposition of $\operatorname{Aut}_{\mathbb{Z}} L(\Phi, R)$ into the semidirect product.

(vi) **Normality.** By virtue of (iii), it is sufficient to prove the normality of $G(\Phi, R)$. For $R = \mathbb{C}$, this property is well known, see, e.g., [VOn88]. Let $F_N(y_{ij}) = 0$ and $F_G(y_{ij}) = 0$ be systems of integer-coefficient polynomial equations that define the groups $N(\Phi, R) = \operatorname{Aut}_R L(\Phi, R)$ and $G(\Phi, R)$ (these systems do not depend on R). We assume that the ideals of $\mathbb{Z}[y_{11}, y_{12}, \ldots, y_{nn}]$ generated by the sets of polynomials $F_N(y_{ij})$ and $F_G(y_{ij})$ are radical. For $R = \mathbb{C}$ we have the quasi-identity

$$F_G(Y) = 0 \quad \& \quad F_N(Z) = 0 \quad \Longrightarrow \quad F_G\bigl(ZYZ^{-1}\bigr) = 0. \qquad (2)$$

Since the ideal of $\mathbb{Z}[y_{11}, y_{12}, \ldots, y_{nn}, z_{11}, z_{12}, \ldots, z_{nn}]$ generated by $F_G(Y)$ and $F_N(Z)$ is radical, Nullstellensatz implies that quasi-identity (2) holds for all rings R without additive torsion. Thus, $G(\Phi, R)$ is a normal subgroup of $N(\Phi, R)$.

Centralisers. For $R = \mathbb{C}$, the centraliser of the set $\{x_i(1)\}$ in $\operatorname{Aut}_R L(\Phi, R)$ is trivial. Therefore, the same is true for any ring R without additive torsion (by Nullstellensatz). Thus, the centraliser of the set $\{x_i(1)\}$ in the group $\operatorname{Aut}_{\mathbb{Z}} L(\Phi, R) = (\operatorname{Aut}_{\mathbb{Z}} R) \leftthreetimes \operatorname{Aut}_R L(\Phi, R)$ coincide with $\operatorname{Aut}_{\mathbb{Z}} R$. On the other hand, each nontrivial ring automorphism $\alpha \in \operatorname{Aut}_{\mathbb{Z}} R$ induces a nontrivial automorphism $x_i(r) \mapsto x_i(\alpha(r))$ of $E(\Phi, R)$. Therefore, the centraliser of $E(\Phi, R)$ in the group $\operatorname{Aut}_{\mathbb{Z}} L(\Phi, R)$ is trivial. This completes the proof of Lemma 2.

Proposition 5. *Let Φ be a reduced irreducible root system of rank $\geqslant 2$ and let R be an associative commutative ring without additive torsion, with unity and $\frac{1}{6}$. Then any retraction $\pi : E(\Phi, R[t]) \to E(\Phi, R)$ (i.e., a homomorphism such that $\pi^2 = \pi$) has the form $E(\Phi, R[t]) \ni a(t) \mapsto a(r) \in E(\Phi, R)$ for some $r \in R$. In other words, $\pi = \mathrm{REP}_r$.*

Proof. According to Lemma 2 (ii),

$$\bigl\langle\!\bigl\langle \{x_i(f);\ f \in J\} \bigr\rangle\!\bigr\rangle_{E(\Phi,R[t])} \subseteq \ker \pi \subseteq E\bigl(\Phi, R[t]\bigr) \cap \bigl(1 + M_n(J)\bigr) \quad \text{for some ideal } J \text{ of } R[t].$$

The right-hand inclusion and the equality $E(\Phi, R[t]) = E(\Phi, R) \leftthreetimes \ker\pi$ show that $t - r \in J$ for some $r \in R$; the left-hand inclusion and the equality $E(\Phi, R) \cap \ker\pi = \{1\}$ show that $J = (t - r)R[t]$. Therefore, $\ker\pi = E(\Phi, R[t]) \cap (1 + M_n(J))$ and $\pi = \mathrm{REP}_r$. $\square$

Thus, we have a natural one-to-one correspondence between the ring R and the set of retractions. Clearly, the ring structure on R can also be described in terms of retractions and integer-coefficient reparametrisations:

$$\begin{aligned} \mathrm{REP}_{r+r'} : E\bigl(\Phi, R[t]\bigr) &\xrightarrow{t \to t+t'} E\bigl(\Phi, R\bigl[t, t'\bigr]\bigr) \xrightarrow[t' \to r']{t \to r} E(\Phi, R), \\ \mathrm{REP}_{rr'} : E\bigl(\Phi, R[t]\bigr) &\xrightarrow{t \to tt'} E\bigl(\Phi, R\bigl[t, t'\bigr]\bigr) \xrightarrow[t' \to r']{t \to r} E(\Phi, R). \end{aligned} \tag{3}$$

Proposition 5 and these formulae imply that any continuous automorphism $\varphi \in \operatorname{Aut} E(\Phi, R)$ induces a ring automorphism $\widehat{\varphi} \in \operatorname{Aut}_{\mathbb{Z}} R$ by the formula $\varphi\,\mathrm{REP}_r\,\widetilde{\varphi}^{-1} = \mathrm{REP}_{\hat{\varphi}(r)}$:

$$\begin{array}{ccc} E(\Phi, R[t]) & \xrightarrow{\ \widetilde{\varphi}\ } & E(\Phi, R[t]) \\ \downarrow{\scriptstyle \mathrm{REP}_r} & & \downarrow{\scriptstyle \mathrm{REP}_{\hat{\varphi}(r)}} \\ E(\Phi, R) & \xrightarrow{\ \varphi\ } & E(\Phi, R). \end{array}$$

For each ideal $J \lhd R$ we have two normal subgroups of $E(\Phi, R)$, namely, $E(J)_{\max} \overset{\text{def}}{=} E(\Phi, R) \cap (1 + M_n(J))$ and $E(J)_{\min} \overset{\text{def}}{=} \langle\langle \{x_i(r);\ r \in J\} \rangle\rangle_{E(\Phi,R)}$.

Lemma 3. *Let Φ be a reduced irreducible root system of rank $\geqslant 2$ and let R be an associative commutative ring without additive torsion, with unity and $\frac{1}{6}$. Then $\varphi(E(J)_{\min}) = E(\widehat{\varphi}(J))_{\min}$ and $\varphi(E(J)_{\max}) = E(\widehat{\varphi}(J))_{\max}$ for any continuous automorphism φ of the group $E(\Phi, R)$.*

Proof. Clearly, it is sufficient to define $E(J)_{\min}$ and $E(J)_{\max}$ in terms of retractions. The subgroup $E_1(\Phi, R) \overset{\text{def}}{=} E(\Phi, R[t]) \cap (1 + tM_n(R[t]))$ can be defined as $E_1(\Phi, R) = \ker \mathrm{REP}_0$ (hence, this subgroup is $\widetilde{\varphi}$-invariant). Then,

$$E(J)_{\min} = \bigl\langle\!\bigl\langle \bigl\{\mathrm{REP}_r\bigl(a(t)\bigr);\ r \in J,\ a(t) \in E_1(\Phi, R)\bigr\} \bigr\rangle\!\bigr\rangle_{E(\Phi,R)}.$$

The inclusion $\supseteq$ follows from the equality $E_1(R) = \langle\langle \{x_i(rt^k);\ \ i \in I,\ \ r \in R,\ \ k = 1, 2, \ldots\} \rangle\rangle_{E(\Phi,R[t])}$, which is valid for any R-group with Property (EX).

$$\begin{gathered} E(J)_{\max} = \text{the (unique) maximal subgroups among all normal subgroups } H \\ \text{such that } E(J)_{\min} \subseteq H \text{ and } E\bigl(J'\bigr)_{\min} \nsubseteq H \text{ for any ideal } J' \nsubseteq J. \end{gathered}$$

The correctness of this definition of $E(J)_{\max}$ follows from Lemma 2 (ii) and the equality $E(J_1 + J_2)_{\min} = E(J_1)_{\min} \cdot E(J_2)_{\min}$. $\square$

Lemma 4. *Let Φ be a reduced irreducible root system of rank $\geqslant 2$ and let R be an associative commutative ring without additive torsion, with unity and $\frac{1}{6}$. Then any continuous automorphism φ of $E(\Phi, R)$ is smooth.*

Proof. We have to prove that the subgroups $E_k(\Phi, R) \overset{\text{def}}{=} E(\Phi, R[t]) \cap (1 + t^k M_n(R[t]))$ are $\widetilde{\varphi}$-invariant. This is true for $k = 1$, because $E_1(\Phi, R) = \ker \mathrm{REP}_0$. On the other hand, $E_1 = E(tR[t])_{\max}$. Hence, the ideal $tR[t] \triangleleft R[t]$ is $\widehat{\widetilde{\varphi}}$-invariant by Lemma 3. Then, the ideal $(tR[t])^k$ is $\widehat{\widetilde{\varphi}}$-invariant and the subgroup $E_k(\Phi, R) = E((tR[t])^k)_{\max}$ is $\widetilde{\varphi}$-invariant. □

Proposition 6. *The tangent module of a Chevalley group coincide with the corresponding Lie algebra*: $T(E(\Phi, R)) = L(\Phi, R)$.

Proof. Suppose that $X \in T(E(\Phi, R))$, i.e., $1 + tX + o(t) \in E(\Phi, R[t])$. Let us express this element via the generators:

$$1 + tX + o(t) = \prod_j x_{i_j}\big(r_j t^{k_j}\big). \tag{4}$$

Clearly, we can assume that $k_j \in \{0, 1\}$. Also, the substitution $t = 0$ shows that

$$\prod_j{}' x_{i_j}(r_j) = 1, \quad \text{where the prime means that the product is taken over all } j \text{ such that } k_j = 0.$$

Therefore, the expression (4) can be rewritten in the form

$$1 + tX + o(t) = \prod_l g_l x_{i_l}(r_l t) g_l^{-1}, \quad \text{where } g_l \in E(\Phi, R).$$

Hence, $X = \sum g_l \circ r_l x_{i_l} \in L(\Phi, R)$ and $T(E(\Phi, R)) \subseteq L(\Phi, R)$.

Let us prove the opposite inclusion. Clearly, $T(E(\Phi, R))$ contains the nilpotent part $\{x_i\}$ of the Chevalley basis: $x_i(t) = \exp(tx_i) = 1 + tx_i + o(t)$. The remaining basis vectors h_i lie in $T(E(\Phi, R))$ also, because $h_i = x_i(1) \circ x_{-i} + x_i - x_{-i}$ (see, e.g., [Bor70]). This completes the proof. □

In particular, Proposition 6 shows that any adjoint Chevalley group is adjoint in the sense of Section 4.

6. Proof of the main theorems

The automorphisms of $E(\Phi, R)$**.** By Lemma 2 (vi), we have the natural injective homomorphism $\Pi : \operatorname{Aut}_{\mathbb{Z}} L(\Phi, R) \to \operatorname{Aut} E(\Phi, R)$. By Proposition 1 and Lemma 2 (i), each automorphism of $E(\Phi, R)$ is unipotent (for a suitably chosen m) and, hence, continuous (by Proposition 4 and Lemma 2 (i) and (iv)) and, therefore, smooth by Lemma 4. Then the map Π is surjective by Propositions 3 and 6. Thus, $\operatorname{Aut} E(\Phi, R) \simeq \operatorname{Aut}_{\mathbb{Z}} L(\Phi, R) \simeq \operatorname{Aut}_{\mathbb{Z}} R \leftthreetimes \operatorname{Aut}_R L(\Phi, R)$ (the latter isomorphism holds by Lemma 2 (v)).

The automorphisms of $G(\Phi, R)$ are the same as of $E(\Phi, R)$. Indeed, each automorphism of $E(\Phi, R)$ is standard and, hence, can be extended to an automorphism of $G(\Phi, R)$ by Lemma 2 (vi). Thus, the natural map $\operatorname{Aut} G(\Phi, R) \to \operatorname{Aut} E(\Phi, R)$ is surjective (and well defined by Lemma 2 (iii)). This map is also injective, because of Lemma 2 (vi) and the following general fact.

Lemma 5. *If A is an automorphism invariant subgroup of a group B and the centraliser of A in B is trivial, then the natural map $\rho : \operatorname{Aut} B \to \operatorname{Aut} A$ is injective.*

Proof. For any $\varphi \in \ker \rho$, $a \in A$, and $b \in B$, we have $bab^{-1} = \varphi(bab^{-1}) = \varphi(b)\varphi(a)\varphi(b^{-1}) = \varphi(b)a\varphi(b^{-1})$. Therefore, $b^{-1}\varphi(b)$ centralises A. Hence, $b = \varphi(b)$ for any $b \in B$. This completes the proof of the automorphism theorem. □

The isomorphism theorem is an easy corollary of the automorphism theorem. Each isomorphism of Chevalley groups $\sigma : G(\Phi, R) \to G(\Phi, R')$ induces an automorphism φ_σ of the group $G(\Phi, R \times R')$, because this group is isomorphic to $G(\Phi, R) \times G(\Phi, R')$ and we can put $\varphi_\sigma(g, g') = (\sigma^{-1}(g'), \sigma(g))$.

The standardness of φ_σ implies that σ is induced by an isomorphism of the corresponding Lie rings. A similar argument applies to elementary subgroups.

7. Automorphisms of Chevalley algebras

Recall that an *inner automorphism* of a Chevalley algebra $L(\Phi, R)$ is a conjugation $x \mapsto gxg^{-1}$ by an element g of the corresponding Chevalley group $G(\Phi, R)$. Clearly, the inner automorphisms form a group isomorphic to $G(\Phi, R)$.

Let $\Delta = \{\delta_1, \dots, \delta_d\}$ be the symmetry group of the Dynkin diagram of Φ (the number d can be 1, 2, or 6, depending on Φ) and let $R = R_1 \oplus \dots \oplus R_d$ be a (possibly trivial) decomposition of the ring R into a direct sum of ideals. Suppose that $f_i \in \operatorname{Aut}_{R_i} L(\Phi, R_i)$ is the automorphism induced by the symmetry δ_i (see [VOn88]). The automorphism f of the algebra $L(\Phi, R) = L(\Phi, R_1) \oplus \dots \oplus L(\Phi, R_d)$ that sends $x_1 + \dots + x_d$ to $f_1(x_1) + \dots + f_d(x_d)$, where $x_i \in L(\Phi, R_i)$, is called a *diagram automorphism* of the algebra $L(\Phi, R)$. Clearly, diagram automorphisms form a group isomorphic to the subgroup

$$D(\Phi, R) = \left\{ \sum e_i \delta_i \;\middle|\; e_i \in R,\ e_i^2 = e_i,\ e_i e_j = 0 \text{ for } i \neq j,\ \sum e_i = 1 \right\}$$

of the group of units of the group algebra $R\Delta$.

Theorem 1. *Let R be an associative commutative ring without additive torsion, with unity and $\frac{1}{6}$ and let Φ be a reduced irreducible root system. Then any automorphism f of the Lie R-algebra $L(\Phi, R)$ can be expressed uniquely as a composition of diagram and inner automorphisms,* $\operatorname{Aut}_R L(\Phi, R) \simeq D(\Phi, R) \leftthreetimes G(\Phi, R)$.

Proof. Let n be the dimension of the Lie algebra $L(\Phi)$. Consider the ideal J in $\mathbb{Z}[x_{11}, x_{12}, \dots, x_{nn}]$ defining the group $\operatorname{Aut}_{\mathbb{C}} L(\Phi)$. The ideal J decomposes into a product $J = J_1 J_2 \cdots J_d$ of prime ideals J_i corresponding to irreducible (= connected) components $h_i G(\Phi)$ of the group $\operatorname{Aut}_{\mathbb{C}} L(\Phi)$, where h_i are integer matrices of diagram automorphisms. Take a matrix $A = (a_{pq}) \in \operatorname{Aut}_R L(\Phi, R)$. Then $f(a_{pq}) = 0$ for $f \in J$. Put $I_i = \{f(a_{pq});\ \ f \in J_i\} \lhd R$. Then

(i) $\prod I_i = \{0\}$;
(ii) $I_i + I_j = R$ for $i \neq j$ (otherwise we take the factor ring by a maximal ideal $M \supseteq I_i + I_j$ and obtain a matrix A_M belonging to the intersection of two irreducible components of the group $\operatorname{Aut}_{R/M} L(\Phi, R/M)$, but this intersection is empty, because R/M is a field).

These conditions (i) and (ii) imply that the ring R is the direct sum $R = \bigoplus R/I_i$ [Bou61, Ch. 2 §1, Proposition 5]. So, $A = \sum A_{I_i}$ and the entries of the matrix $A_{I_i} \in M_n(R/I_i)$ satisfy the equations $f(a_{pq}) = 0$ for $f \in I_i$. Therefore, $A_{I_i} = h_i g_i \in h_i G(\Phi, R/I_i)$ and $A = (\sum e_i h_i)(\sum g_i)$, where e_i is the unity of the ring R/I_i. This completes the proof. □

Another approach to describing the automorphisms of Chevalley algebras was suggested in [Pia02].

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